The Lie theory approach to special functions

Willard Miller
University of Minnesota

November 5, 2010

Contents

0.1 Abstract and introduction

These notes describe some of the most important interrelationships between the theory of Lie groups and algebras, and special functions, with a strong emphasis on results obtained in the 50 years after the publication of the Bateman Project. An informal justification for this treatment is that most functions commonly called “ special ” obey symmetry properties that are best described via group theory (the mathematics of symmetry). In particular, those special functions that arise as explicit solutions of the partial differential equations of mathematical physics, such as via separation of variables, can be characterized in terms of their transformation properties under the Lie symmetry groups and algebras of the differential equations. (The same ideas extend to difference and q-difference equations.) We shall treat, briefly, the following topics:

1. Special functions as matrix elements of Lie group representations. (addition theorems, orthogonality relations)

2. Special functions as basis functions for Lie group representations (generating functions)

3. Special functions as solutions of Laplace-Beltrami eigenvalue problems (with potential) via separation of variables.

4. Special functions as Clebsch-Gordan coefficients for the reduction of tensor products of irreducible group representations (the motivation for Wilson polynomials).

In practice, the first two items involve hypergeometric functions predominantly and are special cases of the third item. The group theoretic basis for variable separation allows treatment of non-hypergeometric functions, such as those of Lamé and Heun. The last item provides an important motivation for the constuction of the Askey-Wilson polynomials.

I conclude with a brief examination of special functions (or functions that deserve to be called "special") that arise when one restricts certain irreducible Lie group representations to a discrete lattice subgroup. The two most important examples are an irreducible representation of the Heisenberg group (and its relation to the windowed Fourier transform, the Weil-Brezin-Zak transform and theta functions), and an irreducible representation of the affine group (and its relation to the continuous and discrete wavelet transforms). We briefly describe the properties of the Daubechies family of scaling functions, a very modern family of "special" functions arising as solutions of two-scale difference equations.

I am omitting some topics of equal or greater importance than the above, such as quantum groups, Askey-Wilson polynomials, Koornwinder's addition theorems for disk and Jacobi polynomials and other polynomials transforming under group actions, special functions related to root systems of semisimple Lie algebras, Dunkl's theory of special funtions related to discrete symmetries on spheres, etc., because they will be treated by other participants in this meeting, for lack of time, or lack of expertise.

Chapter 1

Preliminaries

A group is an abstract mathematical entity which expresses the intuitive concept of symmetry. Here I collect some standard definitions and results from classical group theory.

1.1 Definition of a group

A group is an abstract mathematical entity which expresses the intuitive concept of symmetry.

Definition 1 *A* **group** *G is a set of objects $\{g, h, k, \cdots\}$ (not necessarily countable) together with a binary operation which associates to any ordered pair of elements g, h in G a third element gh. the binary operation (called* **group multiplication***) is subject to the following requirements:*

1. *There exists an element e in G called the* **identity element** *such that $ge = eg = g$ for all $g \in G$.*

2. *For every $g \in G$ there exists in G an* **inverse element** *g^{-1} such that $gg^{-1} = g^{-1}g = e$.*

3. *Associative law. The identity $(gh)k = g(hk)$ is satisfied for all $g, h, k \in G$.*

1.2 Lie groups and algebras, transformation groups

Let W be an open connected set containing $\mathbf{e} = (0, \cdots, 0)$ in the space C_n of all (real or complex) n-tuples $\mathbf{g} = (g_1, \cdots, g_n)$.

Definition 2 *An n-dimensional* **local linear Lie group** *G is a set of $m \times m$ nonsingular matrices $A(\mathbf{g}) = A(g_1, \cdots, g_n)$, defined for each $\mathbf{g} \in W$, such that*

1. *$A(\mathbf{e}) = I_m$ (the identity matrix)*

2. *The matrix elements of $A(\mathbf{g})$ are analytic functions of the parameters $g_1, \cdots, g_n$ and the map $\mathbf{g} \to A(\mathbf{g})$ is one-to-one.*

3. *The n matrices $\frac{\partial A(\mathbf{g})}{\partial g_j}$, $j = 1, \cdots, n$, are linearly independent for each $\mathbf{g} \in \mathbf{W}$. That is, these matrices span an n-dimensional subspace of the m^2-dimensional space of all $m \times m$ matrices.*

4. *There exists a neighborhood W' of $\mathbf{e}$ in C_n, $W' \subseteq W$, with the property that for every pair of n-tuples $\mathbf{g}, \mathbf{h}$ in W' there is an n-tuple $\mathbf{k}$ in W satisfying*
$$A(\mathbf{g})A(\mathbf{h}) = A(\mathbf{k})$$
where the operation on the left is matrix multiplication.

If G be a local linear group of $m \times m$ matrices, we can construct a (connected, global) **linear Lie group** $\tilde{G}$ containing G. Algebraically, $\tilde{G}$ is the abstract subgroup of $GL(m, C)$ generated by the matrices of G. If $B \in \tilde{G}$ we can introduce coordinates in a neighborhood of B by means of the map $\mathbf{g} \to BA(\mathbf{g})$ where $\mathbf{g}$ ranges over a suitably small neighborhood of $\mathbf{e}$. In general an n-dimensional (global) **linear Lie group** K is an abstract matrix group which is also an n-dimensional local linear group G.

Examples: $GL(n, R)$, $SL(n, R)$, $O(n)$, $GL(n, C)$, $SL(n, C)$, $U(n)$), are all linear Lie groups.

Definition 3 *Lie algebra $L(G)$: Tangent space at the identity.*

- *One parameter curve through the identity in G: $A(g(t))$ where $g(0) = e$.*

- *$L(G)$ consists of all matrices $\mathcal{A} = \frac{d}{dt}A(g(t))|_{t=0}$ as $\mathbf{g}$ runs over all differentiable curves through the origin.*

- *$L(G)$ is a vector space: if $\mathbf{g}(t) \to A$ and $\mathbf{h}(t) \to B$ and α, β scalars, then $g(\alpha t)\mathbf{h}(\beta t) \to \alpha A + \beta B$.*

- *$L(G)$ is closed under commutation: if $\mathbf{g}(t) \to \mathcal{A}$ and $\mathbf{h}(t) \to \mathcal{B}$ then $\mathbf{g}(t)\mathbf{h}(t)\mathbf{g}^{-1}(t)\mathbf{h}^{-1}(t) = \mathbf{k}(t^2) \to [\mathcal{A}, \mathcal{B}] = \mathcal{AB} - \mathcal{BA}$.*

- *Properties of the commutator:*

$$[\mathcal{A}, \mathcal{B}] = -[\mathcal{B}, \mathcal{A}] \qquad \text{(skew symmetry)}$$

$$[\alpha\mathcal{A} + \beta\mathcal{B}, \mathcal{C}] = \alpha[\mathcal{A}, \mathcal{C}] + \beta[\mathcal{B}, \mathcal{C}], \qquad \alpha, \beta \text{ scalars} \qquad \text{(linearity)}$$

$$[[\mathcal{A}, \mathcal{B}], \mathcal{C}] + [[\mathcal{B}, \mathcal{C}], \mathcal{A}] + [[\mathcal{C}, \mathcal{A}], \mathcal{B}] = 0, \qquad \text{(Jacobi identity).}$$

  Holds automatically for matrix Lie algebras.

- *Basic relation between Lie algebra and local Lie group: $A(t)$ is a one-parameter subgroup of G, i.e.,*

$$A(t) \in G, A(t_1)A(t_2) = A(t_1+t_2) \Longleftrightarrow A(t) = \exp(t\mathcal{A}) = \sum_{j=0}^{\infty} \frac{(t\mathcal{A})^j}{j!}, \quad \mathcal{A} \in L(G).$$

Definition 4 *Action of a Lie group as a Lie transformation group: Let G be a local Lie group and $\mathcal{M}$ a local coordinate manifold. G acts as a local transformatin group on $\mathcal{M}$ if there is an analytic mapping $\mathcal{M} \times \mathbf{g} \to \mathcal{M}$: $x \times \mathbf{g} \to x\mathbf{g}$ such that*

$$(x\mathbf{g})\mathbf{h} = x(\mathbf{gh}), \qquad x\mathbf{e} = x, \qquad \mathbf{g}, \mathbf{h} \in G, x \in \mathcal{M}.$$

We can transfer this action to functions $f(x)$ on $\mathcal{M}$ by defining operators $T(\mathbf{g})$ such that

$$T(\mathbf{g})f(x) = f(x\mathbf{g}),$$

or, more generally,

$$T(\mathbf{g})f(x) = \nu(\mathbf{g}, x)f(x\mathbf{g}),$$

where the *multiplier* $\nu(\mathbf{g}, x)$ satisfies $\nu(\mathbf{gh}, x) = \nu(\mathbf{g}, x)\nu(\mathbf{h}, x\mathbf{g})$. These operators satify the (local) group law

$$T(\mathbf{g})T(\mathbf{h}) = T(\mathbf{gh}).$$

By transferring this action to the tangent space at the identity,

$$T(\mathcal{A})f(x) = \frac{d}{dt}T(\exp t\mathcal{A})f(x)|_{t=0},$$

one gets a realization of $L(G)$ by first order linear differential operators:

$$T(\alpha\mathcal{A} + \beta\mathcal{B}) = \alpha T(\mathcal{A}) + \beta T(\mathcal{B}),$$

$$T([\mathcal{A}, \mathcal{B}]) = [T(\mathcal{A}), T(\mathcal{B})] = T(\mathcal{A})T(\mathcal{B}) - T(\mathcal{B})T(\mathcal{A}).$$

EXAMPLE: $ax + b$ or "affine" group.

- $\mathbf{g} = (a, b)$, $a > 0, b$ real. $\mathbf{gh} = (a, b) \cdot (c, d) = (ac, ad + b)$.
- Linear Lie group: $g \Leftrightarrow A(\mathbf{g})$, such that $A(\mathbf{g})A(\mathbf{h}) = A(\mathbf{gh})$.

$$(a, b) = \mathbf{g} \Leftrightarrow A(\mathbf{g}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

- A basis for the two-dimensional Lie algebra is given by the matrices

$$\mathcal{L}_1 = \frac{d}{dt}\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}|_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{L}_2 = \frac{d}{dt}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}|_{t=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

  with commutation relation

$$[\mathcal{L}_1, \mathcal{L}_2] = \mathcal{L}_2.$$

- An action on the real line given by
$$x\mathbf{g} = \frac{x-b}{a}$$
or
$$T(\mathbf{g})f(x) = f(\frac{x-b}{a}).$$
- The induced differential operators that represent the Lie algebra acting on functions of x are
$$T(\mathcal{L}_1) = L_1 = -x\frac{d}{dx}, \qquad T(\mathcal{L}_2) = L_2 = -\frac{d}{dx}, \qquad [L_1, L_2] = L_2.$$

EXAMPLE: The (three-dimensional real) Heisenberg group H_R

-
$$H_R = \left\{ g(x_1, x_2, x_3) = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_i \in R \right\}.$$
- This is a subgroup of $GL(3, R)$ with group product
$$g(x_1, x_2, x_3) \cdot g(y_1, y_2, y_3) = g(x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$
The identity element is the identity matrix $g(0, 0, 0)$ and $g(x_1, x_2, x_3)^{-1} = g(-x_1, -x_2, x_1 x_2 - x_3)$.
- A basis for the three-dimensional Lie algebra is given by the matrices
$$\mathcal{L}_1 = \frac{d}{dt}\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}|_{t=0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathcal{L}_2 = \frac{d}{dt}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathcal{L}_3 = \frac{d}{dt}\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}|_{t=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
with commutation relations
$$[\mathcal{L}_1, \mathcal{L}_2] = \mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_3] = [\mathcal{L}_2, \mathcal{L}_3] = 0.$$

- An action on the real line given by

$$x\mathbf{g}[x_1, x_2, x_3] = x + x_1$$

and a family of representations on functions $f(x)$ by

$$T^\lambda(\mathbf{g})f(x) = e^{2\pi i\lambda(x_3 + xx_2)} f(x + x_1)$$

where λ is a constant.

- The induced differential operators that represent the Lie algebra acting on functions of x are

$$T(\mathcal{L}_1) = L_1 = \frac{d}{dx}, \quad T(\mathcal{L}_2) = L_2 = 2\pi i\lambda x, \quad T(\mathcal{L}_3) = L_3 = 2\pi i\lambda.$$

1.3 Group representations

Definition 5 *Let V be a Hilbert space. A* **representation** *of a group G with* **representation space** *V is a homomorphism $\mathbf{T} : g \to \mathbf{T}(g)$ of G into the space of bounded linear operators on V.*

It follows that

$$\begin{aligned} \mathbf{T}(g_1)\mathbf{T}(g_2) &= \mathbf{T}(g_1 g_2), \quad \mathbf{T}(g)^{-1} = \mathbf{T}(g^{-1}), && (1.1)\\ \mathbf{T}(e) &= \mathbf{I}, \quad g_1, g_2, g \in G, && (1.2) \end{aligned}$$

Definition 6 *A* **matrix representation** *of G is a homomorphism $T : g \to T(g)$ of G into $GL(n, C)$ or $GL(\infty, C)$.*

Definition 7 *The representation* $\mathbf{T}$ *is* **reducible** *if there is a proper subspace W of V which is invariant under* $\mathbf{T}$*. Otherwise,* $\mathbf{T}$ *is* **irreducible** *.*

A representation is irreducible if the only invariant subspaces of V are $\{\theta\}$, (the zero vector) and V itself. For large classes of groups and group representations, a reducible representation $\mathbf{T}$ can be decomposed into a direct sum of irreducible representations in an almost unique manner. This is called the Clebsch-Gordan decomposition.

1.4 Orthogonality relations for finite groups

Let G be a finite group and select one irreducible representation $\mathbf{T}^{(\mu)}$ of G in each equivalence class of irreducible representations. Introduction of a basis in each representation space $V^{(\mu)}$ leads to a matrix representation $T^{(\mu)}$. Here $n_\mu = \dim V^{(\mu)}$. We can choose the $T^{(\mu)}$ to be unitary. Then

$$\sum_{g\in G} T^{(\mu)}_{i\ell}(g)\bar{T}^{(\nu)}_{sm}(g) = \frac{N}{n_\mu}\delta_{is}\delta_{\ell m}\delta_{\mu\nu}.$$

These are the **orthogonality relations** for matrix elements of irreducible representations of G. We can write these relations in a basis-free manner.

1.5 Invariant measures on Lie groups

Let G be a real n-dimensional global Lie group of $m \times m$ matrices. There is a unique (up to a constant) volume element dA in G with respect to which the associated integral over the group is left-invariant, i.e.,

$$\int_G f(BA)dA = \int_G f(A)dA, \quad B \in G,$$

where f is a continuous function on G such that either of the integrals converges.

1.6 Orthogonality relations for compact Lie groups

If G is a compact Lie group then the integral of any continuous function over group space converges and the orthogonality relations generalize to

$$\int_G T^{(\mu)}_{i\ell}(A)\overline{T^{(\nu)}_{sk}}(A)\delta A = (\delta_{is}/n_\mu)\delta_{\ell k}\delta_{\mu\nu},\ 1 \le i,\ell \le n_\mu,\ 1 \le s,k \le n_\nu,$$

where $\delta A = V^{-1}\, dA$ and $V = \int_G 1\, dA$.

1.7 The Peter-Weyl theorem

Let $L_2(G)$ be the space of all functions on the compact goup G which are (Lebesgue) square-integrable:

$$L_2(G) = \{f(A) : \int_G |f(A)|^2 \delta A < \infty\}.$$

With respect to the inner product

$$\langle f_1, f_2 \rangle = \int_G f_1(A)\overline{f_2}(A)\delta A.$$

$L_2(G)$ is a Hilbert space. Let

$$\varphi_{ij}^{(\mu)}(A) = n_\mu^{1/2} T_{ij}^{(\mu)}(A).$$

It follows from the orthogonality relations that $\{\varphi_{ij}^{(\mu)}\}$, where $1 \leq i, j \leq n_\mu$ and μ ranges over all equivalence classes of irreducible representations, forms an ON set in $L_2(G)$.

Theorem 1 *(Peter-Weyl). If G is a compact linear Lie group, the set $\{\varphi_{ij}^{(\mu)}\}$ is an ON basis for $L_2(G)$.*

Chapter 2

Special functions as matrix elements

Let T be a representation of the Lie group G on the Hilbert space V and let $\{v_n\}$ be an ON basis for V. (Here $\{v_n\}$ is typically chosen so that it has simple transformation properties with respect to the subgroups in some chain $G \supset G_1 \supset G_2 \supset \cdots \supset \{e\}$.) Then the matrix elements of the operators $\mathbf{T}(g)$ with respect to this basis satisfy the addition theorem

$$T_{km}(g_1 g_2) = \sum_j T_{kj}(g_1) T_{jm}(g_2), \qquad g_1, g_2 \in G.$$

If T is a unitary representation then these matrices are unitary:

$$T_{mn}(g^{-1}) = \overline{T_{nm}(g)}.$$

For important classes of groups G and bases $\{v_n\}$ these matrix elements are familiar special functions. With respect to the inner product on the Hilbert space, the matrix elements can be expressed as

$$T_{km}(g) =< \mathbf{T}(g) v_m, v_k >,$$

which for function space models ot T may provide an integral representation of the matrix elements.

2.1 A classical example: The rotation group and spherical harmonics

A very important example of the orthogonality relations for compact linear Lie groups and the Peter-Weyl theorem is the rotation group $SO(3) = SO(3, R)$. This case was already treated in the Bateman project. Recall that $SO(3)$ has a convenient realization as the group of all 3×3 real matrices A such that $A^t A = I_3$ and $\det A = 1$. This is the natural realization of $SO(3)$ as the group of all rotations in R_3 which leave the origin fixed. One convenient parametrization of $SO(3)$ is in terms of the **Euler angles**. A rotation through angle φ about the z axis is given by

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

and rotations through angle φ about the x and y axis are given by

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \in SO(3),$$

$$R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \in SO(3),$$

respectively. Differentiating each of these curves in $SO(3)$ with respect to φ and setting $\varphi = 0$ we find the following linearly independent matrices in the tangent space at the identity:

$$L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

One can check from the definition $A^t A = E_3$ that the tangent space at the identity is at most three-dimensional, so the matrices L_x, L_y, L_z form a basis for this space.

The Euler angles φ, θ, ψ for $A \in SO(3)$ are given by

$$A(\varphi, \theta, \psi) = R_z(\varphi) R_x(\theta) R_z(\psi)$$

$$= \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta & \sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta & \sin\psi\sin\theta \\ -\cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\theta & \sin\varphi\sin\theta - \sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta & -\cos\varphi\sin\theta \\ \cos\psi\sin\theta & \cos\theta & \end{pmatrix}$$

$$dA = \sin\theta d\varphi d\theta d\psi.$$

Since $SO(3)$ is compact, dA is both left- and right-invariant. The volume of $SO(3)$ is

$$V_{SO(3)} = \int_{SO(3)} dA = \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = 8\pi^2.$$

The irreducible unitary representations of $SO(3)$ are denoted $\mathbf{T}^{(\ell)}, \ell = 0, 1, 2, \cdots$, where $\dim \mathbf{T}^{(\ell)} = 2\ell+1$. Expressed in terms of an ON basis for the representation space $V^{(\ell)}$ consisting of simultaneous eigenfunctions for the operators $\mathbf{T}^{(\ell)}(R_z(\varphi))$, the matrix elements are

$$\begin{aligned} T^{\ell}_{km}(\varphi,\theta,\psi) &= i^{k-m}\left[\tfrac{(\ell+m)!(\ell-k)!}{(\ell+k)!(\ell-m)!}\right]^{1/2} e^{i(k\varphi+m\psi)} \tfrac{[\sin\theta]^{m-k}(1+\cos\theta)^{\ell+k-m}}{2^{\ell}\Gamma(m-k+1)} {}_2F_1\left(\begin{matrix} -\ell, -k \\ m-1 \end{matrix} ; \tfrac{\cos\theta-1}{\cos\theta+1}\right) \\ &= i^{k-m}\left[\tfrac{(\ell+m)!(\ell-k)!}{(\ell+k)!(\ell-m)!}\right]^{1/2} e^{i(k\varphi+m\psi)} P_{\ell}^{-k,m}(\cos\theta), \end{aligned}$$

where $-\ell \le k, m \le \ell$. Here ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}, x\right)$ is the Gaussian hypergeometric function and $\Gamma(z)$ is the gamma function. A generating function for the matrix elements is

$$g(A,z) = \frac{(\beta z + \bar{\alpha})^{\ell-m}(\alpha z - \bar{\beta})^{\ell+m}}{[(\ell-m)!(\ell+m)!]^{1/2}} = \sum_{k=-\ell}^{\ell} T^{\ell}_{km}(A) \frac{(-1)^{k-m} z^{\ell+k}}{[(\ell-k)!(\ell+k)!]^{1/2}}$$

where

$$\alpha = e^{i(\varphi+\psi)/2}\cos\frac{\theta}{2}, \quad \beta = ie^{i(\phi-\psi)/2}\sin\frac{\theta}{2}.$$

The group property

$$T^{\ell}_{km}(A_1A_2) = \sum_{j=-\ell}^{\ell} T^{\ell}_{kj}(A_1)T^{\ell}_{jm}(A_2)$$

defines an **addition theorem** obeyed by the matrix elements. The unitary property of the operator $\mathbf{T}^{(\ell)}(A)$ implies

$$T^{\ell}_{km}(A^{-1}) = \overline{T^{\ell}_{mk}}(A),$$

or in Euler angles,

$$(-1)^{m-k}P_\ell^{-k,m}(\cos\theta) = \frac{(\ell+k)!(\ell-m)!}{(\ell-k)!(\ell+m)!}P_\ell^{-m,k}(\cos\theta).$$

Also, $|T_{km}^\ell(A)| \leq 1$ or

$$|P_\ell^{-k,m}(\cos\theta)| \leq \left[\frac{(\ell+k)!(\ell-m)!}{(\ell+m)!(\ell-k)!}\right]^{1/2}, 0 \leq \theta \leq \pi.$$

The matrix elements $T_{om}^\ell(\varphi,\theta,\psi)$, are proportional to the spherical harmonics $Y_\ell^m(\theta,\psi)$. Indeed

$$T_{om}^\ell(\varphi,\theta,\psi) = i^m\left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_\ell^m(\theta,\psi) = i^m\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)e^{im\psi}$$

where the $P_\ell^m(\cos\theta)$ are the associated Legendre functions. Moreover,

$$T_{oo}^\ell(\varphi,\theta,\psi) = P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta)$ is the Legendre polynomial.

Orthogonality relations:

$$\int_{SO(3)} T_{km}^\ell(A)\overline{T_{k'm'}^{\ell'}}(A)dA = \frac{8\pi^2}{2\ell+1}\delta_{kk'}\delta_{mm'}\delta_{\ell\ell'}.$$

Thus

$$\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^\pi d\theta \; T_{km}^\ell(\varphi,\theta,\psi)\overline{T_{k'm'}^{\ell'}}(\varphi,\theta,\psi)\sin\theta = \frac{8\pi^2}{2\ell+1}\delta_{kk'}\delta_{mm'}\delta_{\ell\ell'}.$$

The ψ and φ integrations are trivial, while the θ integration gives

$$\int_0^\pi P_\ell^{k,m}(\cos\theta)P_{\ell'}^{k,m}(\cos\theta)\sin\theta d\theta = \frac{2}{2\ell+1}\frac{(\ell-k)!(\ell-m)!}{(\ell+k)!(\ell+m)!}\delta_{\ell\ell'}.$$

For $k = m = 0$ these are the orthogonality relations for the Legendre polynomials. (Note: By definition, $P_\ell^{0,-m}(\cos\theta) = P_\ell^m(\cos\theta), P_\ell^{0,0}(\cos\theta) = P_\ell(\cos\theta)$, where P_ℓ^m, P_ℓ are Legendre functions.)

By the Peter-Weyl theorem, the functions

$$\varphi_{km}^\ell(\varphi,\theta,\psi) = (2\ell+1)^{1/2}T_{km}^\ell(\varphi,\theta,\psi),$$
$$-\ell \leq k,m \leq \ell, \quad \ell = 0,1,2,\cdots$$

constitute an ON basis for $L_2(SO(3))$. If $f \in L_2(SO(3))$ then

$$f(\varphi,\theta,\psi) = \sum_{\ell=0}^{\infty} \sum_{k,m=-\ell}^{\ell} a^\ell_{km} \varphi^\ell_{km}(\varphi,\theta,\psi)$$

where

$$a^\ell_{km} = (f, \varphi^\ell_{km}) = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \times$$
$$\times f(\varphi,\theta,\psi) \overline{\varphi^\ell_{km}}(\varphi,\theta,\psi) \sin\theta.$$

2.2 Matrix elements of some other group representations: SL(2,C), SL(2,R), SU(2)

Example 1 *SL(2,C).*

$$SL(2,C) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in C, \quad \det(A) = 1 \right\}.$$

$$L\left(SL(2,C)\right) = s\ell(2,C) = \left\{ \mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha,\beta,\gamma,\delta \in C, \quad \mathrm{trace}(A) = 0 \right\}.$$

Basis for Lie algebra: L^+, L^-, L^3

$$[L^3, L^\pm] = \pm L^\pm, [L^+, L^-] = 2L^3.$$

$$L^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad L^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad L^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Finite-dimensional representations

$$T_u(A) f(z) = (bz+d)^{2u} f(\frac{az+c}{bz+d}), \qquad 2u = 0,1,2,\cdots.$$

Basis for representation space:

$$f_j(z) = z^j, \qquad j = 0,1,\cdots,2u.$$

Matrix elements: $T_u(A) f_j(z) = \sum_{\ell=o}^{2u} D_{\ell j}(A) f_\ell(z)$

$$(az+c)^j (bz+d)^{2u-j} = \sum_{\ell=0}^{2u} D_{\ell j}(A) z^\ell$$

$$D_{\ell j}(A) = \frac{a^\ell d^{2u-j} c^{j-\ell} j!}{\ell!(j-\ell)!} {}_2F_1\left(\begin{array}{cc} -\ell & -2u+j \\ j-\ell+1 & \end{array} ; \frac{bc}{ad} \right).$$

Addition theorem:

$$D_{\ell j}(AB) = \sum_{k=0}^{2u} D_{\ell k}(A) D_{kj}(B), \qquad \ell, j = 0, 1, \cdots, 2u.$$

Restrict to subgroup $SU(2)$ and use Euler angles to parametrize group elements. Then $D_{\ell j}(\phi, \theta, \psi)$ is a Wigner D-function and $D_{0m}(\phi, \theta, \psi) \sim Y_\ell^m(\theta, \phi)$ is a spherical harmonic, where $\ell = 2u$. With respect to the normalized basis, the matrix elements are unitary: $U^u_{nm}(A^{-1}) = \overline{U^u_{mn}(A)}$.

Note: The addition theorem shows that for fixed ℓ the matrix element $D_{\ell j}(A)$ transforms under right multiplication bt B exactly as the bais function f_j. passing to the Lie algebra action we see that the addition theorem for the ${}_2F_1$ polynomials is obtained from exponentiating the differential recurrence relations $E^{\beta\gamma}, E_{\beta\gamma}$ for the

$${}_2F_1\left(\begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; z \right)$$

where $E^{\beta\gamma}$ raises the β and γ parameters by one and $E_{\beta\gamma}$ lowers the β and γ parameters by one.

Some infinite-dimensional representations:

$$T_u(A) f(z) = (bz+d)^{2u} f(\frac{az+c}{bz+d}), \qquad 2u \in C, 2u \neq 0, 1, \cdots, \qquad f \text{ analytic}$$

Basis for representation space:

$$f_j(z) = z^j, \qquad j = 0, 1, \cdots.$$

Matrix elements: $T_u(A) f_j(z) = \sum_{\ell=0}^\infty B_{\ell j}(A) f_\ell(z)$

$$(az+c)^j (bz+d)^{2u-j} = \sum_{\ell=0}^\infty B_{\ell j}(A) z^\ell$$

$$B_{\ell j}(A) = \frac{a^\ell d^{2u-j} c^{j-\ell} j!}{\ell!(j-\ell)!} {}_2F_1\left(\begin{array}{c} -\ell, -2u+j \\ j-\ell+1 \end{array} ; \frac{bc}{ad} \right).$$

Addition theorem:

$$B_{\ell j}(AB) = \sum_{k=0}^\infty B_{\ell k}(A) B_{kj}(B), \qquad \ell, j = 0, 1, \cdots$$

Restricted to the subgroup

$$\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in C, \qquad |a|^2 - |b|^2 = 1 \right\}$$

the representation is unitary and irreducible for $u = n$ where $2n$ is a positive integer. Then the matrix elements with respect to the normalized basis are unitary.

Another realization of this infinite dimensional representation (acting on functions of two complex variables, z, t:

$$T_u(A)f(z,t) = (d+bt)^u (a + \frac{c}{t})^u \exp(\frac{bzt}{d+bt}) f\left(\frac{zt}{(at+c)(bt+d)}, \frac{at+c}{bt+d} \right),$$

$$|\frac{c}{at}| < 1, |\frac{bt}{d}| < 1.$$

Basis: $f_j(z,t) = \frac{\Gamma(-2u)j!}{\Gamma(j-2u)} L_j^{(-2u-1)}(z) t^{j-u}, \qquad j = 0, 1, \cdots$

$$T_u(A) f_j = \sum_{\ell=0}^{\infty} B_{\ell j}(A) f_\ell$$

Special case:

$$(1-b)^{2u} \exp(\frac{-bz}{1-b}) = \sum_{\ell=0}^{\infty} b^\ell L_\ell^{(-2u-1)}(z), \qquad |b| < 1.$$

Example 2 $SL(2,R) = \{a \in SL(2,C) : A \text{ real }\}$

Representation

$$T_u(A)f(x) = (bx+d)^{2u} f(\frac{ax+c}{bx+d}), \qquad u \in C, \qquad f \text{ infinitely differentiable}$$

Compute matrix elements in continuum basis x^λ corresponding to generator $L_3 \sim x\frac{d}{dx} - u$ of noncompact one-parameter subgroup

$$\exp t\mathcal{L}_3 = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Use Mellin transform to map

$$f(x) \Longleftrightarrow (F_+(\lambda), F_-(\lambda))$$

$$F_+(\lambda) = \int_0^\infty x^{\lambda-1} f(x)dx \equiv \int_{-\infty}^\infty x_+^{\lambda-1} f(x)dx$$
$$F_-(\lambda) = \int_0^\infty x^{\lambda-1} f(-x)dx \equiv \int_{-\infty}^\infty x_-^{\lambda-1} f(x)dx.$$

Thus,

$$f(x) = \begin{cases} \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f_+(\lambda)x^{-\lambda}d\lambda, & x > 0, \\ \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f_+(\lambda)(-x)^{-\lambda}d\lambda, & x < 0, \end{cases}$$

where $0 < a < -2\Re u$. *Induce representation operators*

$$T_u(A)\begin{pmatrix} F_+(\lambda) \\ F_-(\lambda) \end{pmatrix} = \int_{a-i\infty}^{a+i\infty} \begin{pmatrix} K_{++}(\lambda,\mu;u;A) & K_{+-}(\lambda,\mu;u;A) \\ K_{-+}(\lambda,\mu;u;A) & K_{--}(\lambda,\mu;u;A) \end{pmatrix} \begin{pmatrix} F_+(\mu) \\ F_-(\mu) \end{pmatrix} d\mu$$

Addition theorems:

$$\begin{pmatrix} K_{++}(\lambda,\mu;u;AB) & K_{+-}(\lambda,\mu;u;A) \\ K_{-+}(\lambda,\mu;u;A) & K_{--}(\lambda,\mu;u;A) \end{pmatrix} = \int_{a-i\infty}^{a+i\infty} \begin{pmatrix} K_{++}(\lambda,\nu;u;A) & K_{+-}(\lambda,\nu;u;A) \\ K_{-+}(\lambda,\nu;u;A) & K_{--}(\lambda,\nu;u;A) \end{pmatrix} \times$$
$$\begin{pmatrix} K_{++}(\nu,\mu;u;B) & K_{+-}(\nu,\mu;u;B) \\ K_{-+}(\nu,\mu;u;B) & K_{--}(\nu,\mu;u;B) \end{pmatrix} d\nu.$$

The $K_{\pm,\pm}$ *are expressible in terms of Gaussian hypergeometric functions. For example, if*

$$A = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

then

$$K_{++}(\lambda,\mu;u;A) = \frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(-\lambda-2u)}{\Gamma(-2u)}\frac{\cosh^{\lambda+\mu+2u}(\theta)}{\sinh^{\lambda+\mu}(\theta)}{}_2F_1\left(\begin{matrix}\lambda,\mu \\ -2u\end{matrix}\,;-\frac{1}{\sinh^2\theta}\right),$$

etc.

Chapter 3

Symmetries of differential equations

Hypergeometric functions and their generalizations arise as solutions of "canonical" systems of partial differential equations via a particularly simple separation of variables (in so-called subgroup coordinates). They can be characterized via the Lie symmetry algebras of the system of differential equations. Their differential recurrence relations correspond to the possible Lie symmetries of the equations. orthgonality relations for polynomial hypergeometric functions can be derived easily from this approach, and the derivations extend to differential and q-difference equations, including Askey-Wilson polynomials.

Let D be a linear partial differential operator in n dimensions (with locally analytic coefficients). Let λ be a parameter.

Definition 8 *The linear partial differential operator S is a* **symmetry operator** *for the equation $D\Phi = \lambda\Phi$ if S maps local solutions Φ to local solutions $S\Phi$. This is basically equivalent to the requirement that $[S, D] = 0$.*

The linear partial differential operator $\tilde{S}$ is a **conformal symmetry operator** *for the equation $D\Phi = 0$ if $\tilde{S}$ maps local solutions Φ of $D\Phi = 0$ to local solutions $S\Phi$.*

The first order symmetry operators for $D\Phi = \lambda\Phi$ form a Lie algebra, the **symmetry algebra** of this equation. The associated local **Lie symmetry group** maps solutions to solutions. The first order conformal symmetry operators for $D\Phi = 0$ form a Lie algebra, the **conformal symmetry algebra** of this equation. The associated local **Lie conformal symmetry group** maps solutions to solutions.

Special functions frequently arise as solutions of the PDEs of mathematical physics, characterized by their transformation properties under the symmetry algebra. This generalizes all of the preceding examples.

3.1 The wave equation and Gaussian hypergeometric functions

Consider the complex wave (or Laplace) equation in four-dimensional space

$$(\partial_{u_1}\partial_{u_2} - \partial_{u_3}\partial_{u_4})\Phi = 0.$$

(A simple complex linear change of coordinates recasts this equation into the more familiar form

$$(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} + \partial^2_{x_4})\Phi = 0.)$$

The functions

$$\Phi\left(\begin{array}{c}\alpha,\beta\\ \gamma\end{array}\right) \equiv {}_2F_1\left(\begin{array}{c}\alpha,\beta\\ \gamma\end{array};\frac{u_3u_4}{u_1u_2}\right)u_1^{-\alpha}u_2^{-\beta}u_3^{\gamma-1}$$

are solutions of the complex wave equation, where ${}_2F_1$ is a Gaussian hypergeometric function. These solutions are easily characterized in terms of the local Lie symmetries of the wave equation. It is evident that certain linear combinations of the dilation generators $D_j = u_j\partial_j$ are symmetries. The hypergeometric solutions are characterized to within a constant factor by the requirements that they are analytic functions of the u_j in a neighborhood of $u_4 = 0$ and that they satisfy the eigenvalue equations (described by dilation symmetries)

$$(D_1 + D_4)\Phi = -\alpha\Phi, \qquad (D_2 + D_4)\Phi = -\beta\Phi, \qquad (D_3 - D_4)\Phi = (\gamma - 1)\Phi.$$

Note that we have a separable solution of the wave equation in terms of the variables

$$z = \frac{u_3u_4}{u_1u_2}, \quad u_1, u_2, u_3.$$

It is evident that the operators ∂_{u_j}, $1 \le j \le 4$ are also symmetries, i.e., they map solutions to solutions. In particular these operators map the basis functions $\Phi\begin{pmatrix}\alpha,\beta\\ \gamma\end{pmatrix}$ into other solutions. By consideration of the comutation relations of the ∂_{u_j} with the dilation symmetries D_ℓ or by direct power series computation one easily obtains the relations

$$\begin{aligned}\partial_{u_1}\Phi\begin{pmatrix}\alpha,\beta\\ \gamma\end{pmatrix} &= \alpha\Phi\begin{pmatrix}\alpha+1,\beta\\ \gamma\end{pmatrix}, \qquad \partial_{u_2}\Phi=\beta\Phi\begin{pmatrix}\alpha,\beta+1\\ \gamma\end{pmatrix}\\ \partial_{u_3}\Phi &= (\gamma-1)\Phi\begin{pmatrix}\alpha,\beta\\ \gamma-1\end{pmatrix}, \qquad \partial_{u_4}\Phi=\frac{\alpha\beta}{\gamma}\Phi\begin{pmatrix}\alpha+1,\beta+1\\ \gamma+1\end{pmatrix}.\end{aligned}$$

Note that upon factoring out the dependence on u_1, u_2, u_3 we obtain well known differential recurrence relations for the functions

$$ {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right).$$

A notation which suggests the raising operator (lowering operator) nature of the ∂_{u_j} is $E^\alpha\equiv\partial_{u_1}$, $E^\beta\equiv\partial_{u_2}$, $E_\gamma\equiv\partial_{u_3}$, $E^{\alpha\beta\gamma}\equiv\partial_{u_4}$. Note that the wave equation is just

$$(E^\alpha E^\beta - E_\gamma E^{\alpha\beta\gamma})\Phi=0.$$

The conformal symmetries of the wave equation yield eight more symmetries which also have a recurrence relation interpretation: E_α, E_β, E^γ, $E^{\alpha\gamma}$, $E_{\alpha\gamma}$, $E^{\beta\gamma}$, $E_{\beta\gamma}$, $E_{\alpha\beta\gamma}$. For example

$$E_{\alpha\beta\gamma}\Phi=(\gamma-1)\Phi\begin{pmatrix}\alpha-1,\beta-1\\ \gamma-1\end{pmatrix}.$$

- The $SL(2)$ representations in the preceding chapter are all special cases of what we have here. For each of those models there was a single pair of raising and lowering operators, correspnding to recurrence relations for hypergeometric series. Here the full local symmetry group is $SL(6)$.

If we make the change of variable $(1-x)/2 = u_3u_4/u_1u_2$ and factor out the dependence on u_1, u_2, u_3 in the recurrence relations for $E^{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma}$ we obtain a pair of well-known recurrence relations for the Jacobi polynomials $P_n^{(\alpha,\beta)}$:

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{\alpha+\beta+n+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

$$[(x^2-1)\frac{d}{dx}+(\alpha-\beta+(\alpha+\beta+2)x)]P_{n-1}^{(\alpha+1,\beta+1)}(x)=nP_n^{(\alpha,\beta)}(x)$$

where

$$P_n^{(\alpha,\beta)}(x) = \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) {}_2F_1 \left(\begin{array}{c} -n, \alpha+\beta+n+1 \\ \alpha+1 \end{array} ; \frac{1-x}{2} \right), \qquad n = 0, 1, 2, \cdots .$$

Note that the polynomials of orders 0 and 1 are given by

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta}{2} + \frac{1}{2}(\alpha+\beta+2)x.$$

Composition of the two recurrence relations leads to the standard Sturm-Liouville eigenvalue equation:

$$(1-x^2)P_n^{(\alpha,\beta)\,\prime\prime} + [\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)\,\prime} = n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}.$$

(This is precisely the ordinary differential equation for the Jacobi polynomials that we would obtain by directly separating variables in the wave equation. However, here we have demonstrated that this equation can be "factored" in terms of the recurrences.)

Now we derive the orthogonality relations for the Jacobi polynomials through a variant of the usual Sturm-Liouville procedure that exploits the factorization. Let $S_{\alpha,\beta}$ be the space of all polynomials in x with complex inner product

$$< g_1, g_2 >_{\alpha,\beta} = \int_{-1}^{1} g_1(x) g_2(x) \rho_{\alpha,\beta}(x)\, dx,$$

$$g_1, g_2 \in S_{\alpha,\beta}.$$

(We will consider the polynomials $P_n^{(\alpha,\beta)}$ to belong to $S_{\alpha,\beta}$. The interval of integration is motivated by the singularities.) Motivated by the recurrence relations we define maps

$$\tau^{(\alpha,\beta)} = \frac{d}{dx} : S_{\alpha,\beta} \to S_{\alpha+1,\beta+1}$$

$$\tau^{*(\alpha+1,\beta+1)} = (x^2-1)\frac{d}{dx} + (\alpha-\beta+(\alpha+\beta+2)x) : S_{\alpha+1,\beta+1} \to S_{\alpha,\beta},$$

and look for density functions $\rho_{\alpha,\beta}(x)$ such that

$$< g, \tau^{(\alpha,\beta)} f >_{\alpha+1,\beta+1} = < \tau^{*(\alpha+1,\beta+1)} g, f >_{\alpha,\beta}, \tag{3.1}$$

for all $f \in S_{\alpha,\beta}, \quad g \in S_{\alpha+1,\beta+1}$. That is, we require that τ^* is the adjoint operator to τ. A straightforward integration by parts argument, using the fact that f and g are arbitrary polynomials, leads to the necessary and sufficient conditions:

$$\rho_{\alpha+1,\beta+1}(x) = (1-x^2)\rho_{\alpha,\beta}(x), \quad \frac{d}{dx}\rho_{\alpha+1,\beta+1}(x) = [\beta-\alpha-(\alpha+\beta+2)x]\rho_{\alpha,\beta}(x)$$

with solution, unique up to a constant multiplicative factor,

$$\rho_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta.$$

This solution will provide a satisfactory weight function for $S_{\alpha,\beta}$ provided $\alpha > -1$, $\beta > -1$, which we now assume.

Since τ and τ^* are mutual adjoints, it follows immediately that

$$\tau^*\tau : S_{\alpha,\beta} \to S_{\alpha,\beta}$$

is a self-adjoint operator with eigenvalues

$$\lambda_n = n(\alpha+\beta+n-1), \quad n = 0, 1, \cdots$$

and eigenfunctions

$$g_n = P_n^{(\alpha,\beta)}(x).$$

It is easy to show that $\lambda_n = \lambda_m$ if and only if $n = m$. Using the well-known fact that eigenfunctions corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal we obtain the orthogonality relations

$$< P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} >_{\alpha,\beta} = \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \, dx$$

$$= \delta_{nm}M_n(\alpha,\beta).$$

The polynomials $\{P_n^{(\alpha,\beta)}\}$ could have been computed from a knowledge of the weight function $\rho_{\alpha,\beta}$ via the Gram-Schmidt process and are uniquely determined once the coefficient of x^n in $P_n^{(\alpha,\beta)}(x)$ is specified. Since the measure is invariant under the interchange $x \leftrightarrow -x$, $\alpha \leftrightarrow \beta$, it follows easily that

$$P_n^{(\alpha,\beta)}(-x) = P_n^{(\beta,\alpha)}(x),$$

a nontrivial identity.

Now we try to determine the normalization of the Jacobi polynomials, i.e., to compute $M_n(\alpha,\beta)$. Identity (3.1) with $g = P_{n-1}^{(\alpha+1,\beta+1)}$, $f = P_n^{(\alpha,\beta)}$ yields the recurrence

$$\frac{1}{2}(\alpha+\beta+n+1)M_{n-1}(\alpha+1,\beta+1) = -nM_n(\alpha,\beta),$$

so it is sufficient to compute

$$||1||_{\alpha,\beta} \equiv M_0(\alpha,\beta) =< 1,1 >_{\alpha,\beta}= \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \, dx.$$

This is the beta-integral and can be easily evaluated by a well-known trick, but for pedagogical purposes we shall determine what a knowledge of the recurrence symmetries and the connection with orthogonality alone will tell us. Using the evident facts that

$$< P_1^{(\alpha,\beta)}, P_0^{(\alpha,\beta)} >_{\alpha,\beta}= 0, \qquad < (1+x), 1 >_{\alpha,\beta}=< 1,1 >_{\alpha,\beta+1},$$

we obtain the recurrence relations

$$\frac{1}{2}(\alpha+\beta+2)||1||_{\alpha,\beta+1} = (\beta+1)||1||_{\alpha,\beta}$$

$$\frac{1}{2}(\alpha+\beta+2)||1||_{\alpha+1,\beta} = (\alpha+1)||1||_{\alpha,\beta}.$$

These recurrence relations have the solution

$$||1||_{\alpha,\beta} = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+2)} 2^{\alpha+\beta} h(\alpha,\beta)$$

where $h(\alpha+1,\beta) = h(\alpha,\beta+1) = h(\alpha,\beta)$. (Here we are using the property $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function. If $f(z+1) = zf(z)$ then $f(z) = \Gamma(z)h(z)$ where $h(z+1) = h(z)$.) This is as far as we can go using recurrence relations alone, since the Gamma function isn't uniquely determined by its fundamental recurrence relation. We need additional facts to compute h.

One way to proceed is to replace α, β by $\alpha+k$, $\beta+k$, $k = 0,1,2,\cdots$, and write the resulting identity in the form

$$\int_{-1}^{1} \left[\left(\frac{1-x}{2}\right)^{\alpha+k} \left(\frac{1+x}{2}\right)^{\beta+k} \frac{\Gamma(\alpha+\beta+2k+2)}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \right] dx = h(\alpha,\beta).$$

Letting $k \to +\infty$ we find from Stirling's formula, that the integrand of the left-hand side converges to 1 and that $h(\alpha, \beta) \equiv 2$.

None of the steps in the foregoing development is very novel and the results are, of course, well known. What is remarkable, is the fact that the same development works for families of polynomials satisfying difference and q-difference equations. Indeed the derivation of the norm is frequently more straightforward than in the differential equations case.

3.2 Canonical equations for generalized hypergeometric functions. Introduction to Gel'fand theory

The above approach generalizes to all N-variable hypergeometric series. For example consider the basis functions

$$\Phi \equiv F_1 \left(\begin{array}{c} \alpha, \beta, \beta' \\ \gamma \end{array} ; \frac{u_3 u_4}{u_1 u_2}, \frac{u_3 u_6}{u_1 u_5} \right) u_1^{-\alpha} u_2^{-\beta} u_5^{-\beta'} u_3^{\gamma-1}$$

where F_1 is the Appell function

$$F_1 \left(\begin{array}{c} \alpha, \beta, \beta' \\ \gamma \end{array} ; x, y \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n, \qquad |x|, |y| < 1.$$

Now Φ satisfies the canonical equations

$$(\partial_{u_1 u_2} - \partial_{u_3 u_4})\Phi = 0, \quad (\partial_{u_1 u_5} - \partial_{u_3 u_6})\Phi = 0, \quad (\partial_{u_5 u_4} - \partial_{u_2 u_6})\Phi = 0,$$

and the eigenvalue equations

$$(D_1 + D_4 + D_6)\Phi = -\alpha \Phi, \quad (D_2 + D_4)\Phi = -\beta \Phi,$$

$$(D_3 - D_4 - D_6)\Phi = (\gamma - 1)\Phi, \quad (D_5 + D_6)\Phi = -\beta' \Phi.$$

A basis of first order symmetries for these equations is given by

$$\partial_{u_k}, \ k = 1, \cdots, 6$$

and the dilations

$$D_1 + D_4 + D_6, \quad D_2 + D_4, \quad D_3 - D_4 - D_6, \quad D_5 + D_6.$$

The 6 symmetries ∂_{u_k} correspond exactly to the 6 differential recursion formulas for F_1:

$$(x\partial_x + y\partial_y + \alpha)F_1 = \alpha F_1(\alpha+1), \qquad (x\partial_x + \beta)F_1 = \beta F_1(\beta+1),$$

$$(x\partial_x + y\partial_y + \gamma - 1)F_1 = (\gamma-1)F_1(\gamma-1), \qquad (y\partial_y + \beta')F_1 = \beta' F_1(\beta'+1),$$

$$\partial_x F_1 = \frac{\alpha\beta}{\gamma}F_1(\alpha+1, \beta+1, \gamma+1), \quad \partial_y F_1 = \frac{\alpha\beta'}{\gamma}F_1(\alpha+1, \beta'+1, \gamma+1).$$

This is closely related to the Gel'fand theory of hypergeometric functions.

Chapter 4

Symmetry characterization of variable separation for the Helmholtz and Schrödinger equations

Separable solutions for Laplace-Beltrami eigenvalue equations can be characterized group theoretically. For real Euclidean n-space, the real n-sphere and the real $(n-1,1)$ hyperboloid, all separable systems are known. If an equation admits multiple separable systems we can expand the special functions corresponding to one eigensystem in terms of the other.

R-separable solutions:

$$\Psi(x) = R(x)\prod_{k=1}^{N}\Psi^{(k)}(x_k,\lambda_j), \qquad \text{separation constants } \lambda_j, j=1,\cdots,N.$$

Theorem 2 *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^i\}$ for the Laplace-Beltrami eigenvalue equation*

$$(\Delta_N + V(x))\,\Psi = E\Psi$$

on an N-dimensional pseudo-Riemannian manifold are that there exists a linearly independent set $\{A_1 = \Delta_N + V, A_2, \cdots, A_N\}$ of second-order differential symmetry operators on the manifold such that:

- $[A_k, A_\ell] = 0, \quad 1 \le k, \ell \le N,$

- *Each A_k is in self-adjoint form,*
- *There is a basis $\{\omega_{(j)} : 1 \le j \le N\}$ of simultaneous eigenforms for the $\{A_k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that:

$$\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$$

The R-separable solutions

$$\Psi(x) = R(x) \prod_{k=1}^{N} \Psi^{(k)}(x_k)$$

are characterized by the eigenvalue equations

$$A_k \Psi = \lambda_k \Psi$$

where $\lambda_1, \cdots, \lambda_N$ are the separation constants. The main point of the theorems is that, under the required hypotheses the eigenforms ω^ℓ of the quadratic forms a^{ij} are normalizable, i.e., that up to multiplication by a nonzero function, ω^ℓ is the differential of a coordinate. This fact permits us to compute the coordinates directly from a knowledge of the symmetry operators.

NOTE:

- If Δ_N is the Laplace-Beltrami operator on flat space the symmetry algebra is the Lie algebra of the Euclidean group $e(N)$ (or a Minkowski group). The universal enveloping algebra maps onto the space of all differential symmetries of Δ_N.
- If Δ_N is the Laplace-Beltrami operator on a space of nonzero constant curvature the symmetry algebra is the Lie algebra $so(N+1)$ (or $so(p,q)$ with $p+q = N+1$). The universal enveloping algebra maps onto the space of all differential symmetries of Δ_N.
- CHESHIRE CAT PHENOMENON: The second order terms in a symmetry operator for the Schrödinger equation

$$(\Delta_N + V)\Psi = \lambda\Psi$$

on a flat space or space of constant curvature are expressible in terms of the second order elements in the enveloping algebras of $e(N)$ (flat space) or $so(N+1)$ (non-zero constant curvature space). Thus even if the potential V breaks the group symmetry, all separable solutions of the Schrödinger equation can be classified in terms of N commuting second-order elements in the enveloping algebras. **The group symmetry is broken but the smile lingers on: $\smile$.**

- Thus, group theoretic methods can be used to study *all* special functions that arise via separation of variables for these equations, e.g. Heun, Lame', spheroidal, toroidal, etc., not just functions of hypergeometric type.

4.1 Constuction of separable coordinate systems for spheres and Euclidean space

A complete construction of separable coordinate systems on the N-sphere and on Euclidean N-space, and a graphical method for constructing these systems is known. Here we mention some of the main ideas.

The basic elliptic coordinate system on the N-sphere is denoted

$$[e_0|e_1|\cdots|e_N].$$

All separable coordinate systems on the N-sphere can be obtained by nesting these basic coordinates for the k-spheres for $k \leq N$. For example we can obtain a separable coordinate system on the N-sphere by starting with a basic elliptic coordinate system on the $(N-k)$-sphere and embedding in it a k-sphere. The k-sphere Cartesian coordinates $(V_0, \cdots, V_k)$ can be attached to any one of the $N-k+1$ Cartesian coordinates $(U_0, \cdots, U_{N-k})$ of the $(N-k)$-sphere. Let us attach it to the first coordinate. Then we have

$$(X_0, \cdots, X_N) = (U_0V_0, \cdots, U_0V_k, U_1, \cdots, U_{N-k}), \qquad \sum_{\ell=0}^{k} V_\ell^2 = 1,$$

$$V_\ell^2 = \frac{\prod_{i=1}^{k}(v_i - f_\ell)}{\prod_{i\neq\ell}(f_i - f_\ell)}, \quad U_0^2 = \frac{\prod_{t=1}^{N-k}(u_t - e_0)}{\prod_{i\neq 0}(e_i - e_0)},$$

$$ds^2 = ds_1^2 + U_0^2 ds_2^2, \quad ds_1^2 = \sum_{h=0}^{N-k} dU_h^2, \quad ds_2^2 = \sum_{\ell=0}^{k} dV_\ell^2.$$

The resulting system is denoted graphically by

$$\begin{array}{ccccccccc}
 & & [& e_0 & \mid & e_1 & \mid \cdots \mid & e_{N-k} &] \\
 & & & \downarrow & & & & & \\
[& f_0 & \mid & \cdots & \mid & f_k &] & &
\end{array}$$

Here is another possibility:

$$\begin{array}{ccccccccccc}
 & & & [& e_0 & \mid & e_1 & \mid \cdots \mid & e_{N-k-\ell-m} & &] \\
 & & & & \downarrow & & & & & & \searrow \\
[& f_0 & \mid & f_1 & \mid & \cdots & \mid & f_k &] & [& g_0 \mid \cdots \mid g_\ell \;] \\
 & & & \downarrow & & & & & & & \\
[& h_0 & \mid & \cdots & \mid & h_m &] & & & &
\end{array}$$

Each separable system can be obtained in this way via embeddings. The graph is a tree whose nodes are basic elliptic coordinate systems.

For the special case of the 2-sphere there are just 2 separable systems, ellipsoidal coordinates

$$[e_0|e_1|e_2]$$

and spherical coordinates

$$\begin{array}{c}
[e_0| \quad e_1] \\
\downarrow \\
[f_0| \quad f_1]
\end{array}$$

For Euclidean space the results are a bit more complicated. The basic ellipsoidal coordinate system on N-space is denoted

$$< e_0|e_1| \cdots |e_{N-1} >,$$

and the parabolic coordinate system is

$$(e_1| \cdots |e_{N-1}).$$

The graphs need no longer be trees; they can have several connected components. Each connected component is a tree with a root node that is either of the above forms. Just as above, spheres can be embedded in the root coordinates or to each other. Here are two examples:

$$< e_0 > \qquad < e_0' >,$$

1) Cartesian coordinates in two-space, and 2) oblate spheroidal coordinates

$$\begin{array}{c} \left\langle \quad e_0 \quad \mid \quad e_1 \quad \right\rangle \\ \downarrow \\ \left[\quad a_1 \quad \mid \quad a_2 \quad \right] \end{array}$$

in three-space.

4.2 Example of separation of variables: A "magic" potential on the n-sphere

MOTIVATION:

The Lauricella functions

$$\Phi = F_A \left[\begin{matrix} M+G-1; & -m_1, & \cdots, & -m_n \\ & \gamma_1, & \cdots, & \gamma_n \end{matrix} ; x_1, \cdots, x_n \right]$$

and

$$(1-x)^M F_A \left[\begin{matrix} -M-\gamma_{n+1}+1; & -m_1, & \cdots, -m_n & \\ & \gamma_1, & \cdots, & \gamma_n \end{matrix} ; \frac{-x_1}{1-x}, \cdots, \frac{-x_n}{1-x} \right]$$

form a biorthogonal polynomial family where $m_i = 0, 1, 2, \cdots$, $M = \sum_{k=1}^n m_i$, $G = \sum_{\ell=1}^{n+1} \gamma_\ell$, $x = \sum_{k=1}^n x_i$ and the γ_ℓ are positive real numbers. The inner product is

$$(\Phi_1, \Phi_2) = \int \cdots \int_{x_i>0, x<1} \Phi_1 \overline{\Phi_2}\, d\tilde{\omega},$$

$$d\tilde{\omega} = x_1^{\gamma_1 - 1} \ldots x_n^{\gamma_n - 1} (1-x)^{\gamma_{n+1}-1} dx_1 \ldots dx_n.$$

The Lauricella function F_A is defined by

$$F_A \left[\begin{matrix} a; & b_1, & \cdots, & b_n \\ & c_1, & \cdots, & c_n \end{matrix} ; x_1, \cdots, x_n \right]$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \cdots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \cdots m_n!},$$

where

$$(a)_m = \begin{cases} 1 & m = 0 \\ a(a+1) \ldots (a+m-1) & m \geq 1. \end{cases}$$

The standard partial differential equations for the F_A are,

$$x_j(1-x_j)\partial_{x_jx_j}F_A - x_j\sum_{k\neq j} x_k\partial_{x_kx_j}F_A + [c_j-(a+b_j+1)x_j]\,\partial_{x_j}F_A$$

$$-b_j\sum_{k\neq j} x_k\partial_{x_k}F_A - ab_jF_A = 0$$

for $j=1,2,\cdots,n$. Adding these equations together, we see that the polynomial functions Φ, satisfy the eigenvalue equation

$$H\Phi = -M(M+G-1)\Phi$$

where

$$H = \sum_{i,j=1}^{n}(x_i\delta_{ij}-x_ix_j)\partial_{x_ix_j} + \sum_{i=1}^{n}(\gamma_i - Gx_i)\partial_{x_i}.$$

PROPERTIES OF H:

1. H maps polynomials of maximum order m_i in x_i to polynomials of the same type.

2. As the m_i range over all nonnegative integers the functions form a basis for the space of all polynomials in variables $x_1,\cdots,x_n$.

3. The eigenvalues of H acting on this space are exactly

$$\{-M(M+G-1) : M = 0,1,2,\cdots\}.$$

RELATION TO N-SPHERE:

Equation is closely related to the Laplace-Beltrami eigenvalue equation on the n-sphere. Consider the contravariant metric determined by the second derivative terms in H:

$$g^{ij} = \delta_{ij}x_i - x_ix_j, \qquad 1\le i,j\le n.$$

Then $\det(g^{ij}) = g^{-1} = x_1x_2\cdots x_n(1-x)$ and

$$g_{ij} = \frac{1}{1-x} + \frac{\delta_{ij}}{x_i}.$$

Thus

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$$

determines a metric with associated Laplace-Beltrami operator

$$\Delta_n = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_{x_i}(g^{ij}\sqrt{g}\ \partial_{x_j}).$$

A straightforward computation yields

$$H = \Delta_n + \Lambda_n$$

where

$$\Lambda_n = \sum_{j=1}^{n} [\gamma_j - \frac{1}{2} + (\frac{n+1}{2} - G)x_j]\partial_{x_j}.$$

If $\gamma_1 = \cdots = \gamma_{n+1} = 1/2$ then $H \equiv \Delta_n$, but in general H differs from Δ_n by the first order differential operator Λ_n.

Introduce Cartesian coordinates $z_0, z_1, \cdots, z_n$ in $n+1$ dimensional Euclidean space and restrict these coordinates by the conditions

$$\begin{aligned} z_0^2 &= 1 - \textstyle\sum_{i=1}^n x_i = 1 - x, \\ z_1^2 &= x_1 \\ z_2^2 &= x_2 \\ &\vdots \\ z_n^2 &= x_n. \end{aligned}$$

Note that $z_0^2 + z_1^2 + \cdots + z_n^2 = 1$. Defining a metric ds^2 by

$$ds^2 = \sum_{m=0}^{n} (dz_m)^2$$

we find

$$ds^2 = \frac{1}{4} \sum_{i,j=1}^{n} (\frac{1}{1-x} + \frac{\delta_{ij}}{x_i}) dx_i dx_j.$$

Thus the space corresponds to a portion of the n-sphere S^n.

Set $\Phi(\mathbf{x}) = \rho(\mathbf{x})\Psi(\mathbf{x})$ for a nonzero scalar function ρ:

$$(\Delta_n + \Lambda_n)\Phi = -M(M + G - 1)\Phi$$

$$\Longleftrightarrow$$

$$(\Delta_n + V_n(\mathbf{x}))\Psi = -M(M+G-1)\Psi,$$

where

$$\rho^{-1} = x_1^{\gamma_1/2-1/4}\cdots x_n^{\gamma_n/2-1/4}(1-x)^{\gamma_{n+1}/2-1/4}.$$

and

$$V_n = -\frac{1}{4}\sum_{i=1}^{n}\frac{(\gamma_i-\frac{1}{2})(\gamma_i-\frac{3}{2})}{x_i}-$$

$$\frac{1}{4}\frac{(\gamma_{n+1}-\frac{1}{2})(\gamma_{n+1}-\frac{3}{2})}{1-x}+\frac{1}{4}\left[(1-G)^2-1-\frac{(n-3)(n+1)}{4}\right].$$

The equation $H'\Psi \equiv (\Delta_n + V_n)\Psi = \lambda\Psi$ has a natural metric

$$d\omega = g^{1/2}dx_1\cdots dx_n = x_1^{-1/2}\cdots x_n^{-1/2}(1-x)^{-1/2}dx_1\cdots dx_n.$$

The operator $H' = \rho^{-1}H\rho = \Delta_n + V_n$ is formally self-adjoint with respect to the inner product

$$< \Psi_1, \Psi_2 >= \int\cdots\int_{x_i>0, x<1}\Psi_1(\mathbf{x})\overline{\Psi_2}(\mathbf{x})d\omega,$$

$$< H'\Psi_1, \Psi_2 >=< \Psi_1, H'\Psi_2 > .$$

This induces an inner product on the space of polynomial functions $\Phi(\mathbf{x}) = \rho\Psi$, with respect to which H is self-adjoint:

$$(\Phi_1, \Phi_2) \equiv< \Psi_1, \Psi_2 >= \int\cdots\int_{x_i>0, x<1}\Phi_1\overline{\Phi_2}\ d\tilde{\omega},$$

$$d\tilde{\omega} = x_1^{\gamma_1-1}\ldots x_n^{\gamma_n-1}(1-x)^{\gamma_{n+1}-1}dx_1\ldots dx_n,$$

$$(H\Phi_1, \Phi_2) = (\Phi_1, H\Phi_2).$$

A *first order symmetry operator* for the equation $H\Phi = \lambda\Phi$ is a differential operator

$$K = \sum_{i=1}^{n} f_i(\mathbf{x})\partial_{x_i} + g(\mathbf{x})$$

such that

$$[H, K] \equiv HK - KH = 0.$$

REMARKS:

1. The first order symmetry operators form a real Lie algebra.

2. If $\gamma_1 = \gamma_2 = \cdots = \gamma_{n+1} = 1/2$ then $H = \Delta_n$ and that the Lie algebra of real symmetry operators of Δ_n is so$(n+1)$, with dimension $n(n+1)/2$ and a basis of the form $\{L_{\ell k}\}$ where $0 \le \ell < k \le n$, and $L_{\ell k} = -L_{k\ell}$. Explicitly,
$$L_{\ell k} = z_\ell \partial_{z_k} - z_k \partial_{z_\ell},$$
and
$$\begin{array}{rl} L_{ij} = & 2\sqrt{x_i x_j}(\partial_{x_j} - \partial_{x_i}), \quad 1 \le i,j \le n \\ L_{0i} = & 2\sqrt{x_i(1-x)}\partial_{x_i}, \quad 1 \le i \le n. \end{array}$$

3. All real second-order differential operators S that commute with Δ_n can be expressed as linear combinations over R of real constants, elements $L_{\ell k}$ and elements $L_{\ell k}L_{\ell' k'}$.

4. For $\gamma_1, \ldots, \gamma_{n+1}$ arbitrary, the only first order symmetry is multiplication by a constant. However, there are second order symmetries:
$$S_{ij} \equiv 4x_i x_j(\partial_{x_i} - \partial_{x_j})^2 + 4(\gamma_i x_j - \gamma_j x_i)(\partial_{x_i} - \partial_{x_j})$$
$$= L_{ij}^2 + 4[(\gamma_i - \frac{1}{2})x_j - (\gamma_j - \frac{1}{2})x_i](\partial_{x_i} - \partial_{x_j}) = S_{ji}, \qquad 1 \le i < j \le n,$$
$$S_{0i} \equiv 4x_i(1-x)\partial_{x_i}^2 + 4[\gamma_i(1-x) - \gamma_{n+1}x_i]\partial_{x_i}$$
$$= L_{0i}^2 + 4[(\gamma_i - \frac{1}{2})(1-x) - (\gamma_{n+1} - \frac{1}{2})x_i]\partial_{x_i} = S_{i0}, \qquad 1 \le i \le n.$$
Also
$$8H \equiv \sum_{i,j=1}^{n} S_{ij} + 2\sum_{i=1}^{n} S_{0i}.$$

4.2.1 Orthogonal bases of separable solutions

All separable coordinates on the n-sphere are known, i.e., all separable coordinates for the Laplace-Beltrami eigenvalue equation $\Delta_n \Psi = \lambda \Psi$. They can be constructed by the graphical procedure given above. We know that:

- All separable coordinates are orthogonal.
- For each separable coordinate system the corresponding separated solutions are characterized as simultaneous eigenfunctions of a set of n second order, self-adjoint, commuting symmetry operators for Δ_n.

- These operators are real linear combinations of the symmetries L_{ij}^2, $1 \leq i < j \leq n+1$, where L_{ij} is a rotational generator in so$(n+1)$.

- For $n = 2$ there are two separable systems (ellipsoidal and spherical coordinates), while for $n = 3$ there are 6 systems. The number of separable systems grows rapidly with n, but all systems can be constructed through a simple graphical procedure. (In general, the possible separable systems are the various polyspherical coordinates, the basic ellipsoidal coordinates, and combinations of polyspherical and ellipsoidal coordinates.)

- The equation $(\Delta_n + V_n)\Psi = \lambda\Psi$ where the scalar potential takes the special form

$$V_n = \sum_{i=1}^{n} \frac{\alpha_i}{z_i^2} + \frac{\alpha_0}{z_0^2}, \quad \alpha_0, \alpha_1, \ldots, \alpha_n \text{ constants.},$$

  is separable in *all* the coordinate systems in which the Laplace-Beltrami eigenvalue equation is separable. Indeed, the equation with this potential is separable in general ellipsoidal coordinates. Since all other coordinates are limiting cases of ellipsoidal coordinates, the conclusion follows.

- The symmetry operators describing the variable separation for the potential are given explicitly as linear combinations of the symmetries S_{ij}. These operators are formally self-adjoint.

These results can now easily be extended to results for

$$(\Delta_n + \Lambda_n)\Phi = \lambda\Phi$$

through the mappings

$$\begin{aligned} \Delta_n + \Lambda_n &= \quad \rho(\Delta_n + V_n)\rho^{-1} \\ S_{ij} &= \quad \rho S'_{ij}\rho^{-1} \\ \Phi &= \quad \rho\Psi. \end{aligned}$$

CONCLUSIONS:

- All separable solutions Ψ map to separable solutions Φ.
- The separable coordinates and solutions are determined by sets of n commuting symmetry operators S of $\Delta_n + \Lambda_n$.
- The defining symmetry operators are all formally self-adjoint with respect to the inner product $(\cdot, \cdot)$.
- Since each S_{ij} maps polynomials of maximum order m_k in x_k to polynomials of the same type, it follows that a basis of separated solutions can be expressed as *polynomials* in the x_i.
- Since the symmetry operators are self-adjoint, the basis of simultaneous eigenfunctions can be chosen to be *orthogonal*.

We conclude from this argument that every separable coordinate system for the Laplace-Beltrami eigenvalue equation on the n-sphere yields an orthogonal basis of polynomial solutions, hence an orthogonal basis for all n-variable polynomials with our inner product.

EXAMPLE: Spherical coordinates $\{u_i\}$ on S^n

$$\begin{array}{ll}
z_0^2 = & 1 - x = 1 - u_n \\
z_1^2 = & x_1 = u_1 u_2 \dots u_n \\
z_2^2 = & x_2 = (1 - u_1) u_2 \dots u_n \\
\vdots & \\
z_{n-1}^2 = & x_{n-1} = (1 - u_{n-2}) u_{n-1} u_n \\
z_n^2 = & x_n = (1 - u_{n-1}) u_n .
\end{array}$$

(Note that in terms of angles $\{\theta_i\}$ one usually sets $u_i = \sin^2 \theta_i$.)

In terms of the $\{u_i\}$,

$$H = \sum_{i=1}^{n} \frac{1}{u_{i+1} \cdots u_n} \left[u_i (1 - u_i) \partial_{u_i}^2 + \left(\sum_{j=1}^{i} \gamma_j - (\sum_{p=1}^{i+1} \gamma_p) u_i \right) \partial_{u_i} \right].$$

Equation is separable in these coordinates with separation equations

$$\begin{array}{rl}
u_1(1-u_1)\partial_{u_1}^2 \Theta_1 + [\gamma_1 - (\gamma_1 + \gamma_2) u_1] \, \partial_{u_1} \Theta_1 & = c_1 \Theta_1, \\
\left[\frac{c_{k-1}}{u_k} + u_k (1 - u_k) \partial_{u_k}^2 \right] \Theta_k + \left[\sum_{j=1}^{k} \gamma_j - (\sum_{p=1}^{k+1} \gamma_p) u_k \right] \partial_{u_k} \Theta_k & = c_k \Theta_k, \\
k = 2, 3, \cdots, n. &
\end{array}$$

Here $\Theta = \prod_{k=1}^{n} \Theta_k(u_k)$ and the c_i are the separation constants, with $c_n = -M(M+G-1)$. Separable solution:

$$\Theta_1(u_1) = {}_2F_1\left(\begin{array}{c} -\ell_1, \ell_1+\gamma_1+\gamma_2-1 \\ \gamma_1 \end{array} ; u_1 \right)$$
$$c_1 = -\ell_1(\ell_1+\gamma_1+\gamma_2-1),$$
$$\Theta_k(u_k) = u_k^{\ell_1+\ell_2+\cdots+\ell_{k-1}} \times$$
$${}_2F_1\left(\begin{array}{c} -\ell_k, 2(\ell_1+\cdots+\ell_{k-1})+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1 \\ 2(\ell_1+\cdots+\ell_{k-1})+\gamma_1+\cdots+\gamma_k \end{array} ; u_k \right),$$
$$c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1),$$
$$k = 2, 3, \cdots, n,$$

where $\sum_{i=1}^{n} \ell_i = M$ and $\ell_i = 0, 1, 2 \cdots$.

We have the eigenvalue equations

$$S_\ell \Theta_\ell = c_\ell \Theta_\ell, \quad \ell = 1, \cdots, n$$

where

$$S_1 = u_1(1-u_1)\partial_{u_1}^2 + [\gamma_1 - (\gamma_1+\gamma_2)u_1]\,\partial_{u_1},$$
$$S_k = \frac{1}{u_k} S_{k-1} + u_k(1-u_k)\partial_{u_k}^2 + [\gamma_1+\cdots+\gamma_k - (\gamma_1+\cdots+\gamma_{k+1})u_k]\,\partial_{u_k},$$

$k = 2, 3, \cdots, n$, and $S_n = H$. Furthermore, $[S_i, S_j] = 0$ and the S_i are self-adjoint with respect to the inner product $(\cdot, \cdot)$.

It follows immediately that

$$(\Theta_\ell, \Theta_{\mathbf{m}}) = 0$$

unless $\ell_1 = m_1$, $\ell_2 = m_2, \ldots, \ell_n = m_n$. The measure $d\tilde{\omega}$ becomes

$$d\tilde{\omega} = u_1^{\gamma_1-1} u_2^{\gamma_1+\gamma_2-1} \ldots u_n^{\gamma_1+\cdots+\gamma_n-1} (1-u_1)^{\gamma_2-1} (1-u_2)^{\gamma_3-1} \ldots$$
$$\times (1-u_n)^{\gamma_{n+1}-1} du_1 \ldots du_n,$$

where $0 < u_i < 1$. In terms of the symmetries S_{ij}, S_{0i} we have:

$$S_k = \frac{1}{8} \sum_{i,j=1}^{k+1} S_{ij}, \qquad k = 1, \ldots, n-1,$$

$$S_n = H = \frac{1}{8}(\sum_{h,p=0}^{n} S_{hp}),$$

where we set $S_{hh} = 0$.

- The number of separable bases is 2 for $n = 2$, 6 for $n = 3$, and grows very rapidly with n.

4.2.2 Relations between bases on the sphere

Take case $n = 2$. Then

$$H\Phi = -j(j + G - 1)\Phi,$$

where

$$H = \sum_{i,k=1}^{2} (x_i \delta_{ik} - x_i x_k) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{2} (\gamma_i - G x_i) \frac{\partial}{\partial x_i}.$$

Here $G = \gamma_1 + \gamma_2 + \gamma_3$.

FACTS:

- H maps polynomials of maximum degree m_i in x_i to polynomials of the same type.
- The polynomial eigenfunctions of H form a basis for the space of all polynomials $f(x_1, x_2)$.
- The spectrum of H acting on this space is exactly $\{-j(j + G - 1) : j = 0, 1, \dots\}$.
- $H = \Delta_2 + \Lambda_2$ where Δ_2 is the Laplace Beltrami operator on S^2 and

  $$\Lambda_2 = \sum_{i=1}^{2} [\gamma_i - \frac{1}{2} + (\frac{3}{2} - G) x_i] \frac{\partial}{\partial x_i}.$$

- H is self-adjoint with respect to the inner product

  $$(f_1, f_2) = \int \cdots \int_{x_1, x_2 > 0, 1 - x_1 - x_2 > 0} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} d\omega$$

  where

  $$dw = x_1^{\gamma_1 - 1} x_2^{\gamma_2 - 1} (1 - x_1 - x_2)^{\gamma_3 - 1} dx_1 dx_2, \quad (Hf_1, f_2) = (f_1, Hf_2),$$

  and $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers. Here f_1, f_2 are polynomials in $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$.

- There are exactly two separable coordinate systems for this equation: spherical coordinates and ellipsoidal coordinates.

Spherical coordinates:
For fixed j the polynomials

$$\begin{aligned}\psi_{jm}(x_1,x_2) \quad &= (x_1+x_2)^m P_{j-m}^{\gamma_1+\gamma_2+2m-1,\gamma_3-1}(2x_1+2x_2-1)\\ &\times P_m^{\gamma_2-1,\ \gamma_1-1}\left(\tfrac{2x_1}{x_1+x_2}-1\right),\ m=0,1,\dots,j\end{aligned}$$

form an orthogonal basis for the eigenspace corresponding to eigenvalue $-j(j+G-1)$. In terms of spherical coordinates

$$z_1=\sin\theta\cos\phi,\quad z_2=\sin\theta\sin\phi,\quad z_3=\cos\theta,$$

and with $\psi_{jm}(x_1,x_2)\equiv\psi_{jm}[\theta,\phi]$, this basis reads

$$\psi_{jm}[\theta,\phi]=(\sin\theta)^{2m}P_{j-m}^{\gamma_1+\gamma_2+2m-1,\gamma_3-1}(\cos 2\theta)P_m^{\gamma_2-1,\ \gamma_1-1}(\cos 2\phi).$$

Ellipsoidal coordinates:
For the case of ellipsoidal coordinates $\{x,y\}$ we have

$$z_i^2=\frac{(x-e_i)(y-e_i)}{(e_j-e_i)(e_k-e_i)},\quad i=1,2,3,\quad i,j,k \text{ pairwise distinct.}$$

The metric on the 2-sphere is, in terms of these coordinates,

$$ds^2=\frac{y-x}{4}\left[\frac{dx^2}{(x-e_1)(x-e_2)(x-e_3)}-\frac{dy^2}{(y-e_1)(y-e_2)(y-e_3)}\right].$$

The separation equations are

$$\begin{aligned}&[-(\lambda-e_1)(\lambda-e_2)(\lambda-e_3)\left[\tfrac{d^2}{d\lambda^2}+\left(\tfrac{\gamma_1}{\lambda-e_1}+\tfrac{\gamma_2}{\lambda-e_2}\right.\right.\\ &\left.\left.+\tfrac{\gamma_3}{\lambda-e_3}\right)\tfrac{d}{d\lambda}\right]+j(j+G-1)\lambda+q]\Phi_{jq}^{\epsilon}(\lambda)=0\end{aligned}$$

where $\lambda=x,y$ according as $\epsilon=1,2$, respectively. **This is Heun's equation, the Fuchsian equation of second order with four singularities.**

The solutions for the functions $\Phi_{jq}^{\epsilon}(\lambda)$ are Heun polynomials which for fixed j will form a complete set of basis functions once the eigenvalues q have been calculated. To calculate the eigenvalues it is convenient to observe that in the coordinate system x_1,x_2 the operator $\mathcal{M}$ whose eigenvalues u are

$$u=-4q-(e_1+e_2+e_3)j(j+G-1)$$

is given by

$$\mathcal{M} = (e_1 + e_2)S_{12} + (e_2 + e_3)S_{23} + (e_1 + e_3)S_{13}$$

where the S_{ik} are the symmetry operators. That is, $\mathcal{M}$ is the second order symmetry operator for the Laplacian ($[\mathcal{M}, \Delta] = 0$) which corresponds to the separable coordinates x, y, and the Heun basis $\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y)$ is characterized as the set of eigenfunctions $\mathcal{M}\psi = u\psi$.

Now we consider the problem of expanding the Heun basis $\Phi^1_{jq}(x)\Phi^2_{jq}(y)$ in terms of the Jacobi polynomial basis:

$$\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y) = \sum_{m=0}^{j} \xi_m \psi_{jm}[\theta, \phi].$$

Three term recurrence relations for the expansion coefficients ξ_m can be deduced by requiring that

$$\mathcal{M}\psi = u\psi.$$

To carry out the computation we need the action of the various pieces S_{ik} of $\mathcal{M}$ on the Jacobi bases $\psi_{jm}[\theta, \phi]$. We find

$$\mathcal{M}\psi_{jm}[\theta, \phi] = \sum_{r=-1}^{+1} X_r \psi_{j,m+r}[\theta, \phi]$$

where

$$X_1(m, j) = \frac{4(e_1 - e_2)(\gamma_1 + \gamma_2 + \gamma_3 + m + j - 1)(\gamma_3 - m + j - 1)(m + 1)}{(\gamma_1 + \gamma_2 + 2m - 1)(\gamma_1 + \gamma_2 + 2m)}(\gamma_1 + \gamma_2 + m - 1),$$

$$X_{-1}(m, j) = \frac{4(e_1 - e_2)(\gamma_1 + \gamma_2 + m + j - 1)(-m + j + 1)(\gamma_2 - 1)(\gamma_1 - 1)}{(\gamma_1 + \gamma_2 + 2m - 1)(\gamma_1 + \gamma_2 + 2m - 2)},$$

$$\begin{aligned} X_0(m, j) - u = {} & \frac{2(e_1 - e_2)[m^2 + m(\gamma_1 + \gamma_2 - 1) - j^2 - j(\gamma_1 + \gamma_2 + \gamma_3 - 1)]}{(\gamma_1 + \gamma_2 + 2m - 2)(\gamma_1 + \gamma_2 + 2m)}(\gamma_1 + \gamma_2 - 2)(\gamma_1 - \gamma_2) \\ & + 4\frac{(e_1 - e_2)m\gamma_3(\gamma_1 - \gamma_2)(m + \gamma_2)}{(\gamma_1 + \gamma_2 + 2m - 2)(\gamma_1 + \gamma_2 + 2m)} \\ & + 2(e_1 + e_2)[-m^2 - m(\gamma_1 + \gamma_2 - 1) + j^2 + j(\gamma_1 + \gamma_2 + \gamma_3 - 1)] \\ & + 4e_3[m^2 + m(\gamma_1 + \gamma_2 - 1)] + 4q. \end{aligned}$$

Substituting this expansion into the eigenvalue equation $\mathcal{M}\psi = u\psi$ we find the three term recurrence relation

$$X_1(m-1,j)\xi_{m-1} + (X_0(m,j) - u)\,\xi_m + X_{-1}(m+1,j)\xi_{m+1} = 0$$

where $m = 0, 1, \ldots j$. Consequently the $j+1$ independent eigenvalues q are calculated from the determinant

$$\begin{vmatrix} X_0(j,j)-u & X_1(j-1,j) & & & \\ X_{-1}(j,j) & X_0(j-1,j)-u & X_1(j-2,j) & & \\ \ddots & \ddots & \ddots & & \\ & & & & \\ & & & X_{-1}(1,j) & X_0(0,j)-u \end{vmatrix}$$

$$= 0.$$

Chapter 5

Clebsch-Gordan coefficients and orthogonal polynomials

I will briefly indicate some relations between Clebsch-Gordan series for the decomposition of tensor products of group representations and special function theory. I will limit myself to a single example. The generalizations are obvious.

Example 3 *SU(2).*

$$SU(2) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in C, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

$$L\,(SU(2)) = s\ell(2, R) = \left\{ \mathcal{A} = \begin{pmatrix} ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & -ix_3 \end{pmatrix} : x_j \in R \right\}.$$

Basis for complexification of Lie algebra: L^+, L^-, L^3

$$[L^3, L^{\pm}] = \pm L^{\pm}, [L^+, L^-] = 2L^3.$$

$$L^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad L^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad L^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

This algebra is isomorphic to $so(3)$.

Finite-dimensional representations

$$T_u(A)f(z) = (\beta z + \bar{\beta})^{2u} f(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}), \qquad 2u = 0, 1, 2, \cdots,$$

Orthonormal basis for representation space:

$$f_m(z) = \frac{(-z)^{u+m}}{\sqrt{(u-m)!(u+m)!}}, \qquad m = -u, -u+1, \cdots, u-1, u.$$

Action of Lie algebra:

$$L^3 f_m = m f_m, \qquad L^{\pm} f_m = \sqrt{(u \pm m + 1)(u \mp m)} f_{m\pm 1}.$$

Note that the Casimir operator $C = L^+L^- + L^3L^3 - L^3$ is a multiple of the identity on the representation space: $C = u(u+1)I$.

Matrix elements: $T_u(A)f_m(z) = \sum_{n=-u}^{u} U_{nm}(A) f_n(z)$

Addition theorem for unitary matrix elements:

$$U_{nm}(AB) = \sum_{k=-u}^{u} U_{nk}(A) D_{km}(B), \qquad n, m = -u, \cdots, u.$$

Decompose tensor product $T_u \otimes T_v$ (Clebsch-Gordan series):

$$T_u \otimes T_v \equiv \sum_{w=|u-v|}^{u+v} \oplus T_w.$$

Here,

$$L_1^3 f_m^{(u)} = m f_m^{(u)}, \qquad L_1^{\pm} f_m^{(u)} = \sqrt{(u \pm m + 1)(u \mp m)} f_{m\pm 1}^{(u)}.$$

$$L_2^3 f_m^{(v)} = m f_m^{(v)}, \qquad L_2^{\pm} f_m^{(v)} = \sqrt{(v \pm m + 1)(v \mp m)} f_{m\pm 1}^{(v)}.$$

Orthonormal basis for left-hand side: $f_m^{(u)} \otimes f_n^{(v)}$.

Orthonormal basis for right-hand side:

$$h_k^{(w)}, \qquad w = |u-v|, \cdots, u+v, \quad -w \le k \le w,$$

where

$$J^3 h_k^{(w)} = k h_k^{(w)}, \qquad J^{\pm} h_k^{(w)} = \sqrt{(w \pm k + 1)(w \mp k)} h_{k\pm 1}^{(w)}$$

and

$$J^3 = L_1^3, \qquad J^+ = L_1^+ + L_2^+, \qquad J^- = L_1^- + L_2^-.$$

The Clebsch-Gordan coefficients are the coefficients of the unitary matrix relating these two bases:

$$h_k^{(w)} = \sum_{m,n} C(u,m;v,n|w,k) f_m^{(u)} \otimes f_n^{(v)}.$$

Relating the matrix elements of the group operators in the two bases, we find the important identity

$$U^u_{mm'}(A) U^v_{nn'}(A) = \sum_{w,k,k'} C(u,m;v,n|w,k) C(u,m';v,n'|w,k') U^w_{kk'}(A).$$

The C-G coefficients can be computed directly from our models, and are expressible in terms of hypergeometric functions ${}_3F_2(1)$. They have important symmetries that follow from the group theory. These symmetries lead to transformation formulas for the ${}_3F_2(1)$.

To decompose this representation we compute the common eigenfunctions of J^3 and $C = J^+J^- + J^3J^3 - J^3$. Clearly, eigenfunctions of $J^3 = L_1^3 + L_2^3$ with eigenvalue s are just those linear combinations of the basis vectors $F^\alpha_{u+m} = f_m^{(u)} \otimes f_n^{(v)}$ where $n = \alpha - m$. To be definite, suppose $u \leq v, \alpha \geq 0$. Applying C to the ON set $\{F^\alpha_k\}$ we find a three term recurrence relation of the form

$$CF^\alpha_k = a_k F^\alpha_{k+1} + b_k F^\alpha_k + c_k F_{k-1}$$

where a_k, b_k, c_k are explicit. The operator C is self-adjoint with discrete eigenvalues $w(w+1)$. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then this expression takes the form of a three term recurrence relation for orthogonal polynomials $F^\alpha_k(x)$ of order k in x. The functions $F^\alpha_k(x)$ are, essentially, the Clebsch-Gordan coefficients for this decomposition; the orthogonality and completeness relations for the polynomials are the unitarity conditions for the C-G coefficients.

Insights along these lines lead to the Wilson polynomials, true generalizations of the classical orthogonal polynomials, and finally to the Askey-Wilson polynomials. Indeed, the starting point for the Wilson polynomials is the Racah coefficients. These coefficients relate the basis vectors $h_k^{(s)}$ on the two sides of the expression

$$(T_u \otimes T_v) \otimes T_w \equiv T_u \otimes (T_v \otimes T_w).$$

The irreducible representation T_s may appear more than once so the Racah coefficients are by no means trivial. They can be expressed as sums of products of 4 C-G coefficients and then shown to be expressible in terms of hypergeometric functions ${}_4F_3(1)$. They satisfy unitarity, symmetry and recurrence relations, and one of those recurrence relations can be reinterpreted as a three term recurrence relation for a family of polynomials, the Racah polynomials.

Chapter 6

Harmonic analysis and lattice subgroups

We study two procedures for the analysis of time-dependent signals, locally in both frequency and time. The first procedure, the "windowed Fourier transform" is associated with the Heisenberg group while the second, the "wavelet transform" is associated with the affine group.

6.1 Harmonic analysis: windowed Fourier transforms

Let $g \in L_2(R)$ with $||g|| = 1$ and define the time-frequency translation of g by

$$g^{[x_1,x_2]}(t) = e^{2\pi i t x_2} g(t + x_1) = \mathbf{T}^1[x_1, x_2, 0]g(t)$$

where $\mathbf{T}^1$ is the unitary irreducible representation

$$\mathbf{T}^1[x_1, x_2, x_3]g(t) = e^{2\pi i x_3 + 2\pi i t x_2} g(t + x_1)$$

of the Heisenberg group H_R. Now suppose g is centered about the point (t_0, ω_0) in phase (time-frequency) space, i.e., suppose

$$\int_{-\infty}^{\infty} t|g(t)|^2 dt = t_0, \quad \int_{-\infty}^{\infty} \omega|\hat{g}(\omega)|^2 d\omega = \omega_0$$

where $\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i\omega t}dt$ is the Fourier transform of $g(t)$. Then

$$\int_{-\infty}^{\infty} t|g^{[x_1,x_2]}(t)|^2 dt = t_0 - x_1, \quad \int_{-\infty}^{\infty} \omega|\hat{g}^{[x_1,x_2]}(t)|^2 d\omega = \omega_0 + x_2$$

so $g^{[x_1,x_2]}$ is centered about $(t_0 - x_1, \omega_0 + x_2)$ in phase space. To analyze an arbitrary function $f(t)$ in $L_2(R)$ we compute the inner product

$$F(x_1, x_2) = \langle f, g^{[x_1,x_2]} \rangle = \int_{-\infty}^{\infty} f(t) \bar{g}^{[x_1,x_2]}(t) dt$$

with the idea that $F(x_1, x_2)$ is sampling the behavior of f in a neighborhood of the point $(t_0 - x_1, \omega_0 + x_2)$ in phase space. As x_1, x_2 range over all real numbers the samples $F(x_1, x_2)$ give us enough information to reconstruct $f(t)$.

However, the set of basis states $g^{[x_1,x_2]}$ is overcomplete: the coefficients $\langle f, g^{[x_1,x_2]} \rangle$ are not independent of one another, i.e., in general there is no $f \in L_2(R)$ such that $\langle f, g^{[x_1,x_2]} \rangle = F(x_1, x_2)$ for an arbitrary $F \in L_2(R^2)$. The $g^{[x_1,x_2]}$ are examples of **coherent states**, continuous overcomplete Hilbert space bases which are of interest in quantum optics and quantum field theory, as well as gropup representation theory. Thus it isn't necessary to compute the inner products $\langle f, g^{[x_1,x_2]} \rangle = F(x_1, x_2)$ for every point in phase space. In the windowed Fourier approach one typically samples F at the lattice points $(x_1, x_2) = (ma, nb)$ where a, b are fixed positive numbers and m, n range over the integers. Here, a, b and $g(t)$ must be chosen so that the map $f \longrightarrow \{F(ma, nb)\}$ is one-to-one; then f can be recovered from the lattice point values $F(ma, nb)$. The study of when this can happen is the study of *Weyl-Heisenberg frames.* It is particularly useful when g can be chosen such that $g^{[ma,nb]}$ is an ON basis for L^2. This leads (in the case $a = b$) to the lattice Hilbert space, the Weil-Brezin-Zak transform and important applications to theta functions.

6.2 Harmonic analysis: continuous wavelets

Here we work out the analog for the affine group of the Weyl-Heisenberg frame for the Heisenberg group. Let $\phi \in L_2(R)$ with $||g|| = 1$ and define the affine translation of ϕ by

$$\phi^{(a,b)}(t) = a^{-1/2} \phi\left(\frac{t+b}{a}\right) = \mathbf{L}_0[a, b]\phi(t)$$

where $a > 0$ and $\mathbf{L}_0$ is the unitary representation

$$\mathbf{L_0}[a, b]\phi(t) = a^{-1/2} \phi\left(\frac{t+b}{a}\right)$$

of the affine group.

Can assume that $\int_{-\infty}^{\infty} t|\phi(t)|^2 dt = 0$. Let $k = \int_0^\infty y|\hat{\phi}(y)|^2 dy$. Then ϕ is centered about the origin in position space and about k in momentum space. It follows that

$$\int_{-\infty}^{\infty} t|\phi^{(a,b)}(t)|^2 dt = -b, \quad \int_0^\infty y|\hat{\phi}^{(a,b)}(y)|^2 dy = a^{-1}k.$$

To define a lattice in the affine group space we choose two nonzero real numbers $a_0, b_0 > 0$ with $a_0 \neq 1$. Then the lattice points are $a = a_0^m, b = nb_0 a_0^m$, $m, n = 0, \pm 1, \cdots$, so

$$\phi^{mn}(t) = \phi^{(a_0^m, nb_0 a_0^m)}(t) = a_0^{-m/2}\phi(a_0^{-m}t + nb_0).$$

Thus ϕ^{mn} is centered about $-nb_0 a_0^m$ in position space and about $a_0^{-m}k$ in momentum space. (Note that this behavior is very different from the behavior of the Heisenberg translates $g^{[ma,nb]}$. In the Heisenberg case the support of g in either position or momentum space is the same as the support of $g^{[ma,nb]}$. In the affine case the sampling of position-momentum space is on a logarithmic scale. There is the possibility, through the choice of m and n, of sampling in smaller and smaller neighborhoods of a fixed point in position space.)

The affine translates $\phi^{(a,b)}$ are called **wavelets** and the function ϕ is a **father wavelet**. The map $\mathbf{T} : f \longrightarrow \int f(t)\phi^{mn}(t)dt$ is the **continuous wavelet transform**

Again the continuous wavelet transform is overcomplete. The question is whether we can find a subgroup lattice and a function ϕ for which the functions

$$\phi^{mn}(t) = \phi^{(a_0^m, nb_0 a_0^m)}(t) = a_0^{-m/2}\phi(a_0^{-m}t + nb_0)$$

generate an ON basis. We will choose $a_0 = 1/2, b_0 = 1$ and find conditions such that the functions

$$\phi^{mn} = 2^{m/2}\phi(2^m t + n), \qquad m, n = 0, \pm 1, \pm 2, \cdots$$

span L^2. In particular we will require that the set $\phi^{0n}(t) = \phi(t + n)$ be orthonormal.

6.3 Harmonic analysis: discrete wavelets and the multiresolution structure

From the discussion of the last section, we want to find a scaling function (or father wavelet) ϕ such that the functions $\phi^{mn}(t) = 2^{m/2}\phi(2^m t + n)$ will generate an ON basis for L^2. In particular we require that the set $\phi^{0n}(t) = \phi(t+n)$ be orthonormal. Then for each fixed m we will have that $\{\phi^{mn}\}$ is ON in n.

EXAMPLE: The Haar scaling function

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1. \\ 0 & \text{otherwise} \end{cases}$$

Here the set $\{\phi(t+n) = \phi^{0n} : n = 0, \pm 1, \cdots\}$ is ON. Let V_m be the space of piecewise constant functions in $L_2(R)$ with possible discontinuities only at the gridpoints $t_k = \frac{k}{2^{mj}}$, $k = 0, \pm 1, \pm 2 \cdots$. Note that

1. $\{\phi^{mn}(t) : n = 0, \pm 1, \cdots\}$ is an ON basis for V_m.
2. $V_m \subset V_{m+1}$
3. $\overline{\cup_m V_m} = L_2(R)$
4. $\phi(t) = \phi(2t) + \phi(2t-1)$.

This example leads naturally to the concept of a multiresolution structure on L^2.

Definition 9 *Let $\{V_j : j = \cdots, -1, 0, 1, \cdots\}$ be a sequence of subspaces of $L^2[-\infty, \infty]$ and $\phi \in V_0$. This is a multiresolution analysis for $L^2[-\infty, \infty]$ provided the following conditions hold:*

1. *The subspaces are nested: $V_j \subset V_{j+1}$.*
2. *The union of the subspaces generates L^2 : $\overline{\cup_{j=-\infty}^{\infty} V_j} = L^2[-\infty, \infty]$. (Thus, each $f \in L^2$ can be obtained a a limit of a Cauchy sequence $\{s_n : n = 1, 2, \cdots\}$ such that each $s_n \in V_{j_n}$ for some integer j_n.)*
3. *Separation: $\cap_{j=-\infty}^{\infty} V_j = \{0\}$, the subspace containing only the zero function. (Thus only the zero function is common to all subspaces V_j.)*

4. *Scale invariance:* $f(t) \in V_j \Longleftrightarrow f(2t) \in V_{j+1}$.

5. *Shift invariance of* V_0*:* $f(t) \in V_0 \Longleftrightarrow f(t-k) \in V_0$ *for all integers* k.

6. *ON basis: The set* $\{\phi(t-k) : k = 0, \pm 1, \cdots\}$ *is an ON basis for* V_0.

Here, the function $\phi(t)$ *is called the scaling function (or the father wavelet).*

Of special interest is a multiresolution analysis with a scaling function $\phi(t)$ on the real line that has *compact support*. The functions $\phi(t+k)$ will form an ON basis for V_0 as k runs over the integers, and their integrals with any polynomial in t will be finite.

- **Can we find continuous scaling functions with compact support?**

Given $\phi(t)$ we can define the functions

$$\phi_{jk}(t) = 2^{\frac{j}{2}}\phi(2^j t - k), \qquad k = 0, \pm 1, \pm 2, \cdots$$

and for fixed integer j they will form an ON basis for V_j. Since $V_0 \subset V_1$ it follows that $\phi \in V_1$ and ϕ can be expanded in terms of the ON basis $\{\phi_{1k}\}$ for V_1. Thus we have the *dilation equation*

$$\phi(t) = \sqrt{2}\sum_k \mathbf{c}(k)\phi(2t-k),$$

or, equivalently,

$$\phi(t) = 2\sum_{k=0}^{N} \mathbf{h}(k)\phi(2t-k)$$

where $\mathbf{h}(k) = \frac{1}{\sqrt{2}}\mathbf{c}(k)$. Since the ϕ_{jk} form an ON set, the coefficient vector $\mathbf{c}$ must be a unit vector in ℓ^2,

$$\sum_k |\mathbf{c}(k)|^2 = 1.$$

Since $\phi(t) \perp \phi(t-m)$ for all nonzero m, the vector $\mathbf{c}$ satisfies the orthogonality relation:

$$(\phi_{00}, \phi_{0m}) = \sum_k \mathbf{c}(k)\overline{\mathbf{c}(k-2m)} = \delta_{0m}.$$

Lemma 1 *If the scaling function is normalised so that*

$$\int_{-\infty}^{\infty} \phi(t)dt = 1,$$

then $\sum_{k=0}^{N} \mathbf{c}(k) = \sqrt{2}$.

We can introduce the orthogonal complement W_j of V_j in V_{j+1}.

$$V_{j+1} = V_j \oplus W_j.$$

We start by trying to find an ON basis for the wavelet space W_0. Associated with the father wavelet $\phi(t)$ there must be a mother wavelet $w(t) \in W_0$, with norm 1, and satisfying the *wavelet equation*

$$w(t) = \sqrt{2} \sum_k \mathbf{d}(k)\phi(2t-k),$$

and such that w is orthogonal to all translations $\phi(t-k)$ of the father wavelet. We further require that w is orthogonal to integer translations of itself. Since the ϕ_{jk} form an ON set, the coefficient vector $\mathbf{d}$ must be a unit vector in ℓ^2,

$$\sum_k |\mathbf{d}(k)|^2 = 1.$$

Moreover since $w(t) \perp \phi(t-m)$ for all m, the vector $\mathbf{d}$ satisfies so-called double-shift orthogonality with $\mathbf{c}$:

$$(w, \phi_{0m}) = \sum_k \mathbf{c}(k)\overline{\mathbf{d}(k-2m)} = 0. \tag{6.1}$$

The requirement that $w(t) \perp w(t-m)$ for nonzero integer m leads to double-shift orthogonality of $\mathbf{d}$ to itself:

$$(w(t), w(t-m)) = \sum_k \mathbf{d}(k)\overline{\mathbf{d}(k-2m)} = \delta_{0m}. \tag{6.2}$$

However, if the unit coefficient vector $\mathbf{c}$ is double-shift orthogonal then the coefficient vector $\mathbf{d}$ defined by

$$\mathbf{d}(n) = (-1)^n \overline{\mathbf{c}(N-n)}. \tag{6.3}$$

automatically satisfies the conditions (6.1) and (6.2).

Theorem 3

$$L^2[-\infty,\infty] = V_j \oplus \sum_{k=j}^{\infty} W_k = V_j \oplus W_j \oplus W_{j+1} \oplus \cdots ,$$

so that each $f(t) \in L^2[-\infty,\infty]$ can be written uniquely in the form

$$f = f_j + \sum_{k=j}^{\infty} w_k, \qquad w_k \in W_k,\ f_j \in V_j. \tag{6.4}$$

To find compact support wavelets must find solutions $\mathbf{c}(k)$ of the orthogonality relations above, nonzero for a *finite* range $k = 0, 1, \cdots, N$. Then given a solution $\mathbf{c}(k)$ must solve the dilation equation

$$\phi(t) = \sqrt{2}\sum_k \mathbf{c}(k)\phi(2t-k). \tag{6.5}$$

to get $\phi(t)$. Can show that the support of $\phi(t)$ must be contained in the interval $[0, N)$.

One way to try to determine a scaling function $\phi(t)$ from the impulse response vector $\mathbf{c}$ is to iterate the dilation equation. That is, we start with an initial guess $\phi^{(0)}(t)$, the Haar scaling function on $[0,1)$, and then iterate

$$\phi^{(i+1)}(t) = \sqrt{2}\sum_{k=0}^{N} \mathbf{c}(k)\phi^{(i)}(2t-k) \tag{6.6}$$

This is called the *cascade algorithm.*

The frequency domain formulation of the dilation equation is :

$$\hat{\phi}(\omega) = \left(\sum_k \mathbf{h}(k)e^{-i\omega k/2}\right)\hat{\phi}(\frac{\omega}{2})$$

where $\mathbf{c}(k) = \sqrt{2}\mathbf{h}(k)$. Thus

$$\hat{\phi}(\omega) = H(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}).$$

where

$$H(\omega) = \sum_{k=0}^{N} \mathbf{h}(k)e^{-i\omega k}$$

Iteration yields the explicit infinite product formula: $\hat{\phi}(\omega)$:

$$\hat{\phi}(\omega) = \Pi_{j=1}^{\infty} H(\frac{\omega}{2^j}). \tag{6.7}$$

$\phi(t)$ IS A SPECIAL FUNCTION

- Daubechies has found a solution $\mathbf{c}(k)$ and the associated scaling function for each $N = 1, 3, 5, \cdots$. (There are no solutions for even N.) Denote these solutions by $D_M = D_{N+1} = D_{2p}$. D_2 is just the Haar function. Daubechies finds the unique solutions for which the Fourier transform of the impulse response vector $C(\omega)$ has a zero of order p at $\omega = \pi$, where $2p = N + 1$. (At each N this is the maximal possible value for p.)

- Can compute the values of $\phi(t)$ exactly at all dyadic points $t = \sum_n \frac{j_n}{2^n}$, $j_n = \pm 1$.

- $\sum_k \phi(\frac{k}{2^j}) = 2^j$ for $j = 0, 1, 2, \cdots$.

- Can find explicit expressions

  $$\sum_k \mathbf{y}_{\ell k} \phi(t + k) = t^{\ell}, \qquad \ell = 0, 1, \cdots, p - 1,$$

  so polynomials in t of order $\leq p - 1$ can be expressed in V_0 with no error.

- The support of $\phi(t)$ is contained in $[0, N)$, and $\phi(t)$ is orthogonal to all integer translates of itself. The wavelets $\{w^{mn}\}$ form an ON basis for L^2.

- B-splines fit into this multiresolution framework, though more naturally with biorthogonal wavelets.

- There are matrices

  $$\mathbf{T} = (\downarrow 2) 2 \mathbf{H} \overline{\mathbf{H}}^{\mathrm{tr}} = \mathbf{M} \overline{\mathbf{H}}^{\mathrm{tr}}.$$

  associated with each of the Daubechies solutions whose eigenvalue struture determines the convergence properties of the wavelet expansions. These matrices have beautiful eigenvalue structures.

- There is a smoothness theory for Daubechies D_M. Recall $M = N+1 = 2p$. The smoothness grows with p. For $p = 1$ (Haar) the scaling function is piecewise continuous. For $p = 2$, (D_4) the scaling function is continuous but not differentiable. For $p \geq 3$ we have $s = 1$ (one derivative). For $p = 5, 6, 7, 8$ we have $s = 2$. For $p = 9, 10$ we have $s = 3$. Asymptotically s grows as $0.2075p +$ constant.

- The constants $\mathbf{c}$ are explicit for $N = 1, 3$. For $N = 5, 7, \cdots$ they must be computed numerically.

Example 4 *The nonzero Daubechies filter coefficients for D_4 ($N = 3$) are $4\sqrt{2}\mathbf{c}(k) = 1+\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3}$. With the normalization $\phi(0) + \phi(1) + \phi(2) = 1$ we have, uniquely,*

$$\phi(0) = 0, \quad \phi(1) = \frac{1}{2}(1+\sqrt{3}) \quad \phi(2) = \frac{1}{2}(1-\sqrt{3}).$$

From these three values the values of $\phi(t)$ at any dyadic point can be computed explicitly.

Wavelets are extremely useful in signal analysis, data compression (including image compression), edge detection, noise removal, etc.

Chapter 7

Significant research opportunities and challenges

- The study of many-variable hypergeometric functions via the canonical equations/ Gel'fand approach. q-analogs.
- Study of the analogs of windowed Fourier transforms and wavelet transforms in higher dimensions.
- Study of the relationship between Clebsch-Gordon coefficients for semisimple Lie algebras and multivariable orthogonal polynomials. q-analogs.
- The "magic" potentials discussed earlier are examples of superintegrable systems, Hamiltonian systems in N variables that admit $2N-1$ independent second order constants of the motion. These are integrable systems with the maximum possible symmetry and are intimately related to special function theory. The complete classification and analysis of these systems is an important task.
- In contrast to the theory of orthogonal variable separation for constant curvature spaces, classification of nonorthogonal separable systems is not well understood.
- Study of the special functions arising as solutions of the spin equations of general relativity. Teukolsky functions.