Superintegrability in a non-conformally-flat space

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Abstract

Superintegrable systems in two- and three-dimensional spaces of constant curvature have been extensively studied. From these, superintegrable systems in conformally flat spaces can be constructed by Stäckel transform. In this paper a method developed to establish the superintegrability of the Tremblay-Turbiner-Winternitz system in two dimensions is extended to higher dimensions and a superintegrable system on a non-conformally-flat four-dimensional space is found. In doing so, curvature corrections to the corresponding classical potential are found to be necessary. It is found that some subalgebras of the symmetry algebra close polynomially.

A maximally superintegrable quantum system on a n-dimensional manifold is an integrable Hamiltonian system of n mutually commuting differential operators and an additional $n-1$ differential operators so that the full $2n-1$ are algebraically independent and commute with one distinguished operator, the Hamiltonian, which we will take to have the form $H = \nabla^2 + V$. In all previously know quantun superintegrable systems of this form, ∇^2 is the the natural Laplacian of a constant curvature manifold. A Stäckel transform can be used to construct systems on conformally-flat manifolds from systems of constant curvature manifolds [3, 6], but no systems have been previously exhibited on a non-conformally-flat manifold.

In [4] a classical superintegrable system on a non-conformally flat 4-dimensional space was found by generalising the two-dimensional Tremblay-Turbiner-Winternitz (TTW) system [8] and here we will show that this system can be quantised in a way that preserves its superintegrability.

The two-dimensional TTW system is integrable by virtue of a second order operator associated with its separability. Its superintegrability has been demonstrated by constructing a symmetry from raising and lowering operators built out of special function recurrence relations that act on an eigenbasis of separated solutions [5]. Here we extend raising and lowering operator method used on the quantum TTW system to higher dimensions and construct sufficient additional algebraically independent operators to show that it is superintegrable. Furthermore, we find that some subalgebras of symmetry operators close polynomially as is common with previously known superintegrable systems.

An interesting feature encountered below is that in order to construct the additional symmetries required for superintegrability the potential must be deformed by the addition of curvature terms that make the Hamiltonian conformally covariant. These terms are not simply the usual minimal choice for a conformally covariant Laplacian, namely, $-\mathcal{R}/6$, where $\mathcal{R}$ is the scalar curvature associated with the underlying metric, but also include an invariant constructed from the Weyl conformal curvature.

The system considered below is 4-dimensional only so as to provide the simplest non-conformally-flat example. The procedure can be extended to higher dimensions with no greater difficulty and a number of other systems similar to the TTW system [5] could also be extended.

1   The classical 4D non-conformally-flat system

The Tremblay-Turbiner-Winternitz (TTW) system [8] sparked great interest because it provided an infinite family of superintegrable systems and examples of systems with arbitrarily high degree symmetries. The system in polar coordinates is given by

$$H_{TTW} = p_r^2 + \alpha r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{\beta_1}{\cos^2(k\theta)} + \frac{\beta_2}{\sin^2(k\theta)} \right).$$

The superintegrability of both classical [1] and quantum versions [2] were established for all positive rational values of the parameter k along with the polynomial closure of symmetry algebra and the general approach was quickly extended to other families of systems in two dimensions such as

$$H = \cosh^2 \psi \left(p_\psi^2 + p_\varphi^2 + \frac{\alpha}{\cos^2 k\varphi} + \frac{\beta}{\sin^2 k\varphi} + \frac{\gamma}{\sinh^2 \psi} \right).$$

In the classical case, this approach as also been extended to higher dimensions and the superintegrability of the 4-dimensional generalisation of the classical TTW system,

$$H = L_1 \;=\; p_r^2 + \alpha r^2 + \frac{L_2}{r^2} \tag{1}$$

$$L_2 \;=\; p_{\theta_1}^2 + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_3}{\sin^2(k_1\theta_1)}$$

$$L_3 \;=\; p_{\theta_2}^2 + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_4}{\sin^2(k_2\theta_2)}$$

$$L_4 \;=\; p_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)},$$

was shown to be superintegrable for all positive rational k_1, k_2 and k_3 [4]. Here, the underlying manifold, on which this is a natural Hamiltonian system, has metric

$$g = e_0 \otimes e_0 + e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3, \tag{2}$$

$$e_0 = dr, \quad e_1 = r d\theta_1, \quad e_2 = r\sin(k_1\theta_1)d\theta_2, \quad e_3 = r\sin(k_1\theta_1)\sin(k_2\theta_2)d\theta_3,$$

and is no longer flat unless $k_1 = k_2 = 1$. Furthermore, in these coordinates, each component of the Weyl conformal tensor is a constant multiple of

$$\frac{k_1^2 - k_2^2}{r^2 \sin^2(k_1\theta_1)}$$

and so the underlying manifold is only conformally flat when this quantity vanishes, that is, $k_1 = k_2$. In [4] it was found that (1) is superintegrable with the 4 second order constants given above as well as an additional cubic and two quartic constants.

A natural question to ask is whether there is a corresponding quantum system with the same or minimally modified potential. It is straightforward to check that

$$H \;=\; \partial_r^2 + \frac{3}{r}\partial_r - \omega^2 r^2 + \frac{L_1}{r^2}$$

$$\frac{1-k_1^2}{r^2}$$

$$\frac{k_1^2 - k_2^2}{4r^2\sin^2(k_1\theta_1)}L_1 \;=\; \partial_{\theta_1}^2 + 2k_1\cot(k_1\theta_1)\partial_{\theta_1} + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_2}{\sin^2(k_1\theta_1)}$$

$$L_2 \;=\; \partial_{\theta_2}^2 + k_2\cot(k_2\theta_2)\partial_{\theta_2} + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_3}{\sin^2(k_2\theta_2)}$$

$$L_3 \;=\; \partial_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)}$$

are four mutually commuting differential operators and H is a Hamiltonian of the form $H = \nabla^2 + V_0$ where ∇^2 is the Laplacian on a four-dimensional manifold with metric (2) and

$$V_0 = \alpha r^2 + \frac{\beta_1}{r^2\cos^2(k_1\theta_1)} + \frac{\beta_2}{r^2\sin^2(k_1\theta_1)\cos^2(k_2\theta_2)} + \frac{\beta_3}{r^2\sin^2(k_1\theta_1)\sin^2(k_2\theta_2)\cos^2(k_3\theta_3)}$$

3

$$+ \frac{\beta_4}{r^2 \sin^2(k_1\theta_1)\sin^2(k_2\theta_2)\sin^2(k_3\theta_3)}. \tag{3}$$

However, it is not a simple matter to quantize the additional classical constants found in [4], and so to investigate whether this system remains superintegrable in the quantum case we attempt to adapt the raising and lowering operator methods from [5].

In a two-dimensional examples, with parameter k, the essence of the method is that solutions can be found by separation of variables with the separated eigenfunctions enumerated by positive integers n_0 and n_1. The energy eigenvalue of each sparated solution depended only on combination $n_0 + kn_1$. Differential operators were then constructed to raise or lower n_0 or n_1 by integer amounts and so that for rational k, compositions of these operators could be found that left $n_0 + k_1 n_1$ unchanged and hence preserved the energy. While these additional operators were constructed to act on a separated eigenbasis, they were found to in fact be expressible as differential operators. It should be noted here that the linear dependence of the energy on the quantum numbers n_0 and n_1 is crucial to the method and maintaining this below leads to the need for quantum corrections to the potential.

In order to extended this approach to our present example we must first solve the system by separation of variables and so we postulate a solution to $H\Psi = E\Psi$ of the form

$$\Psi = \Psi_0(r)\Psi_1(\theta_1)\Psi_2(\theta_2)\Psi_3(\theta_3),$$

with

$$L_3\Psi_3 = \ell_3\Psi_3, \qquad L_2\Psi_2\Psi_3 = \ell_2\Psi_2\Psi_3, \qquad L_1\Psi_1\Psi_2\Psi_3 = \ell_1\Psi_1\Psi_2\Psi_3.$$

While the solution of the separated equations is unremarkable, the details are written out at length so as to expose the point at which the quantum corrections (9) and (12) to the potential become necessary.

We find that each angular equation is, up to a gauge scaling, of the form

$$u''(y) + \left(\frac{\frac{1}{4} - \alpha^2}{\sin^2 y} + \frac{\frac{1}{4} - \beta^2}{\cos^2 y} + (2n + \alpha + \beta + 1)^2 \right) u(y) = 0,$$

which has solution

$$u(y) = (\sin y)^{\alpha + \frac{1}{2}} (\cos y)^{\beta + \frac{1}{2}} P_n^{(\alpha,\beta)}(\cos 2y)$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi function [7].

Starting with the θ_3 equation, we make the replacements

$$\beta_3 = k_3^2 \left(\frac{1}{4} - a_4^2 \right), \qquad \beta_4 = k_3^2 \left(\frac{1}{4} - a_3^2 \right),$$

and find the separated equation is

$$L_3\Psi_3(\theta_3) = \Psi_3''(\theta_3) + \left(\frac{k_3^2 \left(\frac{1}{4} - a_4^2 \right)}{\cos^2(k_3\theta_3)} + \frac{k_3^2 \left(\frac{1}{4} - a_3^2 \right)}{\sin^2(k_3\theta_3)} \right) \Psi_3(\theta_3) = \ell_3\Psi_3(\theta_3)$$

4

which has solutions

$$\Psi_{3,n_3}^{a_3,a_4}(\theta_3) = (\sin(k_3\theta_3))^{a_3+\frac{1}{2}}(\cos(k_3\theta_3))^{a_4+\frac{1}{2}} P_{n_3}^{(a_3,a_4)}(\cos(2k_3\theta_3))$$

with eigenvalues

$$L_3\Psi_{3,n_3}^{a_3,a_4}(\theta_3) = \ell_3\Psi_{3,n_3}^{a_3,a_4}(\theta_3), \qquad \ell_3 = -k_3^2(2n_3 + a_3 + a_4 + 1)^2. \tag{4}$$

The separated equation $L_2\Psi_2(\theta_2) = \ell_2\Psi_2(\theta_2)$ is now

$$\Psi_2''(\theta_2) + k_2\cot(k_2\theta_2)\Psi_2'(\theta_2) + \left(\frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{\ell_3}{\sin^2(k_2\theta_2)}\right)\Psi_2(\theta_2) = \ell_2\Psi_2(\theta_2)$$

which we transform with

$$\Psi_2(\theta_2) = (\sin(k_2\theta_2))^{-\frac{1}{2}}\psi_2(\theta_2)$$

to absorb the first derivative term to give

$$\psi_2''(\theta_2) + \left(\frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{\ell_3 + \frac{1}{4}k_2^2}{\sin^2(k_2\theta_2)} + \frac{1}{4}k_2^2 - \ell_2\right)\psi_2(\theta_2) = 0$$

and we make the replacements

$$\beta_2 = k_2^2\left(\frac{1}{4} - a_2^2\right), \qquad \ell_3 + \frac{1}{4}k_2^2 = k_2^2\left(\frac{1}{4} - A_2^2\right),$$

which when combined with (4) gives

$$A_2 = \frac{k_3}{k_2}(2n_3 + a_3 + a_4 + 1). \tag{5}$$

The separated θ_2 equation becomes

$$\psi_2''(\theta_2) + \left(\frac{k_2^2\left(\frac{1}{4} - a_2^2\right)}{\cos^2(k_2\theta_2)} + \frac{k_2^2\left(\frac{1}{4} - A_2^2\right)}{\sin^2(k_2\theta_2)} + \frac{k_2^2}{4} - \ell_2\right)\psi_2(\theta_2) = 0$$

where

$$\frac{k_2^2}{4} - \ell_2 = k_2^2(2n_2 + a_2 + A_2 + 1)^2 \tag{6}$$

and has solution

$$\Psi_{2,n_2}^{A_2,a_2}(\theta_2) = (\sin(k_2\theta_2))^{A_2}(\cos(k_2\theta_2))^{a_2+\frac{1}{2}} P_{n_2}^{(A_2,a_2)}(\cos(2k_2\theta_2)).$$

The separated θ_1 equation $L_1\Psi_1(\theta_1) = \ell_1\Psi_1(\theta_1)$ is

$$\Psi_1''(\theta_1) + 2k_1\cot(k_1\theta_1)\Psi_1'(\theta_1) + \left(\frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{\ell_2}{\sin^2(k_1\theta_1)}\right)\Psi_1(\theta_1) = \ell_1\Psi_1(\theta_1), \tag{7}$$

5

which we transform with

$$\Psi_1(\theta_1) = (\sin(k_1\theta_1))^{-1}\psi(\theta_1)$$

to absorb the first order term to give

$$\psi_1''(\theta_1) + \left(\frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{\ell_2}{\sin^2(k_1\theta_1)} + k_1^2 - \ell_1 \right)\psi_1(\theta_1) = 0 \qquad (8)$$

and we make the replacements

$$\beta_1 = k_1^2\left(\frac{1}{4} - a_1^2\right), \qquad \ell_2 = k_1^2\left(\frac{1}{4} - A_1^2\right).$$

Combining this with (6) gives

$$A_1 = \sqrt{\frac{1}{4}\left(1 - \frac{k_2^2}{k_1^2}\right) + \frac{k_2^2}{k_1^2}(2n_2 + a_2 + A_2 + 1)^2}.$$

This does not have the same form as (5) and will not lead to an energy eigenvalue that depends linearly on n_2. Hence, we instead propose an additional quantum correction in the potential of

$$\hat{V}_1 = \frac{\frac{1}{4}(k_1^2 - k^2)}{r^2\sin^2(k_1\theta_1)}, \qquad (9)$$

which in turn leads to a modified (8),

$$\psi_1''(\theta_1) + \left(\frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{\ell_2 + \frac{1}{4}(k_1^2 - k_2^2)}{\sin^2(k_1\theta_1)} + k_1^2 - \ell_1 \right)\psi_1(\theta_1) = 0.$$

Now, making the replacements

$$\beta_1 = k_1^2\left(\frac{1}{4} - a_1^2\right), \qquad \ell_2 + \frac{1}{4}(k_1^2 - k_2^2) = k_1^2\left(\frac{1}{4} - A_1^2\right)$$

gives

$$A_1 = \frac{k_2}{k_1}(2n_2 + A_2 + a_2 + 1). \qquad (10)$$

The separated θ_1 equation is now

$$\psi_1''(\theta_1) + \left(\frac{k_1^2\left(\frac{1}{4} - a_1^2\right)}{\cos^2(k_1\theta_1)} + \frac{k_1^2\left(\frac{1}{4} - A_1^2\right)}{\sin^2(k_1\theta_1)} + k_1^2 - \ell_1 \right)\psi_1(\theta_1) = 0$$

where

$$k_1^2 - \ell_1 = k_1^2(2n_1 + A_1 + a_1 + 1)^2$$

and has solutions

$$\Psi_{1,n_1}^{A_1,a_1}(\theta_1) = (\sin(k_1\theta_1))^{A_1 - \frac{1}{2}}(\cos(k_1\theta_1))^{a_1 + \frac{1}{2}}P_{n_1}^{(A_1,a_1)}(\cos(2k_1\theta_1)).$$

6

Finally, the separated radial equation is

$$H\Psi_0(r) = \partial_r^2 \Psi_0(r) + \frac{3}{r}\partial_r\Psi_0(r) + \left(-\omega^2 r^2 + \frac{\ell_1}{r^2}\right)\Psi_0(r) = E\Psi_0(r). \tag{11}$$

In a similar way to above, in order that E depend linearly on n_1, we propose the addition of a quantum correction to the potential of

$$\hat{V}_2 = \frac{1 - k_1^2}{r^2} \tag{12}$$

which leads to a modified version of (11),

$$H\Psi_0(r) = \partial_r^2 \Psi_0(r) + \frac{3}{r}\partial_r\Psi_0(r) + \left(-\omega^2 r^2 + \frac{\ell_1 - k_1^2 + 1}{r^2}\right)\Psi_0(r) = E\Psi_0(r).$$

We remove the first order terms with the transformation

$$\Psi_0(r) = r^{-\frac{3}{2}}\psi_0(r)$$

to give

$$\partial_r^2 \psi_0(r) + \left(-\omega^2 r^2 + \frac{\frac{1}{4} - k_1^2 + \ell_1}{r^2} - E\right)\psi_0(r) = 0.$$

Now,

$$u''(x) + \left(-x^2 + \frac{\frac{1}{4} - A_0^2}{x^2} + 4n + 2A_0 + 2\right)u(x) = 0$$

has solution

$$u(x) = e^{-\frac{x^2}{2}}x^{A_0 + \frac{1}{2}}L_n^{(A_0)}(x^2),$$

where $L_n^{(A_0)}(x)$ is a Laguerre function [7].

We needed $A_0^2 = k_1^2 - \ell_1$ and we already have $k_1^2 - \ell_1 = k_1^2(2n_1 + A_1 + a_1 + 1)^2$ so

$$\Psi_{0,n_0}^{A_0}(r) = \omega^{A_0/2}e^{-\frac{\omega r^2}{2}}r^{A_0 - 1}L_{n_0}^{(A_0)}(\omega r^2), \tag{13}$$

where the mulitplicative factor of $\omega^{A_0/2}$ is chosen for later convenience, and

$$A_0 = k_1(2n_1 + a_1 + A_1 + 1), \qquad E = -\omega(4n_0 + 2A_0 + 2). \tag{14}$$

Note that with the quantum deformation (12) the relationship of A_0 to n_1 is similar to that seen in (5) and (10).

Now, putting together (5), (10) and (14) we find

$$E = -2\omega(2n_0 + 2k_1 n_1 + 2k_2 n_2 + 2k_3 n_3 + k_1 a_1 + k_2 a_2 + k_3 a_3 + k_3 a_4 + k_1 + k_2 + k_3 + 1) \tag{15}$$

7

for a solution of the form

$$\Psi_{n_0,n_1,n_2,n_3} = \Psi_{0,n_0}^{A_0}(r)\Psi_{1,n_1}^{A_1,a_1}(\theta_1)\Psi_{2,n_2}^{A_2,a_2}(\theta_2)\Psi_{3,n_3}^{a_3,a_4}(\theta_3).$$

With the quantum corrections (9) and (12) added to the potential we have the following set of mutually commuting differential operators.

$$
\begin{aligned}
H = L_0 &= \partial_r^2 + \frac{3}{r}\partial_r - \omega^2 r^2 + \frac{L_1}{r^2} + \frac{1-k_1^2}{r^2}\\
L_1 &= \partial_{\theta_1}^2 + 2k_1\cot(k_1\theta_1)\partial_{\theta_1} + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_2}{\sin^2(k_1\theta_1)} + \frac{k_1^2-k_2^2}{4\sin^2(k_1\theta_1)}\\
L_2 &= \partial_{\theta_2}^2 + k_2\cot(k_2\theta_2)\partial_{\theta_2} + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_3}{\sin^2(k_2\theta_2)}\\
L_3 &= \partial_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)}.
\end{aligned}
$$

$H\Psi = E\Psi$ remains separable with the additional terms. In the following, we use these redefined H and L_1.

For the metric (2), the scalar curvature is

$$\mathcal{R} = -\frac{6}{r^2} + k_1^2\left(\frac{6}{r^2} - \frac{2}{r^2\sin^2(k_1\theta_1)}\right) + \frac{2k_2^2}{r^2\sin^2(k_1\theta_1)},$$

and with Weyl conformal tensor W_{abcd}, if we define

$$\mathcal{W} = \sqrt{3W_{abcd}W^{abcd}} = \frac{2(k_1^2-k_2^2)}{r^2\sin^2(k_1\theta_1)}.$$

then

$$H = \nabla^2 + V_0 + \hat{V}_1 + \hat{V}_2 = \nabla^2 + V_0 - \frac{1}{6}\mathcal{R} - \frac{1}{24}\mathcal{W}.$$

Note that $\nabla^2 + \hat{V}_1 + \hat{V}_2$ is a conformally covariant Laplacian and the metric g is conformally flat if and only if $k_1 = k_2$.

2 Raising and lowering operators

Our aim is now to use special function identities to raise and lower the n_i while preserving E and produce new operators commuting with H.

Using differential identities for Laguerre functions [7] we construct the operators that act on the radial part of the separated solutions,

$$
\begin{aligned}
K_{0\ n_0}^{+\ A_0} &= \frac{1-A_0}{r}\partial_r + (2n_0 + A_0 + 1)\omega + \frac{1-A_0^2}{r^2},\\
K_{0\ n_0}^{-\ A_0} &= \frac{1+A_0}{r}\partial_r + (2n_0 + A_0 + 1)\omega + \frac{1-A_0^2}{r^2}.
\end{aligned}
$$

These raise or lower n_0 by 1 while simultaneously lowering or raising and A_0 by 2, that is,

$$K_0^{+\ A_0}{}_{n_0}\Psi_{n_0}^{A_0} = -2\omega(n_0+1)(n_0+A_0)\Psi_{n_0+1}^{A_0-2},$$
$$K_0^{-\ A_0}{}_{n_0}\Psi_{n_0}^{A_0} = -2\omega\Psi_{n_0-1}^{A_0+2}.$$

Note that the constant multiplicative factor of $\omega^{A_0/2}$ in (13) was chosen so that both of these have a factor of ω on the right hand side.

For the angular functions, we can use Jacobi function identities [7] to make operators that raise and lower n alone,

$$
\begin{aligned}
J_n^+ &= -\frac{(N+1)\sin(2k\theta)}{2k}\partial_\theta - \frac{1}{2}\Big((N+1)(N+1-c-d)\cos(2k\theta)\\
&\quad - (N+1)(c-d) + a^2 - b^2\Big),\\
J_n^- &= \frac{(N-1)\sin(2k\theta)}{2k}\partial_\theta - \frac{1}{2}\Big((N-1)(N-1+c+d)\cos(2k\theta)\\
&\quad + (N-1)(c-d) + a^2 - b^2\Big),
\end{aligned}
$$

where $N = 2n + a + b + 1$ and their action on

$$\Theta_n^{(a,b)} = \sin^{a+c}(k\theta)\cos^{b+d}(k\theta)P_n^{(a,b)}\cos(2k\theta)$$

is given by

$$J_n^+\Theta_n^{(a,b)} = -2(n+1)(n+a+b+1)\Theta_{n+1}^{(a,b)},$$
$$J_n^-\Theta_n^{(a,b)} = -2(n+a)(n+b)\Theta_{n-1}^{(a,b)}.$$

The operators above are essentially those used in [5] and the analysis used to show superintegrability for the TTW system immediately carries over the the current example and so we obtain a symmetry operator by raising and lowering the functions associated with r and θ_1.

Notice that the operators above that raise or lower the radial eigenfunctions change both n_0 and A_0. In order to extend this approach and construct a symmetry operator by raising and lowering the θ_1 and θ_2 functions we will need operators with a similar effect on these functions. This can be achieved using Jacobi function identities that raise and lower n and a simultaneously when applied to

$$\Theta_n^{(a,b)} = \sin^{a+c}(k\theta)\cos^{b+d}(k\theta)P_n^{(a,b)}\cos(2k\theta).$$

We find

$$
\begin{aligned}
K_n^{+\ a} &= -\frac{(1-a)\cos(k\theta)}{k\sin(k\theta)}\partial_\theta - 2(n(n+a+b+1)+a(a+b))\\
&\quad - (1-a)(a+c+b+d) - \frac{(1-a)(a-c)}{\sin^2(k\theta)},\\
K_n^{-\ a} &= -\frac{(1+a)\cos(k\theta)}{k\sin(k\theta)}\partial_\theta - 2n(n+a+b+1)\\
&\quad - (1+a)(a+c+b+d) + \frac{(1+a)(a+c)}{\sin^2(k\theta)},
\end{aligned}
$$

9

with action,

$$K_n^+ \, {}^a\Theta_n^{(a,b)} = 2(n+1)(n+a)\Theta_{n+1}^{(a-2,b)},$$
$$K_n^- \, {}^a\Theta_n^{(a,b)} = 2(n+a+b+1)(n+b)\Theta_{n-1}^{(a+2,b)}.$$

3 Constructing the symmetries

For $k_1 = p_1/q_1$ with $\gcd(p_1, q_1) = 1$ the operator

$$\Xi_1^+ = \underbrace{K_0^{-\ \ A_0+2(p_1-1)}_{n_0-(p_1-1)} \cdots K_0^{-\ \ A_0}_{n_0}}_{p_1 \text{ terms}} \underbrace{J_1^+ {}_{n_1+q_1-1} \cdots J_1^+ {}_{n_1}}_{q_1 \text{ terms}} \tag{16}$$

has the effect on a basis function of

$$n_0 \to n_0 - p_1, \quad n_1 \to n_1 + q_1, \quad A_0 \to A_0 + 2p_1,$$

and so

$$E = -2\omega(2n_0 + 2k_1 n_1 + \cdots) \quad \to \quad -2\omega(2(n_0 - p_1) + 2k_1(n_1 + q_1) + \cdots)$$
$$= -2\omega(2n_0 + 2k_1 n_1 + \cdots),$$

that is, E is unchanged. A similar lowering operator is

$$\Xi_1^- = \underbrace{K_0^{+\ \ A_0-2(p_1-1)}_{n_0+(p_1-1)} \cdots K_0^{+\ \ A_0}_{n_0}}_{p_1 \text{ terms}} \underbrace{J_1^- {}_{n_1-(q_1-1)} \cdots J_1^- {}_{n_1}}_{q_1 \text{ terms}}, \tag{17}$$

which also leaves E unchanged, has the effect on a basis function of

$$n_0 \to n_0 + p_1, \quad n_1 \to n_1 - q_1, \quad A_0 \to A_0 - 2p_1.$$

Explicitly, the action of $\Xi_1^\pm$ on a basis function is

$$\Xi_1^+ \Psi_{0,n_0}^{A_0} \Psi_{1,n_1}^{A_1,a_1} = (-2)^{p_1} \omega^{p_1} (n_1+1)_{q_1} (n_1 + A_1 + a_1 + 1)_{q_1} \Psi_{0,n_0-p_1}^{A_0+2p_1} \Psi_{1,n_1+q_1}^{A_1,a_1},$$
$$\Xi_1^- \Psi_{0,n_0}^{A_0} \Psi_{1,n_1}^{A_1,a_1} = (-2)^{p_1} \omega^{p_1} (-n_1 - A_1)_{q_1} (-n_1 - a_1)_{q_1} (n_0+p_1)_{p_1} (n_0+A_0)_{p_1} \Psi_{0,n_0+p_1}^{A_0-2p_1} \Psi_{1,n_1-q_1}^{A_1,a_1}.$$

This is exactly like the TTW raising and lowering operators from [5].

For $k_2/k_1 = p_2/q_2$ with $\gcd(p_2, q_2) = 1$ the operator

$$\Xi_2^+ = \underbrace{K_1^{-\ \ A_1+2(p_2-1)}_{n_1-(p_2-1)} \cdots K_1^{-\ \ A_1}_{n_1}}_{p_2 \text{ terms}} \underbrace{J_2^+ {}_{n_2+q_2-1} \cdots J_2^+ {}_{n_2}}_{q_2 \text{ terms}} \tag{18}$$

has the effect on a basis function of

$$n_1 \to n_1 - p_2, \quad n_2 \to n_2 + q_2, \quad A_1 \to A_1 + 2p_2$$

and so

$$E = -2\omega(2n_0 + 2k_1 n_1 + 2k_2 n_2 + \cdots) \to -2\omega(2n_0 + 2k_1(n_1 - p_2) + 2k_2(n_2 + q_2) + \cdots)$$

$$= -2\omega(2n_0 + 2k_1 n_1 + 2k_2 n_2 + \cdots),$$

that is, E is unchanged. A similar lowering operator is

$$\Xi_2^- = \underbrace{K_{1\ n_1+(p_2-1)}^{+\ A_1-2(p_2-1)} \cdots K_{1\ n_1}^{+\ A_1}}_{p_2 \text{ terms}} \underbrace{J_{2\ n_2-(q_2-1)}^{-} \cdots J_{2\ n_2}^{-}}_{q_2 \text{ terms}}. \tag{19}$$

This is similar to the two-dimensional TTW procedure, but different operators are required.

The operators given so far are only well defined on the separated basis functions and they contain the quantum numbers in their defintions. We now must show that we can construct pure differential operators. The argument is only sketched here as the details are essentially the same as those in [5].

The transformation $n_1 \to -n_1 - A_1 - a_1 - 1$ while holding E constant has the effect of changing the sign of A_0. It is then straightforward to check from the explicit expressions for the operators that

$$L_1^+ = \Xi_1^+ + \Xi_1^- \quad \text{and} \quad L_1^- = k_1 \frac{\Xi_1^+ - \Xi_1^-}{A_0}$$

are polynomials in E, A_0^2 and A_1^2. Since

$$A_0^2 = k_1^2 - \ell_1 \quad \text{and} \quad A_1^2 = \frac{\frac{1}{4}k_2^2 - \ell_2}{k_1^2}$$

we can replace E, A_0^2 and A_1^2 with second order differential operators where ever they appear in these expressions.

Similarly, the transformation $n_2 \to -n_2 - A_2 - a_2 - 1$ while holding L_1 constant has the effect of changing the sign of A_1. It is then straightforward to check from the explicit expressions for the operators that

$$L_2^+ = \Xi_2^+ + \Xi_2^- \quad \text{and} \quad L_2^- = k_2 \frac{\Xi_2^+ - \Xi_2^-}{A_1} \tag{20}$$

are polynomials in L_1, A_1^2 and A_2^2. Since

$$A_1^2 = \frac{\frac{1}{4}k_2^2 - \ell_2}{k_1^2} \quad \text{and} \quad A_2^2 = -\frac{\ell_3}{k_2^2}$$

we can replace L_1, A_1^2 and A_2^2 with a second order differential operators where ever they appear in these expressions.

11

In a similar way, we can define $\Xi_3^\pm$ and $L_3^\pm$ with the label replacements $1 \to 2$ and $2 \to 3$ in (18), (19) and (20) and show that they are in also differential operators.

It is clear from the construction that $\{H, L_1, L_1^+, L_2, L_2^+, L_3, L_3^+\}$ forms an algebraically independent set of differential operators and hence the system is superintegrable.

4   The symmetry algebra

A common feature of superintegrable systems is a polynomially closed symmetry algebra. By direct calcuation, we find some polynomially closed subalgebras of the symmetry algebra.

Adapting the an argument from [5] we find that, for $i = 1, 2, 3$,

$$P_i^{(+)}(L_{i-1}, L_i, A_i^2) = \Xi_i^- \Xi_i^+ + \Xi_i^+ \Xi_i^- \quad \text{and} \quad P_i^{(-)}(L_{i-1}, L_i, A_i^2) = k_i \frac{\Xi_i^+ \Xi_i^- + \Xi_i^- \Xi_i^+}{A_{i-1}},$$

are differential operators that are polynomial in their arguments.

By comparing the action of brackets of the operators $\{L_i, L_i^+, L_i^-\}$ with symmetrised products of the operators we find the following explicit identities for $i = 1, 2, 3$.

$$\begin{aligned}
[L_i, L_i^-] &= -4k_i^2 q_i^2 L_i^- - 4\alpha_i k_i^2 q_i L_i^+ \\
[L_i, L_i^+] &= 2q_i\{L_i, L_i^-\} - 4k_i^2 q_i L_i^+ + 4k_i^2 q_i^2 L_i^- + 8q_i^3 k_i^2 L_i^- \\
[L_i^+, L_i^-] &= 2q_i(L_i^-)^2 - 2P_i^{(-)}(L_{i-1}, L_i, A_i^2)
\end{aligned}$$

and

$$\{L_i, L_i^-, L_i^-\} + 2k_i^2(14q_i^2 - 3\alpha_i)(L_i^-)^2 + 6k_i^2(L_i^+)^2 + 6k_i^2 q_i\{L_i^+, L_i^-\} - 12k_i^2 P_i^{(+)} + 4k_i^2 q_i P_i^{(-)} = 0,$$

where $\alpha_1 = 1$, $\alpha_2 = 1/4$ and $\alpha_3 = 0$. These hold as operator identities on general functions. Furthermore, $[L_j, L_i^\pm] = 0$ for $i \neq j$ and $[L_j^\pm, L_i^\pm] = [L_j^\pm, L_i^\mp] = 0$ for $|i - j| > 1$.

As was found for the TTW operators, the symmetries constructed from raising and lowering operators are not necessarily of minimal order [5]. The same technique for finding lower order operators can be used for the current system.

For example, starting from $L_1^\pm$ we look for $M_1^\pm$ satisfying

$$[L_1, M_1^\pm] = L_1^\pm,$$

$$[H, M_1^\pm] = 0, \qquad [L_2, M_1^\pm] = 0, \qquad [L_3, M_1^\pm] = 0.$$

Find M_1^- is

$$M_1^- = -\frac{1}{4q_1}\left(\frac{L_1^-}{A_0(A_0 + p_1)} + \frac{L_1^+}{A_0(A_0 - p_1)}\right) + \frac{S_1(H, L_2)}{A_0^2 - p_1^2} \tag{21}$$

where $S_1(H, L_2)$ is a polynomial in H and L_2 that can be determined by the methods used in [5].

5   An example

Explicit computations can be performed for particular choices of the k_i. For example, with $k_1, k_2, k_3 = 2, 1, 1$, the operator L_1^- is 6th and L_1^+ is the 5th order operator,

$$L_1^+ = \left(-\frac{2}{r^3}\partial_r + \frac{6}{r^4}\right) A_0^2 A_1^2 + \left(-\frac{\cos(4\theta_1)}{2r^3}\partial_r + \frac{\sin(4\theta_1)}{4r^4}\partial_{\theta_1} + \frac{1 + 5\cos(4\theta_1)}{2r^4}\right) A_0{}^4$$

$$+ \left(-\frac{1}{r}\partial_r + \frac{2}{r^2}\right) E A_1{}^2 + \left(\frac{\sin(4\theta_1)}{16}\partial_{\theta_1} + \frac{\cos(4\theta_1)}{4} + \frac{1}{8}\right) E^2$$

$$+ \left(-\frac{\cos(4\theta_1)}{4r}\partial_r + \frac{\sin(4\theta_1)}{4r^2}\partial_{\theta_1} + \frac{3\cos(4\theta_1) + 1}{2r^2}\right) E A_0{}^2 + \left(-\frac{10}{r^3}\partial_r + \frac{4}{r^2}\partial_r^2 - \frac{6}{r^4}\right) A_1{}^2$$

$$- \left(\frac{\sin(4\theta_1)}{r}\partial_r\partial_{\theta_1} + \frac{3\cos(4\theta_1) + 2 - a_1^2}{r}\partial_r - \frac{5\sin(4\theta_1)}{4r^2}\partial_{\theta_1} - \frac{(6\cos(4\theta_1) + 5 - 4a_1^2)}{2r^2}\right) E$$

$$+ \left(-\frac{\sin(4\theta_1)}{4r^2}\partial_r^2\partial_{\theta_1} + \frac{13\sin(4\theta_1)}{4r^3}\partial_r\partial_{\theta_1} - \frac{(4\cos(4\theta_1) + 1)}{2r^2}\partial_r^2 + \frac{13 + 27\cos(4\theta_1) - 4a_1^2}{2r^3}\partial_r\right.$$

$$\left. - \frac{5\sin(4\theta_1)}{r^4}\partial_{\theta_1} - \frac{25\cos(4\theta_1) + 20 - 12a_1^2}{2r^4}\right) A_0{}^2$$

$$+ \frac{11\sin(4\theta_1)}{4r^2}\partial_r^2\partial_{\theta_1} + \frac{(11 - 8a_1^2 + 14\cos(4\theta_1))}{2r^2}\partial_r^2 - \frac{23\sin(4\theta_1)}{4r^3}\partial_r\partial_{\theta_1}$$

$$- \frac{26\cos(4\theta_1) + 23 - 20a_1^2}{2r^3}\partial_r - \frac{21\sin(4\theta_1)}{4r^4}\partial_{\theta_1} - \frac{3(10\cos(4\theta_1) + 7 - 4a_1^2)}{2r^4}$$

with the replacements $E \to H$, $A_0^2 \to k_1^2 - L_1$ and $A_1^2 \to (k_2^2 - 4L_2)/(4k_1^2)$.

A 4th order operator can also be constructed using (21) and in this case,

$$S_1(H, L_2) = -\frac{(H^2 - 4\omega)(A_1^2 - a_1^2)}{16}$$

and M_1^+ is a polynomial in E and even powers of A_0 and A_1.

6   Conclusion

The methods developed in [5] have been extended to demonstrate the superintegrability of a 4-dimensional quantum Hamiltonian system on a non-conformally-flat space. The system discussed is a quantisation of a previously described classical superintegrable system and in order to maintain superintegrability in the quantisation, correction terms were required to be added to the potential. These correction terms make the Hamiltonian conformally covariant Laplacian, but are not the usual minimal conformally covariant correction of $-\mathcal{R}/6$ as they depend

on the conformal curvature. The 3-dimensional analogue of this system also requires the addition of the term $-\mathcal{R}/8$ to maintain superintegrability and this too gives a conformally covariant Hamiltonian. While many previously known superintegrable systems possess a polynomially closed symmetry algebra, here we have only found some polynomially closed subalgebras. An investigation of a closely related c lassical system found that in general the the symmetry algebra will close rationally rather than polynomially [9]. It seems reasonable to conjecture that, except in some special cases, the symmetry algebra of the 4-dimensional system considered here does not close polynomially, but rather it obeys an appropriate quantum analogue of rational closure.

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