A Geometrical Perspective on the Coherent Multimode Optical Field and Mode Coupling Equations

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Abstract—The generalization of the Poincaré sphere to $N \geq 2$ modes is the $(N − 1)$-dimensional complex projective space $CP(N − 1)$. There is a minimal set of $2N − 2$ Stokes vector components that determine the coherent multimode optical field. These are obtained from the inverse stereographic projection of coordinate hyperplanes in $CP(N−1)$ into a $2N−2$ sphere, just as in the $N = 2$ case. We derive N-mode analogs of Poole's optical fiber polarization-mode dispersion (PMD) equations that involve only $2N − 2$ independent variables. This is achieved by means of an explicit generalized coherent state representation of the optical field, which enables the components of the PMD vector to be expressed in terms of the optical state and its frequency derivatives. Poole's equations describe mode coupling as a flow on $CP(N−1)$. We give general constraints on the mode-coupling matrix and Stokes vector components. The group delay operator is shown to be a rank-2 perturbation of a diagonal matrix.

Index Terms—Multimode optical fiber, mode coupling, Poincaré sphere, complex projective space.

I. INTRODUCTION

THE "Poincaré sphere" representation of the polarization of a plane-wave [1] is well-known and widely valued because of its direct connection to measurements. Poole's polarization mode dispersion (PMD) equations [2], [3] describe mode coupling in optical fiber as a flow on the Poincaré sphere. It is natural that with the heightened interest in few-mode and multimode optical fiber, researchers would attempt to generalize the Poincaré sphere description and Poole's equations to a larger number of modes. Indeed, several interesting candidate geometries have already appeared in the literature, including very-large-dimensional spheres [4], disjoint collections of spheres [5], and special unitary groups [6]. However, these models can suffer from unphysical degrees of freedom in the geometry or insufficiently constrained coupling matrices. In particular, as acknowledged in [4], models involving "generalized Stokes parameters" can involve significantly more parameters than the optical field they are meant to describe.

Most models are based in some way on the symmetries of the special unitary matrix groups SU(N), but these groups are too large to provide an efficient description of modal

dynamics. To see this, choose any unit vector $|\Omega>$. The action of SU(N) is transitive: any other unit vector can be obtained by group action on $|\Omega>$. However, on any vector $|\Omega>$ there is a large subgroup of matrices in SU(N) that leaves it invariant. This subgroup is isomorphic to U(N−1). In the basis where $|\Omega> = [0\ 0\ 0\ 0 \ldots 1]^{T}$, this becomes obvious. The parameters of this subgroup are therefore immaterial to a physical description of the fully coherent optical field. (Partial coherence is not considered in this article.) Hence the most efficient generalization of the Poincaré sphere is the coset space SU(N)/U(N−1) of physically distinguishable group actions on a reference vector. For N=2 this is the usual Poincaré sphere but in general it is not a sphere. Instead SU(N)/U(N−1) is isomorphic to the $(N−1)$-dimensional complex projective space $CP(N−1)$ [7]. The number of real parameters needed to specify a point in $CP(N−1)$ is $2N−2$, the same as the number of parameters needed to specify an N-mode complex optical field, once the overall phase and amplitude of the field are fixed. The coset spaces above consist of the "generalized coherent states" [7], [8] of the group SU(N). That SU(3)/U(2) $\approx$ CP(2) is the most appropriate generalization of the Poincaré sphere to three modes was already stated in [9].

In this article we show by explicit construction how the coset action in SU(N) can be implemented as a unitary matrix parameterized by coordinates in CP(N−1). Generalized Stokes parameters are defined, and their relationship to CP(N−1) coordinates is demonstrated. We explicitly compute the derivatives of the matrix representing the coset action with respect to position and frequency. Using the special form of these derivatives, we are able to present generalized Poole's equations with the physical number of degrees of freedom. The connection to previous work is discussed and an algorithm for eliminating dependent variables is given. Examples are provided for N = 2, 3. We show that the mode-coupling matrix elements cannot all be arbitrarily chosen. Finally the group delay matrix is considered. It is shown to be a rank-two perturbation of the diagonal, uncoupled group delay matrix.

We note that coset spaces have already been used in atomic physics to describe the dynamics of higher order Bloch spheres in N-level atomic systems possessing SU(N) symmetries, such as nuclear magnetic resonance or laser-atom interactions [10], [11]. These treatments typically exploit decompositions of SU(N) that represent general group elements as products of several simpler elements, such as rotations about coordinate axes. In our case, this approach would lead to an expression of the generalized displacement

matrix described below as a product of simpler matrices. Instead, we use a result, due to Gilmore, that expresses the generalized displacement matrix as the exponential of a single matrix [7]. In this paper, we find an explicit expression (6) for this matrix exponential. This formula enables the derivation of mode coupling equations and constraints on the mode coupling matrix and group delay operator that would not be immediately obvious using previously published techniques.

II. REPRESENTATION OF THE N-MODE OPTICAL FIELD

Let the N-mode optical field be given by:

$$E = \exp(i\theta(\omega)) M |s\rangle, \tag{1}$$

where θ is an overall frequency-dependent phase,

$$M = \mathrm{diag}\big(\exp[ik_1(\omega)z], \\ \exp[ik_2(\omega)z], \ldots, \exp[ik_N(\omega)z]\big) \tag{2}$$

is a diagonal matrix, and the complex N-vector

$$|s\rangle = [s_1, s_2, \cdots, s_N]^T \tag{3}$$

is slowly varying in position z and frequency ω. We choose $<s|s> = 1$, and assume $0 < s_N \leq 1$ is real and positive. With these constraints, the set of complex (N−1)-tuples $\{(s_1/s_N, s_2/s_N, \ldots, s_{N-1}/s_N)\}$ comprise a coordinate chart on CP(N−1). (The $s_N = 0$ case requires the use of a different coordinate chart.) We can represent a general state |s> as the coset action on |Ω> as follows:

$$|s\rangle = R_s|\Omega\rangle = R_s \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \text{ where } R_s = \begin{pmatrix} * & * & s_1 \\ * & * & s_2 \\ \vdots & \vdots & \vdots \\ * & * & s_N \end{pmatrix}. \tag{4}$$

R_s is the generalized displacement operator. Following [7], a unitary representation of R_s in the form

$$R_s = \exp\left[i\,\tilde{A}\right] = \exp\left[i\left(\begin{array}{c|c} 0 & A \\ \hline A^* & 0 \end{array}\right)\right] \tag{5}$$

may be obtained with a complex (N–1)-vector $A = [a_1, a_2, \ldots, a_{N-1}]^T$. (The asterisk on A denotes Hermitian transpose.) The components of A are related to the vector $S = [s_1, s_2, \ldots, s_{N-1}]^T$ in [7]. However, we have found that the exponential in (5) may be explicitly computed. One may express R_s directly in terms of S: denoting by I_{N-1} the (N–1)-dimensional unit matrix, one has

$$R_s = \left(\begin{array}{c|c} I_{N-1} - \frac{SS^*}{1+s_N} & S \\ \hline -S^* & s_N \end{array}\right), \quad s_N = \sqrt{1 - S^*S} > 0. \tag{6}$$

Equation (6) makes clear that while R_s is in SU(N), the parameters in S alone are sufficient to describe the coherent optical field.

III. THE STOKES VECTOR

We can define $N^2 - 1$ real Stokes vector components by proceeding as in [3], [12],

$$\hat{s}_q = \sqrt{\frac{N}{2N-2}}\langle s| L^{(q)}|s\rangle = \sqrt{\frac{N}{2N-2}}\langle \Omega| R_s^{-1} L^{(q)} R_s|\Omega\rangle \tag{7}$$

where $q = 1, 2, \ldots, N^2 - 1$ and the $L^{(q)}$ are $N \times N$ generators of the real su(N) Lie algebra. This definition is normalized such that

$$\sum_{q=1}^{N^2-1} \hat{s}_q^2 = 1 \tag{8}$$

regardless of N. (Reference [12] discusses the problem of measuring these parameters for the case N = 4.) Particular expressions for Stokes vector components depends on the Lie algebra basis chosen. We choose a specific trace-orthogonal set of generators $L^{(q)}$ such that the first 2N−2 of them are:

$$L_{m,n}^{(2q-1)} = \delta_{m,q}\,\delta_{n,N} + \delta_{m,N}\,\delta_{n,q} \\ L_{m,n}^{(2q)} = -i\,(\delta_{m,q}\,\delta_{n,N} - \delta_{m,N}\,\delta_{n,q}) \\ q = 1, 2, \ldots, N-1 \tag{9}$$

Importantly, none of the other generators have any non-zero entries in the Nth row or column, except for possibly the (N, N) element. These first 2N−2 generators implement the coset action. We will also augment (9) with the choice:

$$L^{(N^2-1)} = \sqrt{\frac{2}{N(N-1)}}\mathrm{diag}(1, 1, \ldots, 1, 1-N). \tag{10}$$

We leave the remaining generators unspecified, as their specific form is not important for the general analysis. Substituting (6) and (9) into (7) shows that the first 2N−2 of these components are proportional to the real and imaginary parts of the components of S and are thus independent. By virtue of (10), the last Stokes parameter depends only on s_N:

$$\hat{s}_{N^2-1} = \frac{1}{N-1}\left(1 - Ns_N^2\right) \tag{11}$$

In the case N=2, (7) yields:

$$\hat{s}_1 = 2s_2\mathrm{Re}\,\bar{s}_1 \\ \hat{s}_2 = 2s_2\mathrm{Im}\,\bar{s}_1 \\ \hat{s}_3 = |s_1|^2 - s_2^2. \tag{12}$$

One readily checks that these are the usual values for the Stokes parameters. In (12) and below, the overbar denotes complex conjugation. The Stokes vector components are the Cartesian coordinates of the Poincaré sphere—they are the inverse stereographic projection of the point $\{\bar{s}_1/s_2\}$ in the complex plane (a coordinate chart in CP(1)) into the unit sphere in three real dimensions. The projection axis is in the direction $\hat{s}_3$.

For general $N \geq 2$, (7) gives, for $q = 1, 2, \ldots, N-1$:

$$\hat{s}_{2q-1} = \sqrt{\frac{2N}{N-1}}s_N\mathrm{Re}\,s_q, \quad \hat{s}_{2q} = -\sqrt{\frac{2N}{N-1}}s_N\mathrm{Im}\,s_q \tag{13}$$
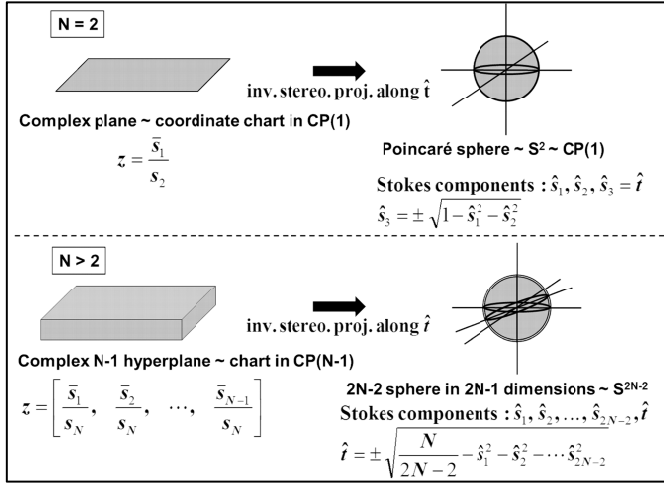
Fig. 1. Generalization of the construction of the Poincaré sphere and Stokes parameters to N>2 modes. The conventional N=2 Poincaré sphere is the inverse stereographic projection of a complex plane, a coordinate chart in CP(1). The Stokes parameters are the Cartesian components of this sphere. In the same way, a (2N–2) sphere may be obtained from a coordinate chart in CP(N–1). The 2N–1 Cartesian components of this sphere are the generalized Stokes parameters for an N-mode field.

Equations (11) and (13) imply:

$$\sum_{k=1}^{2N-2} \hat{s}_k^2 + \hat{t}^2 = \frac{N}{2N-2} \tag{14}$$

where

$$\hat{t} = \sqrt{\frac{2N-2}{N}}\left(\hat{s}_{N^2-1} + \frac{N-2}{2N-2}\right). \tag{15}$$

In other words, as depicted in Fig. 1, the first 2N–2 Stokes parameters lie on a (2N−2)-dimensional sphere. Stereographic projection along the $\hat{t}$-axis shows that these Stokes vector components are exactly the inverse stereographic projection of the point S in the CP(N−1) coordinate chart:

$$
\left(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{2N-2}, \hat{t}\right) \leftrightarrow \left(\text{Re}\left(\frac{\bar{s}_1}{s_N}\right), \text{Im}\left(\frac{\bar{s}_1}{s_N}\right),\right.
$$
$$
\text{Re}\left(\frac{\bar{s}_2}{s_N}\right), \text{Im}\left(\frac{\bar{s}_2}{s_N}\right), \ldots,
$$
$$
\left.\text{Re}\left(\frac{\bar{s}_{N-1}}{s_N}\right), \text{Im}\left(\frac{\bar{s}_{N-1}}{s_N}\right)\right)
$$
$$2N-2 \text{ sphere} \leftrightarrow 2N\text{–}2 \text{ dim coord. patch in CP(N–1)} \tag{16}$$

Combining (8) and (14) shows that the remaining Stokes components also lie on a sphere with radius depending on $\hat{t}$. However, they are not independent and can all be expressed entirely in terms of the first 2N−2 independent components, using (7) and (13). For example, for N = 2, (14) specifies the value of $\hat{t} = \hat{s}_3$ in terms of $\hat{s}_1$ and $\hat{s}_2$ up to a sign.

For N = 3, we have:

$$\hat{s}_5 = \sqrt{3}\frac{\hat{s}_1\hat{s}_3 + \hat{s}_2\hat{s}_4}{1 - 2\,\hat{s}_8}$$

$$\hat{s}_6 = \sqrt{3}\frac{\hat{s}_1\hat{s}_4 - \hat{s}_2\hat{s}_3}{1 - 2\,\hat{s}_8}$$

$$\hat{s}_7 = \frac{\sqrt{3}}{2}\frac{\hat{s}_1^2 + \hat{s}_2^2 - \hat{s}_3^2 - \hat{s}_4^2}{1 - 2\,\hat{s}_8}$$

$$\hat{s}_8 = \frac{\sqrt{3}}{2}\hat{t} - \frac{1}{4}, \hat{t} = \pm\sqrt{\frac{3}{4} - \sum_{k=1}^{4}\hat{s}_k^2} \tag{17}$$

Of course, the precise form of these equations depends on the generators chosen as the basis for expansion. Here (9) and (10) are augmented with the following to obtain (17):

$$L^{(5)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L^{(6)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L^{(7)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

For N > 3, one readily obtains similar equations for the dependent parameters. Because the Stokes parameters arise from an inverse stereographic projection, the sign of $\hat{t}$ is not constrained by the first 2N–2 Stokes components; it must be separately determined. However, it is clearly not necessary to measure all N^2-1 Stokes vector components to completely determine the optical state.

IV. DYNAMICS

Using (6) we obtain the following new formula for the derivative matrix Q_x (where x = ω frequency or x = z spatial derivative):

$$\partial_x|s\rangle = Q_x|s\rangle, \quad \text{where } Q_x = \partial_x R_s R_s^{-1} = \left(\begin{array}{c|c} D_x & B_x \\ \hline -B_x^* & F_x \end{array}\right). \tag{19}$$

The vector B_x is given by

$$B_x = \partial_x S - \frac{S^*\partial_x S}{1 + s_N}S + \frac{\partial_x(S^*S)}{2s_N}S \tag{20}$$

and the matrix D_x and scalar F_x are given by:

$$D_x = \frac{B_x S^* - S B_x^*}{1 + s_N} + \frac{\left(S^*B_x - B_x^*S\right)SS^*}{2s_N\left(1 + s_N\right)^2}$$

$$F_x = \frac{B_x^*S - S^*B_x}{2s_N} \tag{21}$$

The key new observation here is that (21) gives both D_x and F_x as linear functions of B_x and B_x^*, with coefficients that depend only on S and S^*. The local dependence of |s> on the variables ω and z is entirely determined by the projection of the Lie algebra on a 2(N−1)-dimensional subspace.

Equation (20) may be inverted to give the CP(N–1) coordinate derivatives in terms of B_x as follows:

$$\partial_x S = B_x + \frac{S^* B_x - B_x^* S}{2s_N} S - \frac{S^* B_x}{1 + s_N} S \qquad (22)$$

We make the following definitions to better distinguish between ω and z:

$$\tilde{\alpha}_1 = \alpha_2 + i\alpha_1 = B_{\omega,1} \qquad \tilde{\beta}_1 = \beta_2 + i\beta_1 = B_{z,1}$$
$$\tilde{\alpha}_2 = \alpha_4 + i\alpha_3 = B_{\omega,2} \qquad \tilde{\beta}_2 = \beta_4 + i\beta_3 = B_{z,2}$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\tilde{\alpha}_{N-1} = \alpha_{2N-2} \qquad \tilde{\beta}_{N-1} = \beta_{2N-2}$$
$$\qquad + i\alpha_{2N-3} = B_{\omega,N-1} \qquad + i\beta_{2N-3} = B_{z,N-1}.$$
$$(23)$$

The α's and β's with tildes are complex, the others real. Then taking derivatives of (20) and performing some straightforward algebraic manipulations yields the following:

$$\frac{\partial \tilde{\alpha}}{\partial z} - \frac{\partial \tilde{\beta}}{\partial \omega} = \frac{\left(S^* \tilde{\beta}\right)\left(\tilde{\alpha}^* S\right) - \left(S^* \tilde{\alpha}\right)\left(\tilde{\beta}^* S\right)}{2s_N \left(1 + s_N\right)^2} S$$
$$+ \frac{(s_N - 1)}{2s_N (1 + s_N)} \left(\left(S^* \tilde{\alpha}\right)\tilde{\beta} - \left(S^* \tilde{\beta}\right)\tilde{\alpha}\right)$$
$$+ \frac{1}{2s_N}\left(\left(\tilde{\alpha}^* S\right)\tilde{\beta} - \left(\tilde{\beta}^* S\right)\tilde{\alpha}\right) - \frac{\tilde{\beta}^* \tilde{\alpha} - \tilde{\alpha}^* \tilde{\beta}}{1 + s_N} S$$
$$(24)$$

This is our reframing and generalization of Poole's equations [3] for all N$\geq$2. It describes the dispersive evolution of the (N–1)-dimensional complex vector $\tilde{\alpha}$ along the fiber length in terms of the frequency derivative of the vector $\tilde{\beta}$ and a generalized cross product of $\tilde{\alpha}$ and $\tilde{\beta}$, with coefficients that depend only on S. As discussed in the next section, $\tilde{\alpha}$ is related to the "PMD vector," which derives from group delay, and $\tilde{\beta}$ is related to the "birefringence vector" in the standard terminology [3]. When N=2, $\tilde{\alpha}$ and $\tilde{\beta}$ are just complex numbers, i.e., they have only two real components each.

V. CONNECTION TO PAST WORK

Poole's equations have been generalized to the N-mode case previously [4]. This was accomplished by expanding derivatives of the displacement operator R_s in terms of generators of the Lie algebra su(N). There are N^2-1 of these and all must be included in the expansion. However, we have already shown that only 2N−2 of the coefficients in the expansions can be independent. The rest are linearly dependent on the first 2N−2.

The results of [4] can be obtained by expanding the derivative operators as follows:

$$Q_\omega = i \sum_{q=1}^{N^2-1} \alpha_q L^{(q)}, \quad Q_z = i \sum_{q=1}^{N^2-1} \beta_q L^{(q)} \qquad (25)$$

where the $N^2 - 1$ coefficients α_q and β_q are real-valued. They are referred to as components of the "PMD vector" and "birefringence vector," respectively. Then by taking

second derivatives and equating mixed partials [4] obtains a generalization of Poole's equations:

$$\partial_z \alpha_q - \partial_\omega \beta_q = \sum_{a,b=1}^{N^2-1} \alpha_a \beta_b C_q^{ab}$$
$$q = 1, 2, \ldots N^2 - 1 \qquad (26)$$

where the C_q^{ab} are the Lie algebra structure constants for the chosen basis. The conventional Poole's equations are recovered when N = 2. Unfortunately, this formulation presents more components than required to describe the evolution of the optical field.

We will use the basis in (9) to illustrate how to reduce the system of equations. Starting with

$$\alpha_q = \frac{-i}{2} tr\left(Q_\omega L^{(q)}\right), \quad \beta_q = \frac{-i}{2} tr\left(Q_z L^{(q)}\right) \qquad (27)$$

with $q = 1, 2, \ldots, N^2-1$, one shows by direct calculation using (19) that the first (2N−2) α_q components are just the real and imaginary parts of the components of the B_ω vector, exactly as in (23). Likewise for the first (2N−2) β_q components. Thus (19) gives the first (2N−2) α's and β's explicitly in terms of the optical state and its derivatives. Therefore all elements of the matrices Q_ω and Q_z are linear functions of the first (2N−2) α's and β's, respectively, with coefficients that depend only on S. Applying (27) then gives explicit linear relations for each α_q (q > 2N−2) in terms of the first (2N−2) α_q's and likewise for the β_q's. Only the first (2N−2) α's and β's are independent variables, and the rest are linear functions of these. Thus we have a reduced set of equations involving independent variables exclusively:

$$\partial_z \alpha_q - \partial_\omega \beta_q = \sum_{a,b=1}^{2N-2} \alpha_a \beta_b C_q^{ab}(S)$$
$$q = 1, \ldots, 2N - 2 \qquad (28)$$

In contrast to (26), the coefficients $C_q^{ab}(S)$ are no longer constant but depend (exclusively) on S. Equation (28) describes the evolution of the independent components of the PMD vector as a flow on CP(N−1). It is equivalent to (24) but has the disadvantage that it is basis-dependent and the coefficients $C_q^{ab}(S)$ are difficult to write down succinctly. Because the components are S dependent, either (24) or (28) must be integrated together with the propagation equation (19 or 31 below).

For a fixed N, it is straightforward to calculate expressions for the dependent coefficients using the above-described procedure. For example, when N=2, using (27) and (19) and choosing $(s_1, s_2) = \left(e^{-i\phi} \sin \rho, \cos \rho\right)$ leads to

$$\alpha_3 = -(\alpha_2 \sin \phi + \alpha_1 \cos \phi) \tan \rho. \qquad (29)$$

An analogous constraint equation holds for β_3. For N=3, there are four independent variables, $\alpha_1, \ldots, \alpha_4$, and four dependent ones, $\alpha_5, \ldots, \alpha_8$, and likewise for the β's. Using (9), (10) and (18), and choosing

$$(s_1, s_2, s_3) = \left(e^{-i\chi} \sin \rho \sin \eta, \, e^{-i\phi} \sin \rho \cos \eta, \, \cos \rho\right)$$

we have:

$$\alpha_8 = \frac{\sqrt{3}}{2}\alpha_1 \sin\eta\cos\chi\tan\rho + \frac{\sqrt{3}}{2}\alpha_2 \sin\eta\sin\chi\tan\rho$$
$$+ \frac{\sqrt{3}}{2}\alpha_3 \cos\eta\cos\phi\tan\rho + \frac{\sqrt{3}}{2}\alpha_4 \cos\eta\sin\phi\tan\rho \quad (30)$$

with three similar equations expressing α_5, α_6 and α_7 in terms of $\alpha_1, \ldots, \alpha_4$. For any N, one may apply (27) to compute linear equations like (29) and (30) for the dependent components. The advantage of (24) is that, from the outset, no extraneous components are involved.

VI. THE MODE COUPLING MATRIX

As explained in [3], the connection of Poole's equation to the physics of the glass medium is established by relating the β_q components that appear in Poole's equation to the physical, symmetric dielectric tensor ε that appears in the multimode Maxwell equation for the E-field in (1)

$$\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2}\varepsilon E = 0 \quad (31)$$

by expanding the dielectric tensor as follows:

$$\frac{\omega^2}{c^2}\varepsilon = \frac{\omega^2}{c^2}\left(\varepsilon^{(0)} + \delta\varepsilon\right) \equiv K_0\left(K_0 + 2\sum_{q=1}^{N^2-1}\gamma_q L^{(q)}\right), \quad (32)$$

where the γ_q's are the expansion coefficients. Here $\varepsilon^{(0)}$ is the uncoupled, diagonal dielectric tensor, $\delta\varepsilon$ is the coupling matrix, and $K_0 = K_0(\omega) = \mathrm{diag}(k_1, k_2, \ldots k_N)$ is the diagonal matrix of uncoupled propagation constants for the N modes.

The adiabatic approximation [3] then leads to

$$\partial_z|s\rangle = \frac{i\omega}{2nc}M^*\delta\varepsilon\, M|s\rangle = i\sum_{q=1}^{N^2-1}\gamma_q M^* L^{(q)} M|s\rangle \quad (33)$$

Equation (33) is consistent with (19) and (25) only if the following linear relationship holds:

$$\gamma_k = \sum_{q=1}^{N^2-1}\beta_q \sigma_{k,q}, \quad k = 1, \ldots, N^2-1 \quad (34)$$

where the coefficients

$$\sigma_{k,q} = \frac{1}{2}tr\left\{L^{(k)} M L^{(q)} M^*\right\} \quad (35)$$

describe the elements of a Hermitian, unitary matrix. The phase factors that arise in (35) account for the fact that the vector $|s\rangle$ is defined to be independent of rapidly varying propagation phases. Because, for a given optical state S, the matrix Q_z depends on only $2N-2$ real parameters (the components of the vector B_z), the same is true for $\delta\varepsilon$.

If the uncoupled propagation constants are the same for all modes, M is proportional to the identity matrix, $\delta\varepsilon$ is proportional to Q_z, and the γ-coefficients for the dielectric tensor expansion describing mode-coupling are the same as the β-coefficients for the Q_z expansion. This means that only the last column of the coupling matrix, proportional to the B_z vector, can be independently varied. All other matrix elements are determined by this choice and the optical state S. Specifying the coupling of all modes to the Nth mode is a full specification of the entire matrix, to this order in the adiabatic approximation.

Otherwise, if M is not a multiple of the identity, the γ's are linear combinations of the β's, but the situation is otherwise unchanged. Because we have already shown that only $2N-2$ of the β_q components are independent, it follows that only $2N-2$ of the γ_q components may be chosen independently. For the optical state to remain on CP(N−1), the additional γ_q components are constrained by (34). These constraints can be expressed in terms of γ by observing that the (N^2-1)-vector β satisfies an equation $\beta = G\,\beta$, with $G = G(S)$ a matrix having nonzero elements only in the first $2N-2$ columns. Thus $\gamma = \sigma\, G\,\sigma\,\gamma$.

VII. THE GROUP DELAY MATRIX

Finally, we note that (19) constrains frequency derivatives as well: when evaluated at $|s\rangle = |\Omega\rangle$, the matrix Q_ω takes the form:

$$Q_\omega|\Omega\rangle = i\tilde{A}|\Omega\rangle, \quad (36)$$

where $\tilde{A}$ is given in (5). Clearly the maximum rank of $\tilde{A}$ is two. The tangent space to Q_ω at a general $|s\rangle$ consists of the matrices $iR_s\tilde{A}R_s^*$. Because R_s is invertible, the rank of $iR_s\tilde{A}R_s^*$ and hence Q_ω is the rank of $\tilde{A}$. The group delay operator is:

$$\begin{aligned} GD &= -i\partial_\omega\left(MR_s\right)\left(MR_s\right)^{-1} \\ &= -i\left(\partial_\omega M M^* + M\partial_\omega R_s R_s^* M^*\right) \\ &= -iM\left(M^*\partial_\omega M + \partial_\omega R_s R_s^*\right)M^*. \\ &= -i\partial_\omega M M^* - iM Q_\omega M^*. \end{aligned} \quad (37)$$

Hence the group delay operator is a rank-2 perturbation of the diagonal operator $-i\partial_\omega M\, M^*$ regardless of N.

VIII. CONCLUSION

We have shown that the appropriate generalization of the Poincaré sphere to $N \geq 2$ modes is the (N−1)-dimensional complex projective space CP(N−1). The generalized Stokes vector components are the Cartesian coordinates of the inverse stereographic projection of a CP(N−1) coordinate patch into a real 2N−2 sphere in 2N−1 dimensions. We derived N-mode analogs of Poole's PMD equations that involve only the physical number (2N−2) of independent variables, defining a flow on CP(N−1). It may be used to model mode coupling in multimode fiber. The PMD vector components are explicitly related to the optical field and its frequency derivatives. Corresponding general constraints on the mode-coupling matrix are found. The group delay operator was shown to be unitarily equivalent to a rank-2 perturbation of a diagonal matrix.

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