CanCNSM 2008
Toronto
June 25-29, 2008

CONSTITUTIVE RESTRICTIONS FOR A HYPERELASTIC LAMINAR MATERIAL WITH ONE FIBER FAMILY, BASED UPON STRAIN INVARIANTS YIELDING ORTHOGONAL STRESS RESPONSE TERMS

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Key Words: Anisotropic materials, Strong ellipticity, Finite strain

Abstract. *Constitutive restrictions on the strain energy density from the strong ellipticity condition are derived for an anisotropic hyperelastic material which arises in models of myocardium. Following Criscione et al,[1] the strain energy density W is modeled as a function of six rotation invariant scalars, chosen so as to minimize covariance amongst the response terms.*

1 INTRODUCTION

The study of anisotropic materials is of great significance in a lot of applications in industry, engineering and medicine. Although it is recognized that soft tissues are in essence viscoelastic materials, in certain cases they can be effectively modeled within the framework of nonlinear elasticity theory. Myocardium, for example, is often modeled as a laminar elastic material with one family of parallel fibers. The present paper derives constitutive restrictions on the strain energy density for such material models arising from the condition of strong ellipticity.

Convexity properties of the strain-energy function are of utmost importance. In particular, they are central to stability issues of the material model (*strong ellipticity*), and they are crucial to establishing existence theorems for the problem of nonlinear elasticity (*quasiconvexity*). The quasiconvexity condition involves an integral inequality which is difficult to work with. For this reason, the most commonly considered convexity-type conditions are polyconvexity (stronger than quasiconvexity) and strong ellipticity (a weaker condition). It should be noted that there are examples of strongly elliptic functions which are not quasiconvex, but these examples do not satisfy frame indifference. There are strong arguments that strong ellipticity is the physically more compelling condition to consider,[2] despite the fact that existence and regularity of solutions for elastic boundary value problems remain open for strongly elliptic strain energies. This is still an active area of research.

In the case of isotropic hyperelastic materials significant progress has been made in the derivation of necessary and sufficient conditions for strong ellipticity. Among the notable contributions in this area one should mention the works of Knowles and Sternberg,[3,4] Zee and Sternberg,[5] Horgan,[6] Rosakis[7] and Wang and Aron.[8] Unlike in the isotropic case, the study of convexity properties of the constitutive function of anisotropic materials is far from being complete. The work by Walton and Wilber[9,10] offers a step in this direction.

In all of the works cited above the strain energy density is expressed as a function of the principal invariants of the right Cauchy-Green strain tensor. Although this traditional approach is mathematically very elegant, experimentally it presents certain problems. As pointed out by Criscione,[11] the standard invariants lead to models which are ill-suited for fitting parameters to experimental data since they yield highly covariant stress response terms. In a series of papers,[1,12–14] Criscione et al construct invariant sets with orthogonal or nearly orthogonal stress response terms for materials exhibiting various types of behavior - isotropic, transversely isotropic, orthotropic, etc. In addition to their important orthogonality properties, these novel sets of invariants have straightforward physical interpretation. In the case of an orthotropic material considered here, the traditional approach has an additional drawback (first pointed out by Green and Adkins[15]). It uses 7 invariants while strain has only 6 independent components. One usually circumvents this problem by choosing a dependent invariant, but then the stress response terms depend on the choice of dependent invariant which clouds their physical interpretation.

The current paper derives constitutive restrictions on the strain energy function of an anisotropic hyperelastic material. Deriving necessary and sufficient conditions on the strain energy function for strong ellipticity leads to systems of multivariate polynomial equations the complexity of which makes finding all possible solutions a daunting task. Nevertheless, we present various sets of necessary conditions on the strain energy for strong ellipticity. Deriving necessary and sufficient conditions for strong ellipticity leads to interesting questions in algebraic geometry. Recent developments in the area of computational algebraic geometry provide tools to address these questions.

2 PRELIMINARIES

Let Lin denote the space of second order tensors (linear operators from $\mathbb{R}^3$ to $\mathbb{R}^3$). The elements of Lin will be denoted by uppercase boldface letters with the exception of the Cauchy stress tensor, which is denoted by $\mathbf{t}$. For $\mathbf{A}, \mathbf{B} \in$ Lin, $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$ defines an inner product on Lin. For a second order tensor $\mathbf{A}$, $\mathrm{Sym}(\mathbf{A})$ denotes the symmetric part of $\mathbf{A}$, i.e.

$$\mathrm{Sym}(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T).$$

Vectors (elements in $\mathbb{R}^3$) will be denoted by lowercase boldface letters. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ the diad $\mathbf{a} \otimes \mathbf{b}$ is the second order tensor whose action on a first order tensor $\mathbf{v} \in \mathbb{R}^3$ is defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$. $\mathcal{T}^4$ denotes the space of fourth order tensors (linear operators form Lin to Lin) and the action of a fourth order tensor $\mathcal{L} \in \mathcal{T}^4$ on a second order tensor $\mathbf{A} \in$ Lin will be denoted by $\mathcal{L}[\mathbf{A}]$ or equivalently by $\mathcal{L} : \mathbf{A}$.

Let $\mathcal{B}$ be a reference configuration of a given body and let $\mathbf{f}$ be a deformation of $\mathcal{B}$. Let $\mathbf{F} = \nabla \mathbf{f}$ denote the local deformation gradient, $J = \det \mathbf{F}$, and let $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ be the right Cauchy-Green strain tensor. Also, let $\mathbf{T}$ defined on $\mathcal{B}$ be the first Piola-Kirchhoff stress tensor associated with $\mathbf{f}$. A body is said to be *elastic* provided $\mathbf{T}(\mathbf{x}) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{B}$. A material is said to be *hyperelastic* if its material behavior can be characterized by a scalar-valued function W (called the *strain energy density*) which depends on the local deformation gradient $\mathbf{F}$. The constitutive function $\hat{\mathbf{T}}$ is said to be *strongly elliptic* at a deformation $\mathbf{F}_0$ provided the elasticity tensor $\frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}(\mathbf{F}_0)$ satisfies

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\hat{\mathbf{T}}(\mathbf{F}_0) : \mathbf{H} > 0, \qquad \forall\, \mathbf{H} = \mathbf{a} \otimes \mathbf{b}, \text{ with } |\mathbf{a}| = |\mathbf{b}| = 1. \tag{1}$$

For an incompressible material

$$\hat{\mathbf{T}}(\mathbf{F}) = -p\mathbf{F}^{-T} + \hat{\mathbf{T}}_A$$

where $\hat{\mathbf{T}}_A = \frac{\partial W}{\partial \mathbf{F}}$ is the active part of the first Piola-Kirchhoff stress tensor. An incompressible material is strongly elliptic provided

$$\begin{aligned} &\mathbf{H} : \frac{\partial}{\partial \mathbf{F}} \hat{\mathbf{T}}_A(\mathbf{F}_0) : \mathbf{H} > 0, \\ &\forall\ \mathbf{H} = \mathbf{a} \otimes \mathbf{b},\ \text{with } |\mathbf{a}| = |\mathbf{b}| = 1 \text{ such that } \mathbf{F}_0^{-T} : \mathbf{H} = 0. \end{aligned} \tag{2}$$

That is, $(2)_1$ is required to hold for all rank-1 tensors $\mathbf{H}$ that are tangent to the constraint manifold $\{\mathbf{F} : \det(\mathbf{F}) = 1\}$.

We consider a hyperelastic material having one family of parallel fibers arranged in laminae. Let $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{S}$ be three orthonormal vectors in the reference configuration such that $\mathbf{M}$ is parallel to the fiber direction, $\mathbf{S}$ is orthogonal to $\mathbf{M}$ and parallel to the laminar plane, and $\mathbf{N} = \mathbf{M} \times \mathbf{S}$.
Define the following five scalar functions of $\mathbf{M}$, $\mathbf{S}$, $\mathbf{N}$ and the deformation gradient $\mathbf{F}$:

$$\begin{aligned} &\lambda_{\mathbf{M}} = J^{-1/3}\sqrt{\mathbf{M}\cdot\mathbf{CM}} \\ &\zeta = \sqrt{\frac{J\sqrt{\mathbf{M}\cdot\mathbf{CM}}}{(\mathbf{M}\cdot\mathbf{CM})(\mathbf{S}\cdot\mathbf{CS}) - (\mathbf{M}\cdot\mathbf{CS})^2}} \\ &\phi_{\mathbf{MS}} = \frac{\mathbf{M}\cdot\mathbf{CS}}{\mathbf{M}\cdot\mathbf{CM}} \qquad \phi_{\mathbf{MN}} = \frac{\mathbf{M}\cdot\mathbf{CN}}{\mathbf{M}\cdot\mathbf{CM}} \\ &\phi_{\mathbf{SN}} = \frac{(\mathbf{M}\cdot\mathbf{CM})(\mathbf{S}\cdot\mathbf{CN}) - (\mathbf{M}\cdot\mathbf{CS})(\mathbf{M}\cdot\mathbf{CN})}{(\mathbf{M}\cdot\mathbf{CM})(\mathbf{S}\cdot\mathbf{CS}) - (\mathbf{M}\cdot\mathbf{CS})^2}. \end{aligned} \tag{3}$$

For such a material, Criscione et al[1] proposed a strain energy function W, expressed in terms of six strain attributes with minimal covariance

$$W = W(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),$$

where

$$\alpha_1 = \ln J \quad \alpha_2 = \frac{3}{2}\ln\lambda_{\mathbf{M}} \quad \alpha_3 = 2\ln\zeta \quad \alpha_4 = \phi_{\mathbf{MS}} \quad \alpha_5 = \phi_{\mathbf{MN}} \quad \alpha_6 = \phi_{\mathbf{SN}}. \tag{4}$$

The first Piola-Kirchhoff stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}} = \sum_{i=1}^{6} \frac{\partial W}{\partial \alpha_i}\frac{\partial \alpha_i}{\partial \mathbf{F}}.$$

The Cauchy stress $\mathbf{t}$ can be expressed as the sum of six response terms, almost all of which are mutually orthogonal, i.e.

$$\mathbf{t} = \frac{1}{J}\sum_{i=1}^{6}\frac{\partial W}{\partial \alpha_i}\mathbf{A}_i \tag{5}$$

where $\mathbf{A}_i = \dfrac{\partial \alpha_i}{\partial \mathbf{F}}\mathbf{F}^T$, $i = 1..6$, and $\mathbf{A}_i : \mathbf{A}_j = 0$ for $i \neq j$, except for $i = 4$ and $j = 5$ (or vice versa) for which $\mathbf{A}_4 : \mathbf{A}_5 = 2\lambda_{\mathbf{M}}^{-3}\zeta^{-2}\alpha_6$ (cf.[1]). Using (5), $\mathbf{T}$ can be written as

$$\mathbf{T} = J\mathbf{t}\mathbf{F}^{-T} = \sum_{i=1}^{6} \frac{\partial W}{\partial \alpha_i}\mathbf{A}_i\mathbf{F}^{-T}.$$

Consequently,

$$\begin{aligned}\mathbf{H} : \frac{\partial \mathbf{T}}{\partial \mathbf{F}}[\mathbf{H}] = &\sum_{i=1}^{6}\sum_{j=1}^{6} \frac{\partial^2 W}{\partial \alpha_i \partial \alpha_j}(\mathbf{A}_i\mathbf{F}^{-T} : \mathbf{H})(\mathbf{A}_j\mathbf{F}^{-T} : \mathbf{H}) \\ &+ \sum_{i=1}^{6} \frac{\partial W}{\partial \alpha_i}\mathbf{H} : \frac{\partial \mathbf{A}_i}{\partial \mathbf{F}}[\mathbf{H}]\mathbf{F}^{-T} - \sum_{i=1}^{6} \frac{\partial W}{\partial \alpha_i}\mathbf{H} : \mathbf{A}_i\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}}[\mathbf{H}].\end{aligned} \tag{6}$$

Using the definition of $\mathbf{A}_i$, one could easily verify that

$$\begin{aligned}\mathbf{A}_1 &= \mathbf{I} \\ \mathbf{A}_2 &= \frac{3}{2}\mathbf{m}\otimes\mathbf{m} - \frac{1}{2}\mathbf{I} \\ \mathbf{A}_3 &= \mathbf{n}\otimes\mathbf{n} - \mathbf{s}\otimes\mathbf{s} \\ \mathbf{A}_4 &= \lambda_{\mathbf{M}}^{-3/2}\zeta^{-1}(\mathbf{m}\otimes\mathbf{s} + \mathbf{s}\otimes\mathbf{m}) \\ \mathbf{A}_5 &= \lambda_{\mathbf{M}}^{-3/2}\big(\alpha_6\zeta^{-1}(\mathbf{m}\otimes\mathbf{s} + \mathbf{s}\otimes\mathbf{m}) + \zeta(\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m})\big) \\ \mathbf{A}_6 &= \zeta^2(\mathbf{s}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{s}),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathbf{m} &= \frac{\mathbf{F}\mathbf{M}}{\sqrt{\mathbf{M}\cdot\mathbf{C}\mathbf{M}}}, \\ \mathbf{n} &= \frac{\mathbf{F}^{-T}\mathbf{N}}{\sqrt{\mathbf{N}\cdot\mathbf{C}^{-1}\mathbf{N}}}, \\ \mathbf{s} &= \frac{(\mathbf{M}\cdot\mathbf{C}\mathbf{M})\mathbf{F}\mathbf{S} - (\mathbf{M}\cdot\mathbf{C}\mathbf{S})\mathbf{F}\mathbf{M}}{\sqrt{\mathbf{M}\cdot\mathbf{C}\mathbf{M}}\sqrt{(\mathbf{S}\cdot\mathbf{C}\mathbf{S})(\mathbf{M}\cdot\mathbf{C}\mathbf{M}) - (\mathbf{M}\cdot\mathbf{C}\mathbf{S})^2}}.\end{aligned} \tag{8}$$

It can be shown that $\mathbf{m}$, $\mathbf{s}$, and $\mathbf{n}$ are orthonormal.[1]

To represent the quadratic form (6) in terms of the acoustic tensor, let $\mathbf{H} = \mathbf{a}\otimes\mathbf{b}$, then

$$\mathbf{H} : \frac{\partial}{\partial \mathbf{F}}\mathbf{T}[\mathbf{H}] = \mathbf{a}\cdot\mathbf{Q}(\mathbf{b})\mathbf{a}. \tag{9}$$

In what follows we evaluate the acoustic tensor in the case when the deformation is isochoric biaxial stretch.

3 ISOCHORIC BIAXIAL STRETCH

In this section we consider an incompressible material subjected to isochoric biaxial stretch the principal directions of which are aligned with the fiber direction $\mathbf{M}$, the in-sheet plane direction orthogonal to $\mathbf{M}$, $\mathbf{S}$, and $\mathbf{N}$ - the direction orthogonal to the sheet. That is, the deformation gradient is given by

$$\mathbf{F} = \lambda_1 \mathbf{M} \otimes \mathbf{M} + \lambda_2 \mathbf{S} \otimes \mathbf{S} + (\lambda_1 \lambda_2)^{-1} \mathbf{N} \otimes \mathbf{N} \tag{10}$$

where λ_1 and λ_2 are positive real numbers. Using (3), we can evaluate

$$J = 1, \quad \lambda_M = \lambda_1, \quad \zeta = \frac{1}{\sqrt{\lambda_1}\lambda_2}, \quad \phi_{\mathbf{MS}} = \phi_{\mathbf{MN}} = \phi_{\mathbf{SN}} = 0. \tag{11}$$

And consequently,

$$\alpha_1 = 0, \quad \alpha_2 = \frac{3}{2} \ln \lambda_2, \quad \alpha_3 = 2 \ln \frac{1}{\sqrt{\lambda_1}\lambda_2}, \quad \alpha_4 = 0, \quad \alpha_5 = 0, \quad \alpha_6 = 0. \tag{12}$$

From the definition of $\mathbf{m}$, $\mathbf{s}$ and $\mathbf{n}$ (8) it follows that

$$\mathbf{m} = \mathbf{M}, \quad \mathbf{s} = \mathbf{S}, \quad \mathbf{n} = \mathbf{N}. \tag{13}$$

As shown in Creiscione et al (2002),[1] in order for W to be invariant under the orthotropic symmetry group, it has to be of the form

$$W(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \tilde{W}(\alpha_1, \alpha_2, \alpha_3, \alpha_4^2, \alpha_5^2, \alpha_6^2, \alpha_4\alpha_5\alpha_6).$$

This implies that for an incompressible material $W_{,ij}(\alpha_1, \alpha_2, \alpha_3, 0, 0, 0) = 0$ for all $(i, j) \notin \{(2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3)\}$, and the only nonzero first derivatives of W are $W_{,2}$ and $W_{,3}$. Let

$$\mathbf{b} = b_1 \mathbf{M} + b_2 \mathbf{S} + b_3 \mathbf{N}, \qquad \text{where} \quad b_1^2 + b_2^2 + b_3^2 = 1.$$

After some cumbersome but mostly straightforward calculations one concludes that the acoustic tensor $\mathbf{Q}(\mathbf{b})$ can be represented in the following form

$$\begin{aligned}\mathbf{Q}(\mathbf{b}) = {} & \mathbf{m} \otimes \mathbf{m}\Big(b_1^2 Q_{11}^1 + b_2^2 Q_{11}^2 + b_3^2 Q_{11}^3\Big) + \mathbf{s} \otimes \mathbf{s}\Big(b_1^2 Q_{22}^1 + b_2^2 Q_{22}^2 + b_3^2 Q_{22}^3\Big) \\ & + \mathbf{n} \otimes \mathbf{n}\Big(b_1^2 Q_{33}^1 + b_2^2 Q_{33}^2 + b_3^2 Q_{33}^3\Big) + (\mathbf{m} \otimes \mathbf{s} + \mathbf{s} \otimes \mathbf{m}) b_1 b_2 Q_{12} \\ & + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) b_1 b_3 Q_{13} + (\mathbf{s} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{s}) b_2 b_3 Q_{23}\end{aligned} \tag{14}$$

where the coefficients Q_{ij} and Q_{ii}^k depend on the strain energy function W, its derivatives and the principal stretches λ_1 and λ_2, namely

$$
\begin{aligned}
\mathbf{Q}(\mathbf{b}) = \mathbf{m}\otimes\mathbf{m}\Bigg(&\frac{b_1^2}{\lambda_1^2}\big(W_{,22}-W_{,2}\big)+b_2^2\Big(\frac{1}{\lambda_1^2}W_{,44}+(\frac{\lambda_1^2}{\lambda_2^2}-\frac{1}{\lambda_1\lambda_2^2})W_{,3}\Big)+\frac{b_3^2}{\lambda_1^2}W_{,55}\Bigg)\\
+\mathbf{s}\otimes\mathbf{s}\Bigg(&b_1^2\Big(\frac{\lambda_2^2}{\lambda_1^4}W_{,44}+\frac{3}{2\lambda_1^2}W_{,2}+W_{,3}\Big)\\
&+b_2^2\Big(\frac{1}{\lambda_2^2}(\frac{W_{,22}}{4}+W_{,23}+W_{,33}+\frac{W_{,2}}{2}+W_{,3})-\frac{5W_{,3}}{9\lambda_1\lambda_2^2}\Big)+\frac{b_3^2}{\lambda_2^2}W_{,66}\Bigg)\\
+\mathbf{n}\otimes\mathbf{n}\Bigg(&b_1^2\Big(\frac{W_{,55}}{\lambda_1^6\lambda_2^2}+\frac{3W_{,2}}{2\lambda_1^2}-\frac{W_{,3}}{\lambda_1^2}\Big)+b_2^2\Big(\frac{1}{\lambda_1^2\lambda_2^6}W_{,66}-(\frac{1}{\lambda_1\lambda_2^2}+\frac{1}{\lambda_2^2})W_{,3}\Big)\\
&+b_3^2\lambda_1^2\lambda_2^2\Big(\frac{1}{4}W_{,22}-W_{,23}+W_{,33}+\frac{1}{2}W_{,2}-W_{,3}\Big)\Bigg) \qquad (15)\\
+2\mathrm{Sym}(\mathbf{m}\otimes\mathbf{s})b_1b_2\Bigg(&-\frac{W_{,22}}{2\lambda_1\lambda_2}-\frac{W_{,23}}{\lambda_1\lambda_2}+\frac{\lambda_2}{\lambda_1^3}W_{,44}-\frac{\lambda_1\lambda_2}{4}W_{,2}\\
&+\frac{W_{,3}}{2\lambda_1\lambda_2}+\frac{3W_{,2}}{4\lambda_1\lambda_2}+(\frac{2}{9}\frac{1}{\lambda_1^2\lambda_2}+\frac{\lambda_1}{\lambda_2})W_{,3}\Bigg)\\
+2\mathrm{Sym}(\mathbf{m}\otimes\mathbf{n})b_1b_3\Bigg(&-\frac{\lambda_2}{2}W_{,22}+\lambda_2W_{,23}+\frac{1}{\lambda_1^4\lambda_2}W_{,55}+\frac{\lambda_2}{2}W_{,2}-\lambda_2W_{,3}\Bigg)\\
+2\mathrm{Sym}(\mathbf{s}\otimes\mathbf{n})b_2b_3\Bigg(&\frac{\lambda_1}{4}W_{,22}-\lambda_1W_{,33}+\frac{1}{\lambda_1^2\lambda_2^4}W_{,66}+\frac{\lambda_1}{2}W_{,2}-(\frac{\lambda_2}{2}+\frac{7}{9})W_{,3}\Bigg).
\end{aligned}
$$

$\mathbf{Q}(\mathbf{b})$ is positive definite if and only if its principal minors are positive. Thus we require that the first principal minor be positive for all unit vectors $\mathbf{b}$, i.e.

$$\min_{||\mathbf{b}||=1}\left(b_1^2Q_{11}^1+b_2^2Q_{11}^2+b_3^2Q_{11}^3\right)>0. \qquad (16)$$

The critical points of $F(\mathbf{b},\mu)=b_1^2Q_{11}^1+b_2^2Q_{11}^2+b_3^2Q_{11}^3-\mu(b_1^2+b_2^2+b_3^2-1)$ satisfy the following system of equations

$$
\begin{aligned}
2b_1(Q_{11}^1-\mu)&=0\\
2b_2(Q_{11}^2-\mu)&=0\\
2b_3(Q_{11}^3-\mu)&=0\\
b_1^2+b_2^2+b_3^2&=1
\end{aligned}
$$

the solutions of which are $\mathbf{b} = \pm\mathbf{M}$, $\mathbf{b} = \pm\mathbf{S}$, $\mathbf{b} = \pm\mathbf{N}$. By evaluating F at these critical points we obtain the following necessary conditions for $\hat{\mathbf{T}}$ to be strongly elliptic at $\mathbf{F}$ given by (10):

$$\boxed{\begin{aligned} Q_{11}^1 &= \frac{1}{\lambda_1^2}\big(W_{,22} - W_{,2}\big) > 0 \\ Q_{11}^2 &= \frac{1}{\lambda_1^2}W_{,44} + (\frac{\lambda_1^2}{\lambda_2^2} - \frac{1}{\lambda_1\lambda_2^2})W_{,3} > 0 \\ Q_{11}^3 &= \frac{1}{\lambda_1^2}W_{,55} > 0. \end{aligned}} \tag{17}$$

The second principal minor also has to be positive for all $\mathbf{b}$ with $||\mathbf{b}|| = 1$, that is

$$\begin{aligned} &\min_{||\mathbf{b}||=1} \Big(Q_{11}(\mathbf{b})Q_{22}(\mathbf{b}) - (Q_{12}(\mathbf{b}))^2\Big) \\ =& \min_{||\mathbf{b}||=1} \Big(\big(b_1^2Q_{11}^1 + b_2^2Q_{11}^2 + b_3^2Q_{11}^3\big)\big(b_1^2Q_{22}^1 + b_2^2Q_{22}^2 + b_3^2Q_{22}^3\big) - \big(b_1b_2Q_{12}\big)^2\Big) > 0. \end{aligned} \tag{18}$$

In a similar way as before, after rearranging the terms, the critical points satisfy the following system of equations

$$b_1\Big(b_1^2(2Q_{11}^1Q_{22}^1) + b_2^2(Q_{11}^2Q_{22}^1 + Q_{11}^1Q_{22}^2 - (Q_{12})^2) + b_3^2(Q_{11}^3Q_{22}^1 + Q_{11}^1Q_{22}^3) - \mu\Big) = 0 \tag{19}$$

$$b_2\Big(b_1^2(Q_{11}^2Q_{22}^1 + Q_{11}^1Q_{22}^2 - (Q_{12})^2) + b_2^2(2Q_{11}^2Q_{22}^2) + b_3^2(Q_{11}^2Q_{22}^3 + Q_{11}^3Q_{22}^2) - \mu\Big) = 0 \tag{20}$$

$$b_3\Big(b_1^2(Q_{11}^3Q_{22}^1 + Q_{11}^1Q_{22}^3) + b_2^2(Q_{11}^2Q_{22}^3 + Q_{11}^3Q_{22}^2) + b_3^2(2Q_{11}^3Q_{22}^3) - \mu\Big) = 0 \tag{21}$$

$$b_1^2 + b_2^2 + b_3^2 = 1. \tag{22}$$

Clearly $\mathbf{b} = \pm\mathbf{M}$, $\mathbf{b} = \pm\mathbf{S}$, and $\mathbf{b} = \pm\mathbf{N}$ satisfy the above system, from where, as before, we obtain the following additional constitutive restrictions:

$$\boxed{\begin{aligned} Q_{22}^1 &= \frac{\lambda_2^2}{\lambda_1^4}W_{,44} + \frac{3}{2\lambda_1^2}W_{,2} + W_{,3} > 0 \\ Q_{22}^2 &= \frac{1}{\lambda_2^2}(\frac{1}{4}W_{,22} + W_{,23} + W_{,33} + \frac{1}{2}W_{,2} + W_{,3}) + \frac{-5}{9\lambda_1\lambda_2^2}W_{,3} > 0 \\ Q_{22}^3 &= \frac{1}{\lambda_2^2}W_{,66} > 0. \end{aligned}} \tag{23}$$

And finally, $\det \mathbf{Q}(\mathbf{b})$ has to be positive for all vectors $\mathbf{b}$ with unit length, namely

$$\begin{aligned} \min_{||\mathbf{b}||=1} \Big(&Q_{11}(\mathbf{b})Q_{22}(\mathbf{b})Q_{33}(\mathbf{b}) + 2Q_{12}(\mathbf{b})Q_{13}(\mathbf{b})Q_{23}(\mathbf{b}) \\ &- Q_{11}(\mathbf{b})(Q_{23}(\mathbf{b}))^2 - Q_{22}(\mathbf{b})(Q_{13}(\mathbf{b}))^2 - Q_{33}(\mathbf{b})(Q_{12}(\mathbf{b}))^2\Big) \\ > 0 & \end{aligned} \tag{24}$$

As before we conclude that the critical points have to satisfy a non-linear system of homogeneous polynomial equations of fourth order to which $\mathbf{b} = \pm\mathbf{M}$, $\mathbf{b} = \pm\mathbf{S}$, and $\mathbf{b} = \pm\mathbf{N}$ are solutions. Thus we obtain a third set of restrictions on the strain energy function W:

$$\boxed{\begin{aligned} Q^1_{33} &= \frac{1}{\lambda_1^6\lambda_2^2}W_{,55} + \frac{3}{2\lambda_1^2}W_{,2} - \frac{1}{\lambda_1^2}W_{,3} > 0 \\ Q^2_{33} &= \frac{1}{\lambda_1^2\lambda_2^6}W_{,66} + (-\frac{1}{\lambda_1\lambda_2^2} - \frac{1}{\lambda_2^2})W_{,3} > 0 \\ Q^3_{33} &= \lambda_1^2\lambda_2^2\big(\frac{1}{4}W_{,22} - W_{,23} + W_{,33} + \frac{1}{2}W_{,2} - W_{,3}\big) > 0. \end{aligned}} \tag{25}$$

Note: The difficulty here is to prove that these are the only solutions, or to find all other solutions. If we can prove that systems (19)-(22) and the system arising from (24) have no other solutions, besides the ones we have found, then equations (17), (23) and (25) are necessary and sufficient for strong ellipticity of the strain energy function at all isochoric biaxial stretch deformations.

4 CONCLUSIONS

The subject of this paper is the derivation of stability criteria based upon strong ellipticity for certain special classes of orthotropic, hyperelastic bodies of particular relevance to modeling the myocardium and other soft tissues. The strain energy is taken to be a function of a novel set of strain invariants proposed by Criscione et al.[1] While use of these invariants is convenient for experimentally determining material parameters, it poses considerable challenges in deriving criteria on the strain energy guaranteeing strong ellipticity. Presented here are several sets of necessary conditions for strong ellipticity which can be used to guide the determination of material properties from experimental data. It remains an open problem to assess how far these conditions are from also being sufficient for strong ellipticity to hold, either in general or at least for special classes of deformations, and to derive a set of conditions that are both necessary and sufficient.

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