Lecture Notes, December 5, 2003

(Math 2263, Section 010)

- Linear Integrals of Vector Fields
- Surface Integrals of Vector Fields (Sec. 16.7, pp. 1099–1103)
- Stokes' Theorem (Sec. 16.8)
- Divergence Theorem (Sec. 16.9)

1. Line Integrals of Vector Fields

We consider a vector field $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ and a curve C given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$.

Problem 1. The following formulations of the problem are equivalent:
1) Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$;
2) Evaluate $\int_C P dx + Q dy + R dz$;
3) Calculate the work done by the force field $\vec{F}$ when a particle moves under its influence along the curve C.

SOLUTION.

Option 1. If C is <u>closed</u>, you may apply Stokes' Theorem:

Stokes' Theorem. Let S be an oriented surface with <u>unit</u> normal vector $\vec{n}$, bounded by a simple, <u>closed</u> curve C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int\int_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS \tag{1}$$

Note. If $\vec{F}$ is a field in the plane, i.e. $\vec{F} = \langle P, Q\rangle$, and C is <u>closed</u>, you may use Green's theorem.

Option 2. If F is the <u>conservative</u> vector field (curl $\vec{F} = 0$), you may use the Fundamental Theorem for Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)) \tag{2}$$

Option 3. Evaluate the integral directly (see Section 16.2 or *Study Sheet for Sec. 16.2-6.4* for details).

Example 1. Calculate the work done by the force field $\vec{F} = \langle x, y, x^2 + y^2\rangle$ when a particle moves under its influence around the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in first octant, in a counterclockwise direction as viewed from above.

SOLUTION. Since C is <u>closed</u>, we will use Stokes' Theorem (Option 1) to solve the problem. □

2. Surface Integrals of Vector Fields

We consider a vector field $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ and a surface S given by $z = g(x, y)$ or by $\vec{r}(u, v)$.

Problem 2. The following formulations of the problem are equivalent:
1) Calculate the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$;
2) Find the flux of $\vec{F}$ across S.

SOLUTION.

Option 1. If S is <u>closed</u>, you may apply the Divergence Theorem (p. 1111) (Gauss' Theorem):

The Divergence Theorem. Let E be a solid region bounded by a <u>closed</u> surface S oriented by $\vec{n}$ outward from E. Then

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_E \operatorname{div} \vec{F}\, dV \tag{3}$$

Option 2. If S with equation $z = g(x,y)$ is oriented upward, i.e., the $\vec{k}$-th component of the normal vector is positive, then apply the following formula:

$$\int\!\!\int_S \vec{F} \cdot d\vec{S} = \int\!\!\int_{D\,(xy-plane)} \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \tag{4}$$

If S is given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, apply

$$\int\!\!\int_S \vec{F} \cdot d\vec{S} = \pm \int\!\!\int_{D\,(uv-plane)} \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot [\vec{r}_u \times \vec{r}_v]\, dA \tag{5}$$

where the sign $(+)$ or $(-)$ depends on the orientation of the surface.

Example 2. Calculate the outward flux of the vector field

$$\vec{F}(x,y,z) = \langle xy^2 + yz, x^2y + xz^2, 3x^3 + xy^2\rangle$$

across S, where S is the surface of the solid that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3$.

SOLUTION. Since S is closed, we use the Divergence Theorem (Option 1).

Example 3. A fluid is flowing through S according to $\vec{F} = \langle x, y, z\rangle$. Find the flux of $\vec{F}$ across S, where S is the surface given by $z = g(x,y) = 4 - x^2 - y^2$ that lies above the xy-plane and has upward orientation. Note that the xy-plane is not included as part of the surface.

SOLUTION. Since S is not closed, we use Option 2 to find the flux.

$$g_x = -2x \quad g_y = -2y$$

In the example, $P = x$, $Q = y$, and $R = z$, so

$$\begin{aligned} -P\tfrac{\partial g}{\partial x} - Q\tfrac{\partial g}{\partial y} + R \quad &= -x(-2x) - y(-2y) + z \\ &= 2x^2 + 2y^2 + z \\ &= 2x^2 + 2y^2 + (4 - x^2 - y^2) \\ &= 4 + x^2 + y^2 \end{aligned}$$

Then

$$\int\!\!\int_S \vec{F} \cdot d\vec{S} = \int\!\!\int_{D\,(xy-plane)} (4 + x^2 + y^2)\, dA$$

and, by changing to polar coordinates, we obtain that

$$\int\!\!\int_S \vec{F} \cdot d\vec{S} = \int\!\!\int_{D\,(r\theta-plane)} (4 + r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\, dr\, d\theta = 24\pi \quad \square$$