Maximum and minimum values

Problem 1. Find critical points of a function $f(x, y)$.

SOLUTION. To find the critical points, we need to solve the equations

$$f_x(x, y) = 0 \quad f_y(x, y) = 0 \quad \square$$

Problem 1'. Find critical points of a function $f(x, y, z)$.

SOLUTION. To find the critical points, we need to solve the equations

$$f_x(x, y, z) = 0 \quad f_y(x, y, z) = 0 \quad f_z(x, y, z) = 0 \quad \square$$

Problem 2. Find the local maximum and minimum values and saddle point(s) of a function $f(x, y)$.

SOLUTION.
STEP 1. Find the critical points (see Problem 1).
STEP 2. Apply the second derivatives test (box 3, p.940). $\square$

Problem 3. Find the absolute maximum and minimum values of $f(x, y)$ on the set D.

SOLUTION.
STEP 1. Find the critical points of f inside D (see Problem 1). Find the values of f at the critical points.
STEP 2. Find the extreme values of f on the boundary of D.
If D consists of line segments, find the extreme values of f along each segment (see p.282).
STEP 3. The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value. $\square$

Problem 4. Find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ and do NOT use Lagrange multipliers.

SOLUTION.
STEP 1. Solve the equation $g(x, y, z) = k$ for x, y, or for z, and substitute the obtained expression into $f(x, y, z)$. Assume that we solve $g(x, y, z) = k$ for z, then $z = z(x, y)$ and

$$f(x, y, z(x, y)) = h(x, y)$$

STEP 2. Apply Problem 2 to the function $h(x, y)$. $\square$

For example, if $f = xyz$ and $g(x, y, z) = x + 2y + z = 4$, we can substitute $z = 4 - x - 2y$ into f, and so we have

$$f(x, y, z) = xy(4 - x - 2y) = h(x, y)$$

Problem 5. Use *Method of Lagrange multipliers* to find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

SOLUTION.
STEP 1. Solve the equations

$$\begin{aligned} &f_x(x, y, z) = \lambda g_x(x, y, z)\\ &f_y(x, y, z) = \lambda g_y(x, y, z)\\ &f_z(x, y, z) = \lambda g_z(x, y, z)\\ &g(x, y, z) = k \end{aligned}$$

STEP 2. Evaluate f at all points (x, y, z) obtained in the first step. The largest of these values is the maximum value of f; the smallest is the minimum value of f. $\square$

Problem 6. Use *Method of Lagrange multipliers* to find the maximum and minimum values of $f(x, y, z)$ on the region described by the *inequality* $g(x, y, z) \leq k$.

SOLUTION. To solve the problem, we have to find and compare the values of f at the critical points inside the region with values at the points on the boundary.
STEP 1. To find the critical points of f inside the region, we apply problem 1'. Find the values of f at the critical points.
STEP 2. Since the boundary is defined by $g(x, y, z) = k$, we apply the problem 5 to find the maximum and minimum values of $f(x, y, z)$ subject to $g(x, y, z) = k$.
STEP 3. The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value. $\square$

Problem 7. Use *Lagrange multipliers* to find the maximum and minimum values of $f(x, y, z)$ subject to the given constraints $g(x, y, z) = k$ and $h(x, y, z) = c$
SOLUTION.
STEP 1. Solve the equations

$$\begin{aligned} &f_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)\\ &f_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)\\ &f_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)\\ &g(x, y, z) = k\\ &h(x, y, z) = c \end{aligned}$$

STEP 2. Evaluate f at all points (x, y, z) obtained in the first step. The largest of these values is the maximum value of f; the smallest is the minimum value of f. $\square$