Chapter 1

The Fourier Transform

1.1 Fourier transforms as integrals

There are several ways to define the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$. In this section, we proceed to define the Fourier transform using an integral representation and state some basic uniqueness and inversion properties. Thereafter, we will consider the transform as being defined as a suitable limit of Fourier series, and we will prove the results stated here.

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$. We define the Fourier transform of $f \in L^1(\mathbb{R})$, denoted by $\mathcal{F}[f](.)$, as follows:*

$$\mathcal{F}[f](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-ixt) dt$$

for $x \in \mathbb{R}$ for which the integral exists. *

We have the **Dirichlet condition** for inversion of Fourier integrals.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that (1) $\int_{-\infty}^{\infty} |f| \, dt$ converges and (2) In any finite interval, f has at most finitely many maxima/mnima/discontinuities. Let $F = \mathcal{F}[f]$. Then if f is continuous at $t \in \mathbb{R}$, we have*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) \exp(itx) dx.$$

*This definition also makes sense for complex valued f but we stick here to real valued f

Moreover, if f is discontinuous at $t \in \mathbb{R}$ and $f(t+)$ and $f(-)$ denote the right and left limits of f at t, then

$$\frac{1}{2}[f(t+) + f(t-)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} F(x) \exp(itx) dx.$$

From the above, we deduce a uniqueness result:

Theorem 2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If*

$$\mathcal{F}[f](x) = \mathcal{F}[g](x), \ \forall x$$

then

$$f(t) = g(t), \ \forall t.$$

Proof: We have from inversion, easily that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}[f](x) \exp(itx) dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}[g](x) \exp(itx) dx$$
$$= g(t).$$

□

Example 1 *Find the Fourier transform of $f(t) = \exp(-|t|)$ and hence using inversion, deduce that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ and $\int_0^\infty \frac{x\sin(xt)}{1+x^2} dx = \frac{\pi \exp(-t)}{2}$, $t > 0$.*

Solution We write

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-ixt) dt$$
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \exp(t(1-ix)) dt + \int_{0}^{\infty} \exp(-t(1+ix)) \right]$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+x^2}.$$

Now by the inversion formula,

$$\exp(-|t|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) \exp(ixt) dx$$
$$= \frac{1}{\pi} \left[\int_0^\infty \frac{\exp(ixt) + \exp(-ixt)}{1+x^2} dt \right]$$
$$= \frac{2}{\pi} \int_0^\infty \frac{\cos(xt)}{1+x^2} dx.$$

Now this formula holds at $t = 0$, so substituting $t = 0$ into the above gives the first required identity. Differentiating with respect to t as we may, gives the second required identity. $\Box$.

Proceeding in a similar way as the above example, we can easily show that

$$\mathcal{F}[\exp(-\frac{1}{2}t^2)](x) = \exp(-\frac{1}{2}x^2),\ x \in \mathbb{R}.$$

We will discuss this example in more detail later in this chapter.

We will also show that we can reinterpret Definition 1 to obtain the Fourier transform of any complex valued $f \in L^2(\mathbb{R})$, and that the Fourier transform is unitary on this space:

Theorem 3 *If $f, g \in L^2(\mathbb{R})$ then $\mathcal{F}[f], \mathcal{F}[g] \in L^2(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\ dt = \int_{-\infty}^{\infty} \mathcal{F}[f](x)\overline{\mathcal{F}[g](x)}\ dx.$$

This is a result of fundamental importance for applications in signal processing.

1.2 The transform as a limit of Fourier series

We start by constructing the Fourier series (complex form) for functions on an interval $[-\pi L, \pi L]$. The ON basis functions are

$$e_n(t) = \frac{1}{\sqrt{2\pi L}} e^{\frac{int}{L}}, \qquad n = 0, \pm 1, \cdots,$$

and a sufficiently smooth function f of period $2\pi L$ can be expanded as

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(x) e^{-\frac{inx}{L}} dx \right) e^{\frac{int}{L}}.$$

For purposes of motivation let us abandon periodicity and think of the functions f as differentiable everywhere, vanishing at $t = \pm\pi L$ and identically zero outside $[-\pi L, \pi L]$. We rewrite this as

$$f(t) = \sum_{n=-\infty}^{\infty} e^{\frac{int}{L}} \frac{1}{2\pi L} \hat{f}(\frac{n}{L})$$

which looks like a Riemann sum approximation to the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda \qquad (1.2.1)$$

to which it would converge as $L \to \infty$. (Indeed, we are partitioning the λ interval $[-L, L]$ into $2L$ subintervals, each with partition width $1/L$.) Here,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt. \tag{1.2.2}$$

Similarly the Parseval formula for f on $[-\pi L, \pi L]$,

$$\int_{-\pi L}^{\pi L} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi L} |\hat{f}(\frac{n}{L})|^2$$

goes in the limit as $L \to \infty$ to the *Plancherel identity*

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda. \tag{1.2.3}$$

Expression (1.2.2) is called the *Fourier integral* or *Fourier transform* of f. Expression (1.2.1) is called the *inverse Fourier integral* for f. The Plancherel identity suggests that the Fourier transform is a one-to-one norm preserving map of the Hilbert space $L^2[-\infty, \infty]$ onto itself (or to another copy of itself). We shall show that this is the case. Furthermore we shall show that the pointwise convergence properties of the inverse Fourier transform are somewhat similar to those of the Fourier series. Although we could make a rigorous justification of the the steps in the Riemann sum approximation above, we will follow a different course and treat the convergence in the mean and pointwise convergence issues separately.

A second notation that we shall use is

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \hat{f}(\lambda) \tag{1.2.4}$$

$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda t} d\lambda \tag{1.2.5}$$

Note that, formally, $\mathcal{F}^*[\hat{f}](t) = \sqrt{2\pi} f(t)$. The first notation is used more often in the engineering literature. The second notation makes clear that $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators mapping $L^2[-\infty, \infty]$ onto itself in one view [and $\mathcal{F}$ mapping the *signal space* onto the *frequency space* with $\mathcal{F}^*$ mapping the frequency space onto the signal space in the other view. In this notation the Plancherel theorem takes the more symmetric form

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}[f](\lambda)|^2 d\lambda.$$

EXAMPLES:

1. The box function (or rectangular wave)

$$\Pi(t) = \begin{cases} 1 & \text{if } -\pi < t < \pi \\ \frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.6}$$

Then, since $\Pi(t)$ is an even function and $e^{-i\lambda t} = \cos(\lambda t) + i\sin(\lambda t)$, we have

$$\hat{\Pi}(\lambda) = \sqrt{2\pi}\mathcal{F}[\Pi](\lambda) = \int_{-\infty}^{\infty} \Pi(t)e^{-i\lambda t}dt = \int_{-\infty}^{\infty} \Pi(t)\cos(\lambda t)dt$$

$$= \int_{-\pi}^{\pi} \cos(\lambda t)dt = \frac{2\sin(\pi\lambda)}{\lambda} = 2\pi \text{ sinc } \lambda.$$

Thus sinc λ is the Fourier transform of the box function. The inverse Fourier transform is

$$\int_{-\infty}^{\infty} \text{sinc}(\lambda)e^{i\lambda t}d\lambda = \Pi(t),$$

as follows from (**??**). Furthermore, we have

$$\int_{-\infty}^{\infty} |\Pi(t)|^2 dt = 2\pi$$

and

$$\int_{-\infty}^{\infty} |\text{ sinc }(\lambda)|^2 d\lambda = 1$$

from (**??**), so the Plancherel equality is verified in this case. Note that the inverse Fourier transform converged to the midpoint of the discontinuity, just as for Fourier series.

2. A truncated cosine wave.

$$f(t) = \begin{cases} \cos 3t & \text{if } -\pi < t < \pi \\ -\frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

Then, since the cosine is an even function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{i\lambda t}dt = \int_{-\pi}^{\pi} \cos(3t)\cos(\lambda t)dt$$

$$= \frac{2\lambda\sin(\lambda)}{9-\lambda^2}.$$

3. A truncated sine wave.

$$f(t) = \begin{cases} \sin 3t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

Since the sine is an odd function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = -i\int_{-\pi}^{\pi} \sin(3t)\sin(\lambda t)dt$$
$$= \frac{-6i\sin(\lambda)}{9-4\lambda^2}.$$

4. A triangular wave.

$$f(t) = \begin{cases} \pi + t & \text{if } -\pi \le t \le 0 \\ \pi - t & \text{if } 0 \le t \le \pi \\ 0 & \text{otherwise.} \end{cases} \quad (1.2.7)$$

Then, since f is an even function, we have

$$\hat{f}(\lambda) = \sqrt{2\pi}\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = 2\int_{0}^{\pi} (\pi - t)\cos(\lambda t)dt$$
$$= \frac{2-2\cos\lambda}{\lambda^2}.$$

NOTE: The Fourier transforms of the discontinuous functions above decay as $\frac{1}{\lambda}$ for $|\lambda| \to \infty$ whereas the Fourier transforms of the continuous functions decay as $\frac{1}{\lambda^2}$. The coefficients in the Fourier series of the analogous functions decay as $\frac{1}{n}$, $\frac{1}{n^2}$, respectively, as $|n| \to \infty$.

1.2.1 Properties of the Fourier transform

Recall that

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt = \frac{1}{\sqrt{2\pi}}\hat{f}(\lambda)$$
$$\mathcal{F}^*[g](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\lambda)e^{i\lambda t}d\lambda$$

We list some properties of the Fourier transform that will enable us to build a repertoire of transforms from a few basic examples. Suppose that f, g belong to $L^1[-\infty,\infty]$, i.e., $\int_{-\infty}^{\infty}|f(t)|dt < \infty$ with a similar statement for g. We can state the following (whose straightforward proofs are left to the reader):

1. $\mathcal{F}$ and $\mathcal{F}^*$ are linear operators. For $a, b \in C$ we have

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g], \quad \mathcal{F}^*[af + bg] = a\mathcal{F}^*[f] + b\mathcal{F}^*[g].$$

2. Suppose $t^n f(t) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}.$$

3. Suppose $\lambda^n f(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n. Then

$$\mathcal{F}^*[\lambda^n f(\lambda)](t) = i^n \frac{d^n}{dt^n}\{\mathcal{F}^*[f](t)\}.$$

4. Suppose the nth derivative $f^{(n)}(t) \in L^1[-\infty, \infty]$ and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}[f^{(n)}](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda)\}.$$

5. Suppose nth derivative $f^{(n)}(\lambda) \in L^1[-\infty, \infty]$ for some positive integer n and piecewise continuous for some positive integer n, and f and the lower derivatives are all continuous in $(-\infty, \infty)$. Then

$$\mathcal{F}^*[f^{(n)}](t) = (-it)^n \mathcal{F}^*[f](t).$$

6. The Fourier transform of a translation by real number a is given by

$$\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda).$$

7. The Fourier transform of a scaling by positive number b is given by

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f](\frac{\lambda}{b}).$$

8. The Fourier transform of a translated and scaled function is given by

$$\mathcal{F}[f(bt-a)](\lambda) = \frac{1}{b}e^{-i\lambda a/b}\mathcal{F}[f](\frac{\lambda}{b}).$$

EXAMPLES

- We want to compute the Fourier transform of the rectangular box function with support on $[c, d]$:

$$R(t) = \begin{cases} 1 & \text{if } c < t < d \\ \frac{1}{2} & \text{if } t = c, d \\ 0 & \text{otherwise.} \end{cases}$$

  Recall that the box function

$$\Pi(t) = \begin{cases} 1 & \text{if } -\pi < t < \pi \\ \frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

  has the Fourier transform $\hat{\Pi}(\lambda) = 2\pi \text{ sinc } \lambda$. but we can obtain R from Π by first translating $t \longrightarrow s = t - \frac{(c+d)}{2}$ and then rescaling $s \longrightarrow \frac{2\pi}{d-c} s$:

$$R(t) = \Pi(\frac{2\pi}{d-c} t - \pi \frac{c+d}{d-c}).$$

$$\hat{R}(\lambda) = \frac{4\pi^2}{d-c} e^{i\pi\lambda(c+d)/(d-c)} \text{sinc}(\frac{2\pi\lambda}{d-c}). \tag{1.2.8}$$

  Furthermore, from (**??**) we can check that the inverse Fourier transform of $\hat{R}$ is R, i.e., $\mathcal{F}^*(\mathcal{F})R(t) = R(t)$.

- Consider the truncated sine wave

$$f(t) = \begin{cases} \sin 3t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$$

  with

$$\hat{f}(\lambda) = \frac{-6i\sin(\lambda)}{9 - 4\lambda^2}.$$

  Note that the derivative f' of $f(t)$ is just $3g(t)$ (except at 2 points) where $g(t)$ is the truncated cosine wave

$$g(t) = \begin{cases} \cos 3t & \text{if } -\pi < t < \pi \\ -\frac{1}{2} & \text{if } t = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

  We have computed

$$\hat{g}(\lambda) = \frac{2\lambda\sin(\lambda)}{9 - 4\lambda^2}.$$

  so $3\hat{g}(\lambda) = (i\lambda)\hat{f}(\lambda)$, as predicted.

- Reversing the example above we can differentiate the truncated cosine wave to get the truncated sine wave. The prediction for the Fourier transform doesn't work! Why not?

1.2.2 Fourier transform of a convolution

There is one property of the Fourier transform that is of particular importance in this course. Suppose f, g belong to $L^1[-\infty, \infty]$.

Definition 2 *The convolution of f and g is the function $f * g$ defined by*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx.$$

Note also that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$, as can be shown by a change of variable.

Lemma 1 *$f * g \in L^1[-\infty, \infty]$ and*

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} |f(x)|dx \int_{-\infty}^{\infty} |g(t)|dt.$$

SKETCH OF PROOF:

$$\int_{-\infty}^{\infty} |f * g(t)|dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x)g(t-x)|dx \right) dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |g(t-x)|dt \right) |f(x)|dx = \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(x)|dx.$$

Q.E.D.

Theorem 4 *Let $h = f * g$. Then*

$$\hat{h}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

SKETCH OF PROOF:

$$\hat{h}(\lambda) = \int_{-\infty}^{\infty} f * g(t) e^{-i\lambda t} dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)g(t-x)dx \right) e^{-i\lambda t} dt$$

$$= \int_{-\infty}^{\infty} f(x)e^{-i\lambda x} \left(\int_{-\infty}^{\infty} g(t-x)e^{-i\lambda(t-x)}dt \right) dx = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \ \hat{g}(\lambda)$$

$$= \hat{f}(\lambda)\hat{g}(\lambda).$$

Q.E.D.

1.3 L^2 convergence of the Fourier transform

In this course our primary interest is in Fourier transforms of functions in the Hilbert space $L^2[-\infty,\infty]$. However, the formal definition of the Fourier integral transform,

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt \tag{1.3.9}$$

doesn't make sense for a general $f \in L^2[-\infty,\infty]$. If $f \in L^1[-\infty,\infty]$ then f is absolutely integrable and the integral (1.3.9) converges. However, there are square integrable functions that are not integrable. (Example: $f(t) = \frac{1}{1+|t|}$.) How do we define the transform for such functions?

We will proceed by defining $\mathcal{F}$ on a dense subspace of $f \in L^2[-\infty,\infty]$ where the integral makes sense and then take Cauchy sequences of functions in the subspace to define $\mathcal{F}$ on the closure. Since $\mathcal{F}$ preserves inner product, as we shall show, this simple procedure will be effective.

First some comments on integrals of L^2 functions. If $f, g \in L^2[-\infty,\infty]$ then the integral $(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g}(t)dt$ necessarily exists, whereas the integral (1.3.9) may not, because the exponential $e^{-i\lambda t}$ is not an element of L^2. However, the integral of $f \in L^2$ over any finite interval, say $[-N,N]$ does exist. Indeed for N a positive integer, let $\chi_{[-N,N]}$ be the indicator function for that interval:

$$\chi_{[-N,N]}(t) = \begin{cases} 1 & \text{if } -N \le t \le N \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.10}$$

Then $\chi_{[-N,N]} \in L^2[-\infty,\infty]$ so $\int_{-N}^{N} f(t)dt$ exists because

$$\int_{-N}^{N} |f(t)|dt = |(|f|, \chi_{[-N,N]})| \le ||f||_{L^2}||\chi_{[-N,N]}||_{L^2} = ||f||_{L^2}\sqrt{2N} < \infty$$

Now the space of step functions is dense in $L^2[-\infty,\infty]$, so we can find a convergent sequence of step functions $\{s_n\}$ such that $\lim_{n\to\infty} ||f - s_n||_{L^2} = 0$. Note that the sequence of functions $\{f_N = f\chi_{[-N,N]}\}$ converges to f pointwise as $N \to \infty$ and each $f_N \in L^2 \cap L^1$.

Lemma 2 *$\{f_N\}$ is a Cauchy sequence in the norm of $L^2[-\infty,\infty]$ and $\lim_{n\to\infty} ||f - f_n||_{L^2} = 0$.*

PROOF: Given $\epsilon > 0$ there is step function s_M such that $||f - s_M|| < \frac{\epsilon}{2}$. Choose N so large that the support of s_M is contained in $[-N,N]$, i.e.,

$s_M(t)\chi_{[-N,N]}(t) = s_M(t)$ for all t. Then $||s_M - f_N||^2 = \int_{-N}^{N} |s_M(t) - f(t)|^2 dt \leq \int_{-\infty}^{\infty} |s_M(t) - f(t)|^2 dt = ||s_M - f||^2$, so

$$||f - f_N|| - ||(f - s_M) + (s_M - f_N)|| \leq ||f - s_M|| + ||s_M - f_N|| \leq 2||f - s_M|| < \epsilon.$$

Q.E.D.

Here we will study the linear mapping $\mathcal{F} : L^2[-\infty, \infty] \to \hat{L}^2[-\infty, \infty]$ from the signal space to the frequency space. We will show that the mapping is *unitary*, i.e., it preserves the inner product and is 1-1 and onto. Moreover, the map $\mathcal{F}^* : \hat{L}^2[-\infty, \infty] \to L^2[-\infty, \infty]$ is also a unitary mapping and is the inverse of $\mathcal{F}$:

$$\mathcal{F}^*\mathcal{F} = I_{L^2}, \qquad \mathcal{F}\mathcal{F}^* = I_{\hat{L}^2}$$

where $I_{L^2}, I_{\hat{L}^2}$ are the identity operators on L^2 and $\hat{L}^2$, respectively. We know that the space of step functions is dense in L^2. Hence to show that $\mathcal{F}$ preserves inner product, it is enough to verify this fact for step functions and then go to the limit. Once we have done this, we can define $\mathcal{F}f$ for any $f \in L^2[-\infty, \infty]$. Indeed, if $\{s_n\}$ is a Cauchy sequence of step functions such that $\lim_{n\to\infty} ||f - s_n||_{L^2} = 0$, then $\{\mathcal{F}s_n\}$ is also a Cauchy sequence (indeed, $||s_n - s_m|| = ||\mathcal{F}s_n - \mathcal{F}s_m||$) so we can define $\mathcal{F}f$ by $\mathcal{F}f = \lim_{n\to\infty} \mathcal{F}s_n$. The standard methods of Section 1.3 show that $\mathcal{F}f$ is uniquely defined by this construction. Now the truncated functions f_N have Fourier transforms given by the convergent integrals

$$\mathcal{F}[f_N](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t) e^{-i\lambda t} dt$$

and $\lim_{N\to\infty} ||f - f_N||_{L^2} = 0$. Since $\mathcal{F}$ preserves inner product we have $||\mathcal{F}f - \mathcal{F}f_N||_{L^2} = ||\mathcal{F}(f - f_N)||_{L^2} = ||f - f_N||_{L^2}$, so $\lim_{N\to\infty} ||\mathcal{F}f - \mathcal{F}f_N||_{L^2} = 0$. We write

$$\mathcal{F}[f](\lambda) = \text{l.i.m.}_{N\to\infty}\mathcal{F}[f_N](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t) e^{-i\lambda t} dt$$

where 'l.i.m.' indicates that the convergence is in the mean (Hilbert space) sense, rather than pointwise.

We have already shown that the Fourier transform of the rectangular box function with support on $[c, d]$:

$$R_{c,d}(t) = \begin{cases} 1 & \text{if } c < t < d \\ \frac{1}{2} & \text{if } t = c, d \\ 0 & \text{otherwise.} \end{cases}$$

is

$$\mathcal{F}[R_{c,d}](\lambda) = \frac{4\pi^2}{\sqrt{2\pi}(d-c)} e^{i\pi\lambda(c+d)/(d-c)} \text{sinc}(\frac{2\pi\lambda}{d-c}).$$

and that $\mathcal{F}^*(\mathcal{F})R_{c,d}(t) = R_{c,d}(t)$. (Since here we are concerned only with convergence in the mean the value of a step function at a particular point is immaterial. Hence for this discussion we can ignore such niceties as the values of step functions at the points of their jump discontinuities.)

Lemma 3

$$(R_{a,b}, R_{c,d})_{L^2} = (\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2}$$

for all real numbers $a \le b$ and $c \le d$.

PROOF:

$$(\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2} = \int_{-\infty}^{\infty} \mathcal{F}[R_{a,b}](\lambda)\overline{\mathcal{F}}[R_{c,d}](\lambda)d\lambda$$

$$= \lim_{N\to\infty} \int_{-N}^{N} \left(\mathcal{F}[R_{a,b}](\lambda) \int_{c}^{d} \frac{e^{i\lambda t}}{\sqrt{2\pi}} dt \right) d\lambda$$

$$= \lim_{N\to\infty} \int_{c}^{d} \left(\int_{-N}^{N} \mathcal{F}[R_{a,b}](\lambda) \frac{e^{i\lambda t}}{\sqrt{2\pi}} d\lambda \right) dt.$$

Now the inside integral is converging to $R_{a,b}$ as $N \to \infty$ in both the pointwise and L^2 sense, as we have shown. Thus

$$(\mathcal{F}R_{a,b}, \mathcal{F}R_{c,d})_{\hat{L}^2} = \int_{c}^{d} R_{a,b}dt = (R_{a,b}, R_{c,d})_{L^2}.$$

Q.E.D.

Since any step functions u, v are finite linear combination of indicator functions R_{a_j,b_j} with complex coefficients, $u = \sum_j \alpha_j R_{a_j,b_j}$, $v = \sum_k \beta_k R_{c_k,d_k}$ we have

$$(\mathcal{F}u, \mathcal{F}v)_{\hat{L}^2} = \sum_{j,k} \alpha_j \overline{\beta}_k (\mathcal{F}R_{a_j,b_j}, \mathcal{F}R_{c_k,d_k})_{\hat{L}^2}$$

$$= \sum_{j,k} \alpha_j \overline{\beta}_k (R_{a_j,b_j}, R_{c_k,d_k})_{L^2} = (u, v)_{L^2}.$$

Thus $\mathcal{F}$ preserves inner product on step functions, and by taking Cauchy sequences of step functions, we have the

Theorem 5 *(Plancherel Formula) Let $f, g \in L^2[-\infty, \infty]$. Then*

$$(f, g)_{L^2} = (\mathcal{F}f, \mathcal{F}g)_{\hat{L}^2}, \qquad ||f||^2_{L^2} = ||\mathcal{F}f||^2_{\hat{L}^2}$$

In the engineering notation this reads

$$2\pi \int_{-\infty}^{\infty} f(t)\overline{g}(t)dt = \int_{-\infty}^{\infty} \hat{f}(\lambda)\overline{\hat{g}}(\lambda)d\lambda.$$

Theorem 6 *The map* $\mathcal{F}^* : \hat{L}^2[-\infty,\infty] \to L^2[-\infty,\infty]$ *has the following properties:*

1. *It preserves inner product, i.e.,*

$$(\mathcal{F}^*\hat{f}, \mathcal{F}^*\hat{g})_{L^2} = (\hat{f}, \hat{g})_{\hat{L}^2}$$

for all $\hat{f}, \hat{g} \in \hat{L}^2[-\infty,\infty]$.

2. $\mathcal{F}^*$ *is the adjoint operator to* $\mathcal{F} : L^2[-\infty,\infty] \to \hat{L}^2[-\infty,\infty]$, *i.e.,*

$$(\mathcal{F}f, \hat{g})_{\hat{L}^2} = (f, \mathcal{F}^*\hat{g})_{L^2},$$

for all $f \in L^2[-\infty,\infty]$, $\hat{g} \in \hat{L}^2[-\infty,\infty]$.

PROOF:

1. This follows immediately from the facts that $\mathcal{F}$ preserves inner product and $\overline{\mathcal{F}}[\overline{f}](\lambda) = \mathcal{F}^*[f](\lambda)$.

2. $$(\mathcal{F}R_{a,b}, R_{c,d})_{\hat{L}^2} = (R_{a,b}, \mathcal{F}^*R_{c,d})_{L^2}$$

as can be seen by an interchange in the order of integration. Then using the linearity of $\mathcal{F}$ and $\mathcal{F}^*$ we see that

$$(\mathcal{F}u, v)_{\hat{L}^2} = (u, \mathcal{F}^*v)_{L^2},$$

for all step functions u, v. Since the space of step functions is dense in $\hat{L}^2[-\infty,\infty]$ and in $L^2[-\infty,\infty]$

Q.E.D.

Theorem 7 1. *The Fourier transform* $\mathcal{F} : L^2[-\infty,\infty] \to \hat{L}^2[-\infty,\infty]$ *is a unitary transformation, i.e., it preserves the inner product and is 1-1 and onto.*

2. *The adjoint map* $\mathcal{F}^* : \hat{L}^2[-\infty,\infty] \to L^2[-\infty,\infty]$ *is also a unitary mapping.*

3. $\mathcal{F}^*$ *is the inverse operator to* $\mathcal{F}$:

$$\mathcal{F}^*\mathcal{F} = I_{L^2}, \qquad \mathcal{F}\mathcal{F}^* = I_{\hat{L}^2}$$

where $I_{L^2}, I_{\hat{L}^2}$ *are the identity operators on* L^2 *and* $\hat{L}^2$, *respectively.*

PROOF:

1. The only thing left to prove is that for every $\hat{g} \in \hat{L}^2[-\infty, \infty]$ there is a $f \in L^2[-\infty, \infty]$ such that $\mathcal{F}f = \hat{g}$, i.e., $\mathcal{R} \equiv \{\mathcal{F}f : f \in L^2[-\infty, \infty]\} = \hat{L}^2[-\infty, \infty]$. Suppose this isn't true. Then there exists a nonzero $\hat{h} \in \hat{L}^2[-\infty, \infty]$ such that $\hat{h} \perp \mathcal{R}$, i.e., $(\mathcal{F}f, \hat{h})_{\hat{L}^2} = 0$ for all $f \in L^2[-\infty, \infty]$. But this means that $(f, \mathcal{F}^*\hat{h})_{L^2} = 0$ for all $f \in L^2[-\infty, \infty]$, so $\mathcal{F}^*\hat{h} = \Theta$. But then $||\mathcal{F}^*\hat{h}||_{L^2} = ||\hat{h}||_{\hat{L}^2} = 0$ so $\hat{h} = \Theta$, a contradiction.

2. Same proof as for 1.

3. We have shown that $\mathcal{F}\mathcal{F}^* R_{a,b} = \mathcal{F}^*\mathcal{F} R_{a,b} = R_{a,b}$ for all indicator functions $R_{a,b}$. By linearity we have $\mathcal{F}\mathcal{F}^* s = \mathcal{F}^*\mathcal{F} s = s$for all step functions s. This implies that

$$(\mathcal{F}^*\mathcal{F}f, g)_{L^2} = (f, g)_{L^2}$$

for all $f, g \in L^2[-\infty, \infty]$. Thus

$$([\mathcal{F}^*\mathcal{F} - I_{L^2}]f, g)_{L^2} = 0$$

for all $f, g \in L^2[-\infty, \infty]$. Thus $\mathcal{F}^*\mathcal{F} = I_{L^2}$. An analogous argument gives $\mathcal{F}\mathcal{F}^* = I_{\hat{L}^2}$.

Q.E.D.

1.4 The Riemann-Lebesgue Lemma and pointwise convergence

Lemma 4 *(Riemann-Lebesgue) Suppose f is absolutely Riemann integrable in $(-\infty, \infty)$ (so that $f \in L^1[-\infty, \infty]$), and is bounded in any finite subinterval $[a, b]$, and let α, β be real. Then*

$$\lim_{\alpha \to +\infty} \int_{-\infty}^{\infty} f(t) \sin(\alpha t + \beta) dt = 0.$$

PROOF: Without loss of generality, we can assume that f is real, because we can break up the complex integral into its real and imaginary parts.

1. The statement is true if $f = R_{a,b}$ is an indicator function, for

$$\int_{-\infty}^{\infty} R_{a,b}(t) \sin(\alpha t + \beta) dt = \int_a^b \sin(\alpha t + \beta) dt = \frac{-1}{\alpha} \cos(\alpha t + \beta)|_a^b \to 0$$

as $\alpha \to +\infty$.

2. The statement is true if f is a step function, since a step function is a finite linear combination of indicator functions.

3. The statement is true if f is bounded and Riemann integrable on the finite interval $[a, b]$ and vanishes outside the interval. Indeed given any $\epsilon > 0$ there exist two step functions $\overline{s}$ (Darboux upper sum) and $\underline{s}$ (Darboux lower sum) with support in $[a, b]$ such that $\overline{s}(t) \geq f(t) \geq \underline{s}(t)$ for all $t \in [a, b]$ and $\int_a^b |\overline{s} - \underline{s}| < \frac{\epsilon}{2}$. Then

$$\int_a^b f(t)\sin(\alpha t + \beta)dt =$$

$$\int_a^b [f(t) - \underline{s}(t)]\sin(\alpha t + \beta)dt + \int_a^b \underline{s}(t)\sin(\alpha t + \beta)dt.$$

Now

$$|\int_a^b [f(t) - \underline{s}(t)]\sin(\alpha t + \beta)dt| \leq \int_a^b |f(t) - \underline{s}(t)|dt \leq \int_a^b |\overline{s} - \underline{s}| < \frac{\epsilon}{2}$$

and (since $\underline{s}$ is a step function, by choosing α sufficiently large we can ensure

$$|\int_a^b \underline{s}(t)\sin(\alpha t + \beta)dt| < \frac{\epsilon}{2}.$$

Hence

$$|\int_a^b f(t)\sin(\alpha t + \beta)dt| < \epsilon$$

for α sufficiently large.

4. The statement of the lemma is true in general. Indeed

$$|\int_{-\infty}^{\infty} f(t)\sin(\alpha t + \beta)dt| \leq |\int_{-\infty}^{a} f(t)\sin(\alpha t + \beta)dt|$$

$$+|\int_a^b f(t)\sin(\alpha t + \beta)dt| + |\int_b^{\infty} f(t)\sin(\alpha t + \beta)dt|.$$

Given $\epsilon > 0$ we can choose a and b such the first and third integrals are each $< \frac{\epsilon}{3}$, and we can choose α so large the the second integral is $< \frac{\epsilon}{3}$. Hence the limit exists and is 0.

Q.E.D.

Theorem 8 *Let f be a complex valued function such that*

- *$f(t)$ is absolutely Riemann integrable on $(-\infty, \infty)$.*
- *$f(t)$ is piecewise continuous on $(-\infty, \infty)$, with only a finite number of discontinuities in any bounded interval.*
- *$f'(t)$ is piecewise continuous on $(-\infty, \infty)$, with only a finite number of discontinuities in any bounded interval.*
- *$f(t) = \frac{f(t+0)+f(t-0)}{2}$ at each point t.*

Let

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt$$

be the Fourier transform of f. Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda t}d\lambda$$

for every $t \in (-\infty, \infty)$.

PROOF: For real $L > 0$ set

$$f_L(t) = \int_{-L}^{L} \hat{f}(\lambda)e^{i\lambda t}d\lambda = \frac{1}{2\pi}\int_{-L}^{L}\left[\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx\right]e^{i\lambda t}d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left[\int_{-L}^{L} e^{i\lambda(t-x)}d\lambda\right]dx = \int_{-\infty}^{\infty} f(x)\Delta_L(t-x)dx,$$

where

$$\Delta_L(x) = \frac{1}{2\pi}\int_{-L}^{L} e^{i\lambda x}d\lambda = \begin{cases} \frac{L}{\pi} & \text{if } x = 0 \\ \frac{\sin Lx}{\pi x} & \text{otherwise.} \end{cases}$$

Using the integral (**??**) we have,

$$f_L(t) - f(t) = \int_{-\infty}^{\infty} \Delta_L(t-x)[f(x) - f(t)]dx$$

$$\int_0^{\infty} \Delta_L(x)[f(t+x) + f(t-x) - 2f(t)]dx$$

$$= \int_0^{\infty} \{\frac{f(t+x) + f(t-x) - 2f(t)}{\pi x}\} \sin Lx \; dx$$

The function in the curly braces satisfies the assumptions of the Riemann-Lebesgue Lemma. Hence $\lim_{L\to+\infty}[f_L(t) - f(t)] = 0$. Q.E.D

Note: Condition 4 is just for convenience; redefining f at the discrete points where there is a jump discontinuity doesn't change the value of any of the integrals. The inverse Fourier transform converges to the midpoint of a jump discontinuity, just as does the Fourier series.

1.5 Relations between Fourier series and Fourier integrals: sampling

For the purposes of Fourier analysis we have been considering signals $f(t)$ as arbitrary $L^2[-\infty, \infty]$ functions. In the practice of signal processing, however, one can treat only a finite amount of data. Typically the signal is digitally sampled at regular or irregular discrete time intervals. Then the processed sample alone is used to reconstruct the signal. If the sample isn't altered, then the signal should be recovered exactly. How is this possible? How can one reconstruct a function $f(t)$ exactly from discrete samples? The answer is, of course, that this isn't possible for arbitrary functions $f(t)$. The task isn't hopeless, however, because the signals employed in signal processing, such as voice or images, are not arbitrary. The human voice for example is easily distinguished from static or random noise. One distinguishing characteristic is that the frequencies of sound in the human voice are restricted to a narrow frequency band. In fact, any signal that we can acquire and process with real hardware must be restricted to some finite frequency band. In this section we will explore Shannon-Whittaker sampling, one way that the special class of signals restricted in frequency can be sampled and then reproduced exactly. This method is of immense practical importance as it is employed routinely in telephone, radio and TV transmissions, radar, etc. In later chapters we will study other special structural properties of signal classes, such as sparsity, that can be used to facilitate their processing and efficient reconstruction.

Definition 3 *A function f is said to be frequency band-limited if there exists a constant $\Omega > 0$ such that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$. The frequency $\nu = \frac{\Omega}{2\pi}$ is called the Nyquist frequency and 2ν is the Nyquist rate.*

Theorem 9 *(Shannon-Whittaker Sampling Theorem) Suppose f is a function such that*

1. *f satisfies the hypotheses of the Fourier convergence theorem 8.*
2. *$\hat{f}$ is continuous and has a piecewise continuous first derivative on its domain.*
3. *There is a fixed $\Omega > 0$ such that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$.*

Then f is completely determined by its values at the points $t_j = \frac{j\pi}{\Omega}$, $j = 0, \pm 1, \pm 2, \cdots$:

$$f(t) = \sum_{-\infty}^{\infty} f(\frac{j\pi}{\Omega}) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi},$$

and the series converges uniformly on $(-\infty, \infty)$.

(NOTE: The theorem states that for a frequency band-limited function, to determine the value of the function at all points, it is sufficient to sample the function at the Nyquist rate, i.e., at intervals of $\frac{\pi}{\Omega}$. The method of proof is obvious: compute the Fourier series expansion of $\hat{f}(\lambda)$ on the interval $[-\Omega, \Omega]$.)

PROOF: We have

$$\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i\pi k\lambda}{\Omega}}, \qquad c_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\lambda) e^{-\frac{i\pi k\lambda}{\Omega}} d\lambda,$$

where the convergence is uniform on $[-\Omega, \Omega]$. This expansion holds only on the interval: $\hat{f}(\lambda)$ vanishes outside the interval.

Taking the inverse Fourier transform we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\lambda) e^{i\lambda t} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \sum_{k=-\infty}^{\infty} c_k e^{\frac{i(\pi k + t\Omega)\lambda}{\Omega}} d\lambda$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c_k \int_{-\Omega}^{\Omega} e^{\frac{i(\pi k + t\Omega)\lambda}{\Omega}} d\lambda = \sum_{k=-\infty}^{\infty} c_k \frac{\Omega \sin(\Omega t + k\pi)}{\pi(\Omega t + k\pi)}.$$

Now

$$c_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\lambda) e^{-\frac{i\pi k\lambda}{\Omega}} d\lambda = \frac{1}{2\Omega}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-\frac{i\pi k\lambda}{\Omega}} d\lambda = \frac{\pi}{\Omega} f(-\frac{\pi k}{\Omega}).$$

Hence, setting $k = -j$,

$$f(t) = \sum_{j=-\infty}^{\infty} f(\frac{j\pi}{\Omega}) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}.$$

Q.E.D.

There is a trade-off in the choice of Ω. Choosing it as small as possible reduces the sampling rate, hence the amount of data to be processed or stored. However, if we increase the sampling rate, i.e. *oversample*, the series converges more rapidly. Moreover, sampling at exactly the Nyquist rate leads to numerical instabilities in the reconstruction of the signal. This difficulty is related to the fact that the reconstruction is an expansion in sinc sinc $(\Omega t/\pi - j) = (\sin(\Omega t - j\pi)/(\Omega t - j\pi)$. The sinc function is frequency band-limited, but its Fourier transform is discontinuous, see (**??**). This causes the sinc function to decay slowly in time, like $1/(\Omega t - j\pi)$. Summing over j yields the, divergent, harmonic series:$\sum_{j=-\infty}^{\infty} |\text{sinc }(\Omega t/\pi - j)|$. Thus a small error ϵ for each sample can lead to arbitrarily large reconstruction error. Suppose we could replace sinc (t) in the expansion by a frequency band-limed function $g(t)$ such that $\hat{g}(\lambda)$ was infinitely differentiable. (Such C^∞ functions with compact support are constructed, for example, in [].) Since all derivatives $\hat{g}^{(n)}(\lambda)$ have compact support it follows from (**??**) that $t^n g(t)$ is square integrable for all positive integers n. Thus $g(t)$ decays faster than $|t|^{-n}$ as $|t| \to \infty$. This fast decay would prevent the numerical instability.

In order to employ $g(t)$ in place of the sinc function it will be necessary to oversample. Oversampling will provide us with redundant information but also flexibility in the choice of expansion function, and improved convergence properties. We will now take samples $f(j\pi/a\Omega)$ where $a > 1$. (A typical choice is $a = 2$.) Recall that the support of $\hat{f}$ is contained in the interval $[-\Omega, \Omega] \subset [-a\Omega, a\Omega]$. We choose $g(t)$ such that 1) $\hat{g}(\lambda)$ is arbitrarily differentiable, 2) its support is contained in the interval $[-a\Omega, a\Omega]$, and 3) $\hat{g}(\lambda) = 1$ for $\lambda \in [-\Omega, \Omega]$. Note that there are many possible functions g that could satisfy these requirements. Now we repeat the major steps of the proof of the sampling theorem, but for the interval $[-a\Omega, a\Omega]$. Thus

$$\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i\pi k\lambda}{a\Omega}}, \qquad c_k = \frac{1}{2a\Omega} \int_{-a\Omega}^{a\Omega} \hat{f}(\lambda) e^{-\frac{i\pi k\lambda}{a\Omega}} d\lambda.$$

At this point we insert $\hat{g}$ by noting that $\hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$, since $\hat{g}(\lambda) = 1$ on the support of $\hat{f}$. Thus,

$$\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} c_k \hat{g}(\lambda) e^{\frac{i\pi k\lambda}{a\Omega}}, \tag{1.5.11}$$

where from property 6 in Section **??**, $\hat{g}(\lambda)e^{\frac{i\pi k\lambda}{a\Omega}}$, is the Fourier transform of $g(t + \pi k/a\Omega)$. Taking the inverse Fourier transform of both sides (OK since the series on the right converges uniformly) we obtain

$$f(t) = \sum_{j=-\infty}^{\infty} f(\frac{j\pi}{a\Omega}) g(t - \pi j/a\Omega). \tag{1.5.12}$$

Since $|g(t)t^n| \to 0$ as $|t| \to \infty$ for any positive integer n this series converges very rapidly and is not subject to instabilities.

1.6 Relations between Fourier series and Fourier integrals: aliasing

Another way to compare the Fourier transform $(-\infty, \infty)$ with Fourier series is to periodize a function. To get convergence we need to restrict ourselves to functions that decay rapidly at infinity. We could consider functions with compact support, say infinitely differentiable. Another useful but larger space of functions is the Schwartz class. We say that $f \in L^2[-\infty, \infty]$ belongs to the *Schwartz class* if f is infinitely differentiable everywhere, and there exist constants $C_{n,q}$ (depending on f) such that $|t^n \frac{d^q}{dt^q} f| \leq C_{n,q}$ on R for each $n, q = 0, 1, 2, \cdots$. Then the projection operator P maps an f in the Schwartz class to a continuous function in $L^2[0, 2\pi]$ with period 2π. (However, periodization can be applied to a much larger class of functions, e.g. functions on $L^2[-\infty, \infty]$ that decay as $\frac{c}{t^2}$ as $|t| \to \infty$.):

$$P[f](t) = \sum_{m=-\infty}^{\infty} f(t + 2\pi m) \tag{1.6.13}$$

Expanding $P[f](t)$ into a Fourier series we find

$$P[f](t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} P[f](t) e^{-int} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-int} dx = \frac{1}{2\pi} \hat{f}(n)$$

where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$. Thus,

$$\sum_{n=-\infty}^{\infty} f(t + 2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}, \tag{1.6.14}$$

and we see that $P[f](t)$ tells us the value of $\hat{f}$ at the integer points $\lambda = n$, but not in general at the non-integer points. (For $t = 0$, equation (1.6.14) is known as the *Poisson summation formula*. If we think of f as a signal, we see that **periodization** (1.6.13) of f results in a loss of information. However, if f vanishes outside of $[0, 2\pi)$) then $P[f](t) \equiv f(t)$ for $0 \leq t < 2\pi$ and

$$f(t) = \sum_n \hat{f}(n) e^{int}, \quad 0 \leq t < 2\pi$$

without error.)

1.7 The Fourier integral and the uncertainty relation of quantum mechanics

The uncertainty principle gives a limit to the degree that a function $f(t)$ can be simultaneously localized in time as well as in frequency. To be precise, we introduce some notation from probability theory. Every $f \in L^2[-\infty, \infty]$ defines a probability distribution function $\rho(t) = \frac{|f(t)|^2}{||f||^2}$, i.e., $\rho(t) \geq 0$ and $\int_{-\infty}^{\infty} \rho(t)dt = 1$.

Definition 4 • *The mean of the distribution defined by f is*

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t|f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

• *The dispersion of f about $a \in R$ is*

$$\Delta_a f = \frac{\int_{-\infty}^{\infty} (t-a)^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

($\Delta_{\bar{t}} f$ is called the variance of f, and $\sqrt{\Delta_{\bar{t}} f}$ the standard deviation.

The dispersion of f about a is a measure of the extent to which the graph of f is concentrated at a. If $f = \delta(x-a)$ the "Dirac delta function", the dispersion is zero. The constant $f(t) \equiv 1$ has infinite dispersion. (However there are no such L^2 functions.) Similarly we can define the dispersion of the Fourier transform of f about some point $\alpha \in R$:

$$\Delta_\alpha \hat{f} = \frac{\int_{-\infty}^{\infty} (\lambda-\alpha)^2 |\hat{f}(\lambda)|^2 d\lambda}{\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda}.$$

Note: It makes no difference which definition of the Fourier transform that we use, $\hat{f}$ or $\mathcal{F}f$, because the normalization gives the same probability measure.

Example 2 *Let $f_s(t) = (\frac{2s}{\pi})^{1/4} e^{-st^2}$ for $s > 0$, the Gaussian distribution. From the fact that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ we see that $||f_s|| = 1$. The normed Fourier transform of f_s is $\hat{f}_s(\lambda) = (\frac{2}{s\pi})^{1/4} e^{\frac{-\lambda^2}{4s}}$. By plotting some graphs one can see informally that as s increases the graph of f_s concentrates more and more about $t = 0$, i.e., the dispersion $\Delta_0 f_s$ decreases. However, the dispersion of $\hat{f}_s$ increases as s increases. We can't make both values, simultaneously, as small as we would like. Indeed, a straightforward computation gives*

$$\Delta_0 f_s = \frac{1}{4s}, \qquad \Delta_0 \hat{f}_s = s,$$

so the product of the variances of f_s and $\hat{f}_s$ is always $\frac{1}{4}$, no matter how we choose s.

Theorem 10 *(Heisenberg inequality, Uncertainty theorem) If $f(t) \neq 0$ and $tf(t)$ belong to $L^2[-\infty, \infty]$ then $\Delta_a f \Delta_\alpha \hat{f} \geq \frac{1}{4}$ for any $a, \alpha \in R$.*

SKETCH OF PROOF: We will give the proof under the added assumptions that $f'(t)$ exists everywhere and also belongs to $L^2[-\infty, \infty]$. (In particular this implies that $f(t) \to 0$ as $t \to \pm\infty$.) The main ideas occur there.

We make use of the canonical commutation relation of quantum mechanics, the fact that the operations of multiplying a function $f(t)$ by t, $(Tf(t) = tf(t))$ and of differentiating a function $(Df(t) = f'(t))$ don't commute: $DT - TD = I$. Thus

$$\frac{d}{dt}\left[tf(t)\right] - t\left[\frac{d}{dt}f(t)\right] = f(t).$$

Now it is easy from this to check that

$$(\frac{d}{dt} - i\alpha)\left[(t-a)f(t)\right] - (t-a)\left[(\frac{d}{dt} - i\alpha)f(t)\right] = f(t)$$

also holds, for any $a, \alpha \in R$. (The a, α dependence just cancels out.) This implies that

$$\left((\frac{d}{dt} - i\alpha)[(t-a)f(t)], f(t)\right) - \left((t-a)[(\frac{d}{dt}\right.$$

$$\left. -i\alpha)f(t)], f(t)\right) = (f(t), f(t)) = ||f||^2.$$

Integrating by parts in the first integral, we can rewrite the identity as

$$-\left([(t-a)f(t)], [(\frac{d}{dt} - i\alpha)f(t)]\right) - \left([(\frac{d}{dt} - i\alpha)f(t)], [(t-a)f(t)]\right) = ||f||^2.$$

The Schwarz inequality and the triangle inequality now yield

$$||f||^2 \leq 2||(t-a)f(t)|| \cdot ||(\frac{d}{dt} - i\alpha)f(t)||. \qquad (1.7.15)$$

From the list of properties of the Fourier transform in Section 1.2.1 and the Plancherel formula, we see that $||(\frac{d}{dt} - i\alpha)f(t)|| = \frac{1}{\sqrt{2\pi}}||(\lambda - \alpha)\hat{f}(\lambda)||$ and $||f|| = \frac{1}{\sqrt{2\pi}}||\hat{f}||$. Then, dividing by $||f||$ and squaring, we have

$$\Delta_a f \Delta_\alpha \hat{f} \geq \frac{1}{4}.$$

Q.E.D.

NOTE: Normalizing to $a = \alpha = 0$ we see that the Schwarz inequality becomes an equality if and only if $2stf(t) + \frac{d}{dt}f(t) = 0$ for some constant s. Solving this differential equation we find $f(t) = c_0 e^{-st^2}$ where c_0 is the integration constant, and we must have $s > 0$ in order for f to be square integrable. Thus the Heisenberg inequality becomes an equality only for Gaussian distributions.

1.8 Mellin Transforms

Mellin transforms were first used in the 1890's in number theory to solve differential and difference equations. They are defined as follows:

Definition 5 *Let $f : [0, \infty) \to \mathbb{R}$. The* **Mellin transform** *of f is*

$$\mathcal{M}[f](s) := \int_0^\infty f(x)x^{s-1},\ s \in \mathbb{R}$$

whenever the integral on the right hand side exists. Uniqueness of the Mellin Transform is self evident. †

Example 3

- *Let*

$$f_a(x) := \begin{cases} x^a, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

  Here, $\mathcal{M}[f_a](s) = \frac{-1}{a+s}$, $s < -a$.

- *Let $f(t) = \exp(-t)$, $t \geq 0$. Then*

$$\mathcal{M}[f](s) = \int_0^\infty \exp(-t)t^{s-1}ds = \Gamma(s)$$

  In particular, $\mathcal{M}[f](n+1) = \Gamma(n+1) = n!$ when n is a non-negative integer.

Further Properties of the Mellin Transform

The following easily derived further properties of the Mellin Transform are listed below as a theorem.

Theorem 11 *(1)* **Linearity***: Let $a, b \in \mathbb{R}$. Then*

$$\mathcal{M}[af + bg](s) = a\mathcal{M}[f](s) + b\mathcal{M}[g](s).$$

† This definition makes sense for complex valued f but we stick here to real valued f

(2) **Multiplication property***: Let $j \geq 0$.*

$$\mathcal{M}[x^j f(x)](s) = \mathcal{M}[f(x)](s+j).$$

(3) **Derivatives***: Let $s > 1$ or $f(0) = 0$. Then*

$$\mathcal{M}[f'](s) = -(s-1)\mathcal{M}[f](s-1).$$

Example 4 *(i) Suppose formally we have $h(s) = \mathcal{M}[f](s)$. Then*

$$h(s+j) = \mathcal{M}[x^j f(x)](s).$$

Suppose that, in addition, formally

$$\begin{aligned} k(s) &= h(s+3) - 4h(s+2) + 2h(s+1) - h(s) \\ &= \mathcal{M}[x^3 f(x)] - 4\mathcal{M}[x^2 f(x)] + 2\mathcal{M}[xf(x)] - \mathcal{M}[f(x)] \\ &= \mathcal{M}[(x^3 - 4x^2 + 2x - 1)f(x)](s). \end{aligned}$$

This observation is often used in solving a difference equation such as the above when we know that the function involved, namely h, can be expressed as a Mellin transform. Thus, if, $k(s) = \mathcal{M}[g](s)$, then

$$\mathcal{M}[(x^3 - 4x^2 + 2x - 1)f(x)](s) = \mathcal{M}g(x)](s)$$

from which we deduce that formally,

$$f(x) = \frac{g(x)}{x^3 - 4x^2 + 2x - 1}.$$

(ii) Claim: For $0 < \alpha < 1$,

$$\sum_{j=0}^{\infty} \frac{\alpha^j \Gamma(s+j)}{j!} = (1-\alpha)^{-s}\Gamma(s)$$

for all s for which the right hand side makes sense. Indeed,

$$\sum_{j=0}^{\infty} \frac{\alpha^j \Gamma(s+j)}{j!}$$
$$= \sum_{j=0}^{\infty} \frac{\alpha^j \mathcal{M}[\exp(-t)](s+j)}{j!}$$
$$= \sum_{j=0}^{\infty} \frac{\alpha^j \mathcal{M}[t^j \exp(-t)](s)}{j!}$$
$$= \mathcal{M}\left[\left(\sum_{j=0}^{\infty} \frac{\alpha^j t^j}{j!}\right) \exp(-t)\right](s)$$
$$= \mathcal{M}[\exp(\alpha-1)t](s)$$
$$= (1-\alpha)^{-s} \int_0^{\infty} \exp(-u) u^{s-1} du$$
$$= (1-\alpha)^{-s}\Gamma(s).$$

We now define **Mellin Convolution**:

Definition 6 *Let $f, g : [0, \infty) \to \mathbb{R}$. We define the* **Mellin Convolution**

$$(f * g)(x) = \int_0^{\infty} f(x/u) g(u) \frac{du}{u},\ x > 0.$$

Theorem 12

$$\mathcal{M}[f * g] = \mathcal{M}[f]\mathcal{M}[g].$$

Proof

$$\mathcal{M}[f * g](s) = \int_0^{\infty} (f * g)(x) x^{s-1} dx$$
$$= \int_0^{\infty} \left[\int_0^{\infty} f(x/u) g(u) \frac{du}{u}\right] x^{s-1} dx$$
$$= \int_0^{\infty} \left[\int_0^{\infty} f(x/u) x^{s-1} dx\right] \frac{g(u)}{u} du$$

[‡] Now we make the substitution $x = ut$ in the inner integral: (u is fixed in the inner inegral). Then we see that

$$\int_0^{\infty} f(x/u) x^{s-1} dx = u^s \int_0^{\infty} f(t) t^{s-1} dt$$
$$= u^s \mathcal{M}[f](s).$$

[‡]The order interchange can be justified

Then

$$\mathcal{M}[f*g](s) = \mathcal{M}[f](s)\int_0^\infty g(u)u^{s-1}du$$
$$= \mathcal{M}[f](s)\mathcal{M}[g](s).$$

.

Example 5 *(1) Let $a, b \in \mathbb{R}$ and*

$$g_a(x) = \begin{cases} x^a, & x \in (0,1] \\ 0, & x \notin (0,1]. \end{cases}$$

Show that $\mathcal{M}[g_a](s) = \frac{1}{a+s}$, $a + s > 0$.

(2) Show that

$$(g_a * g_b)(x) = \begin{cases} \frac{x^b - x^a}{a-b}, & x \in [0,1) \\ 0, & x \geq 1 \end{cases}$$

and hence deduce that

$$\mathcal{M}[g_a * g_b] = \frac{1}{(a+s)(b+s)}.$$

(3) Use (1-2) to solve the **Euler differential equation**

$$xf'(x) + f(x) = x^a,\ x \in [0,1].$$

Hint: Take Mellin Trasforms of each side, use the linearity and derivative rules for Mellin Transforms and then apply (1-2).

1.9 Additional Exercises

Exercise 1 *Find the Fourier transform of the following functions (a sketch may help!). Also write down the inversion formula for each, taking account of where they are discontinuous.*

(i) Let $A, T > 0$. Let f be the rectangular pulse

$$f(t) = \begin{cases} A, & t \in [0,T] \\ 0, & t \notin [0,T] \end{cases}$$

(ii) Let $A, T > 0$. Let f be the two-sided pulse

$$f(t) = \begin{cases} -A, & t \in [-T,0] \\ A, & t \in (0,T] \\ 0, & t \notin [-T,T] \end{cases}$$

(iii) *Let f be the triangular pulse*

$$f(t) = \begin{cases} t+1, & t \in [-1,0] \\ 1-t, & t \in (0,1] \\ 0, & t \notin [-1,1] \end{cases}$$

Deduce that

$$\int_0^\infty \frac{\sin^2(x/2)}{x^2} dx = \frac{\pi}{4}.$$

(iv) *Let $a > 0$ and*

$$f(t) := \begin{cases} \sin(at), & |t| \leq \pi/a \\ 0, & \text{else} \end{cases}$$

(v) *Let*

$$f(t) := \begin{cases} 0, & t < 0 \\ \exp(-t), & t \geq 0 \end{cases}$$

Deduce that

$$\int_0^\infty \frac{\cos(xt) + x\sin(xt)}{1+x^2} dx = \begin{cases} \pi\exp(-t), & t > 0 \\ \pi/2, & t = 0 \\ 0, & t < 0 \end{cases}$$

(vi) *Let $a, b \in \mathbb{R}$ and $f(t) := \exp^{-|at+b|}$, $t \in \mathbb{R}$.*

(vii) *Let $f(t) := (t^2 - 1)\exp^{-t^2/2}$, $t \in \mathbb{R}$.*

Exercise 2 *Prove the following: If f is even,*

$$\mathcal{F}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(xt)dt$$

and if f is odd,

$$\mathcal{F}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(xt)dt.$$

Exercise 3 *The Fourier Cosine ($\mathcal{F}_c[f](.)$) and Fourier Sine ($\mathcal{F}_s[f](.)$) of $f : \mathbb{R} \to \mathbb{R}$ are defined as follows:*

$$\mathcal{F}_c[f](x) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(xt)dt.$$

$$\mathcal{F}_s[f](x) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(xt)dt.$$

Find the Fourier Cosine and Sine transform of the following funcions:

$$f(t) := \begin{cases} 1, & t \in [0,a] \\ 0, & t > a \end{cases}$$

$$f(t) := \begin{cases} \cos(at), & t \in [0,a] \\ 0, & t > a \end{cases}$$

Exercise 4 *Let $f, g, h : \mathbb{R} \to \mathbb{R}$. Let $a, b \in \mathbb{R}$. The following question deals with (convolution *): Show that:*

(i) ** is linear:*

$$(af + bg) * h = a(f * h) + b(g * h).$$

(ii) ** is commutative:*

$$f * g = g * f.$$

(iii) ** is associative:*

$$(f * g) * h = f * (g * h).$$

Exercise 5 *Let $g : \mathbb{R} \to \mathbb{R}$. Find a function H such that for all x,*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} g(t)dt = (H * g)(x).$$

(H is called the Heaviside function).

Exercise 6 *Let $f, g : \mathbb{R} \to \mathbb{R}$. Let f' exist. Assuming the convergence of the relevent integrals below, show that*

$$(f * g)'(x) = f'(x) * g(x).$$

Exercise 7 *For $a \in \mathbb{R}$, let*

$$f_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Compute $f_a * f_b$ for $a, b \in \mathbb{R}$. Deduce that

$$(f_a * f_{-a})(x) = \frac{x f_0(x)}{\sqrt{2\pi}}.$$

Does $f_a * (1 - f_b)$ exist? For $a \in \mathbb{R}$, let

$$g_a(t) := \begin{cases} 0, & t < 0 \\ \exp(-at), & t \geq 0 \end{cases}$$

Compute $g_a * g_b$.

Exercise 8 *Fourier transforms are useful in "deconvolution" or solving "convolution integral equations". Suppose, that we are given functions g, h and are given that*

$$f * g = h.$$

Our task is to find f in terms of g, h.

(i) Show that

$$\mathcal{F}[f] = \mathcal{F}[h]/\mathcal{F}[g]$$

and hence, if we can find a function k such that

$$\mathcal{F}[h]/\mathcal{F}[g] = \mathcal{F}[k]$$

then $f = k$.

(ii) As an example, suppose that

$$f * \exp(-t^2/2) = (1/2)t\exp(-t^2/4).$$

Find f.

Exercise 9 *(i) We recall that the* **Laplace transform** *of a function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$$\mathcal{L}[f](p) = \int_0^\infty f(t)\exp(-pt)dt$$

whenever the right hand side makes sense. Show formally, that if we set

$$g(x) := \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then

$$\mathcal{L}[f](p) := \sqrt{2\pi}\mathcal{F}[g](-ip).$$

(ii) Let $h : \mathbb{R} \to \mathbb{R}$ and define:

$$h_+(x) := \begin{cases} h(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and

$$h_-(x) := \begin{cases} h(-x), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Show that $h(x) = h_+(x) + h_-(x)$ and express $\mathcal{F}[h]$ in terms of $\mathcal{L}[h_+]$ and $\mathcal{L}[h_-]$.

Exercise 10 *The following question deals with properties of Mellin convolution $*$. Let $a, b \in \mathbb{R}$ and $f, g, h : [0, \infty) \to \mathbb{R}$.*

(i) Linearity: Show that $(af + bg) * h = a(f * h) + b(g * h)$.

(ii) Commutativity: Show that $f * g = g * f$.

(iii) Associativity: Show that $(f * g) * h = f * (g * h)$.

Exercise 11 *Show that if $a > 0$,*

$$\mathcal{M}[f(ax)](s) = a^{-s}\mathcal{M}[f](s).$$

Exercise 12 *Let*

$$h(u) := \begin{cases} 0, & u < 1 \\ \log u, & u \geq 1 \end{cases}$$

$$f(t) := \begin{cases} 0, & t < 1 \\ \exp(-t), & t \geq 0 \end{cases}$$

Compute $h * f$.

Exercise 13 *(i) Show that for $j \geq 1$,*

$$\mathcal{M}[f^(j)](s) = -(s-1)\mathcal{M}[f^{j-1}](s-1)$$

and hence that

$$\mathcal{M}[f^(j)](s) = (-1)^j(s-1)(s-2)...(s-j)\mathcal{M}[f](s-j).$$

(ii) Deduce that

$$\mathcal{M}[x^j f^{(j)}(x)](s) = (-1)^j(s+j-1)(s+j-2)...(s+1)s\mathcal{M}[f](s).$$

Exercise 14 *Consider the* **Euler differential equation**

$$x^2 f''(x) + 3xf'(x) - 3f(x) = g_0(x),\ x \in (0, 1]$$

where for $a \in \mathbb{R}$,

$$g_a(x) := \begin{cases} x^a, & x \in (0, 1] \\ 0, & x \geq 1 \end{cases}$$

Solve this in the following steps:

(i) Take Mellin transforms and show that

$$\mathcal{M}[f](s) = \frac{1}{s(s^2 - 2s - 3)}.$$

(ii) Use partial fractions to show that

$$\mathcal{M}[f](s) = \frac{-1}{3s} + \frac{1}{12(s-3)} + \frac{1}{4(s+1)}.$$

(iii) Recalling that $\mathcal{M}[g_a](s) = \frac{1}{a+s}$*, deduce that*

$$f(x) = \frac{-1}{3} + \frac{1}{12x^3} + \frac{x}{4},\ x \in (0,1].$$

Exercise 15 *Extend in a natural way the definition of the Mellin Transform for* $f : [0, \infty) \to \mathbb{C}$*. Hence solve as in (6) the Euler differential equation*

$$x^2 f''(x) + xf'(x) + f(x) = g_{2i}(x),\ x \in (0,1]$$

where g_a *is extended naturally to* $a \in \mathbb{C}$*.*