

Math 1571H, Fall 2005
Solution to Quiz 6 (November 10)

This quiz is based on our *celebrated* **Fundamental Theorem of Calculus** (FTC):

Question 1. [3 points] Given $f(x)$ is a continuous function such that

$$\int_0^x f(t)dt = e^{x^2} \sin(x) + \int_0^x (t^2 e^t + 3)f(t)dt. \quad (1)$$

By using FTC, differentiate equation (1) with respect to x ,

$$f(x) = 2xe^{x^2} \sin(x) + e^{x^2} \cos(x) + (x^2 e^x + 3)f(x).$$

Therefore,

$$f(x) = -\frac{(2x \sin(x) + \cos(x))e^{x^2}}{x^2 e^x + 2}.$$

Question 2. [4 points] Find a function $f(x)$ and a number β such that

$$6 + \int_{\beta}^x \frac{f(t)}{t^2} dt = 3\sqrt{x}. \quad (2)$$

Again by using FTC, differentiate equation (2) with respect to x

$$\frac{f(x)}{x^2} = \frac{3}{2\sqrt{x}} \Rightarrow f(x) = \frac{3}{2}x^{3/2}.$$

To find β , let $x = \beta$ in equation (2), to obtain

$$6 + \int_{\beta}^{\beta} \frac{f(t)}{t^2} dt = 3\sqrt{\beta} \Rightarrow 6 + 0 = 3\beta \Rightarrow \beta = 4.$$

Question 3. [5 points] Again, we use FTC to evaluate the given expression:

$$\begin{aligned} \frac{d}{dx} \left(\frac{d}{dx} \int_0^x \left[\int_{\sin(t)}^{\ln(t^2+1)} \sqrt{1+u^4} du \right] dt \right) \\ &= \frac{d}{dx} \int_{\sin(x)}^{\ln(x^2+1)} \sqrt{1+u^4} du \\ &= \frac{2x}{1+x^2} \sqrt{1+\ln^4(1+x^2)} - \cos(x) \sqrt{1+\sin^4(x)}. \end{aligned}$$

We proved the following in the discussion:

$$\begin{aligned} \frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt &= \frac{d}{dx} \left[\int_{g(x)}^{\alpha} f(t) dt + \int_{\alpha}^{h(x)} f(t) dt \right], \quad (\text{where } \alpha \text{ in the domain of } f) \\ &= \frac{d}{dx} \left[- \int_{\alpha}^{g(x)} f(t) dt + \int_{\alpha}^{h(x)} f(t) dt \right]. \\ &= f(h(x))h'(x) - f(g(x))g'(x) \end{aligned} \tag{3}$$

I used the substitution $u = g(x)$, and $v = h(x)$ to convert the two terms in equation (3) to the standard form of FTC:

$$\frac{d}{dx} \int_{\alpha}^{g(x)} f(t) dt = \frac{du}{dx} \frac{d}{du} \int_{\alpha}^u f(t) dt = \frac{du}{dx} f(u) = g'(x) f(g(x)).$$

$$\frac{d}{dx} \int_{\alpha}^{h(x)} f(t) dt = \frac{dv}{dx} \frac{d}{dv} \int_{\alpha}^v f(t) dt = \frac{dv}{dx} f(v) = h'(x) f(h(x)).$$