Particle in a field of two centers in prolate spheroidal coordinates: integrability and solvability

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Abstract

We analyze one particle, two-center quantum problems which admit separation of variables in prolate spheroidal coordinates, a natural restriction satisfied by the H_2^+ molecular ion. The symmetry operator is constructed explicitly. We give the details of the Hamiltonian reduction of the 3D system to a 2D system with modified potential that is separable in elliptic coordinates. The potentials for which there is double-periodicity of the Schrödinger operator in the space of prolate spheroidal coordinates, including one for the H_2^+ molecular ion, are indicated. We study possible potentials that admit exact-solvability is as well as all models known to us with the (quasi)-exact-solvability property for the separation equations. We find deep connections between second-order superintegrable and conformally superintegrable systems and these tractable problems. In particular we derive a general 4-parameter expression for a model potential that is always integrable and is conformally superintegrable for some parameter choices.

1 Introduction: Symmetry reduction

Let us consider $3D$ Euclidean space quantum problem in (x, y, z) coordinates with $2D$ potential V which has azimuthal symmetry w.r.t. rotations around z-axis. It is convenient to introduce (or parametrize the $3D$ space) the spherical coordinates (r, θ, φ). In these coordinates the Hamiltonian

$$\mathcal{H}^{(3,r)} \; = \; -\Delta^{(3)} + V(r,\theta) \; , \; x \in \mathbf{R}^3 \; , \tag{1}$$

where $\Delta^{(3)}$ is the $3D$ Laplacian,

$$\Delta^{(3)} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cot\theta}{r^2}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2.$$

The problem (1) admits the symmetry $L_\varphi \; = \; -i\,\partial_\varphi$ and dependence on the azimuthal angle φ can be separated out. Any eigenfunction has a form $\psi(r,\theta)\; e^{im\varphi}$, where m is integer (the separation constant or magnetic quantum number). Separating out the φ-dependence we arrive at the spectral problem for the $2D$ operator,

$$\tilde{\mathcal{H}}^{(3,r)}(r,\theta) \; = \; -\partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2 - \frac{\cot\theta}{r^2}\partial_\theta + \frac{m^2}{r^2\sin^2\theta} + V(r,\theta) \; ,$$

with $\psi(r,\theta)$ as eigenfunction. By making the gauge rotation of this operator,

$$\rho^{1/2}\tilde{\mathcal{H}}^{(3,r)}\rho^{-1/2} = (r\sin\theta)^{1/2}\tilde{\mathcal{H}}^{(3,r)}(r,\theta)(r\sin\theta)^{-1/2}$$

one can arrive at the $2D$ Hamiltonian

$$\mathcal{H}^{(2,r)} \; = \; -\Delta^{(2)} + \frac{m^2 - 1/4}{r^2\sin^2\theta} + V(r,\theta), \tag{2}$$

where $\Delta^{(2)}$ is the $2D$ Laplacian,

$$\Delta^{(2)} = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

with domain $r \in [0,\infty)$ and $\theta \in (0,\pi)$, the half-plane. Of course, similar results would be obtained if we parameterize $3D$ space by ellipsoidal coordinates (ξ, η, φ) or coordinates (r_1, r_2, φ) (see Fig.1). Here the "interaction" plane (x, y) is parameterized by elliptic coordinates (ξ, η) or the coordinates (r_1, r_2), respectively. If prolate spheroidal coordinates are used for $3D$ space parametrization (see below) an important observation should be made: the $3D$ potential depends on α, β only. The gauge rotation is needed to get a $2D$ Hamiltonian. This reduction of a $3D$ spectral problem with azimuthal symmetry to a $2D$ one is a reflection of the representation of $3D$ space as $\mathbf{R}^3 = \mathbf{R}^2_+ \times S^1$, where $\mathbf{R}^2_+$ is half-plane.

Let us consider the $3D$ quantum problem in (x, y, z) coordinates with $2D$ potential $V(x, y)$ which is translation-invariant w.r.t. z-axis, thus, with $3D$ space decomposition $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$. In this case the evident symmetry is the momentum $p_z = -i\partial_z$, its eigenfunction is plane wave $\sim e^{ikz}$, thus, the z-variable in the $3D$ Hamiltonian can be separated out. For restriction to $2D$ space a gauge rotation (see above) is unnecessary and the potential in the $2D$ Hamiltonian coincides with the $3D$ potential up to a constant, k^2.

Such a procedure of relating $3D$ and $2D$ quantum problems can be thought as analogous to the the Hamiltonian reduction method in classical mechanics, e.g. [1], when the underlying (configuration) space contains a symmetric space as subspace, say, $\mathbf{R}^n = \tilde{\mathbf{R}}^m \times Symm^{m-n}$. It is not

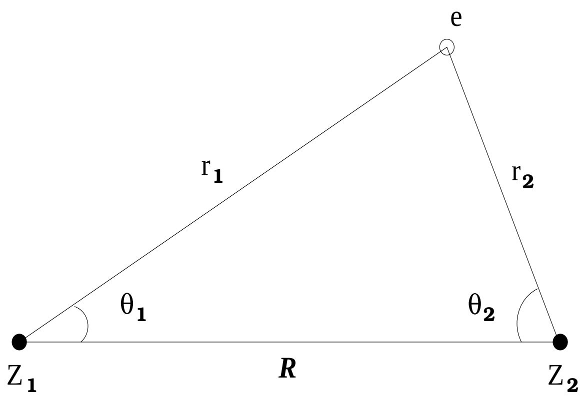

Figure 1: Geometric settings for one-particle two-center problem, the reference point is at the middle of the intercenter distance, the particle position at (x, y, φ)

an instance of the Olshanetsky-Perelomov approach [2]. See [3] and Chapter 4 of [4] for examples related to separation of variables.

In general, on many occasions the dynamics develops separately, in two different configuration subspaces V_1, V_2 which form the total configuration space $V = V_1 \times V_2$. In this case it seems natural to separate one group of variables from another one (if possible) and consider two different, disconnected dynamics. Such separation of variables can lead to appearance of extra terms in potential which have to be calculated. A long time ago it was realized in [5] for the case of four-dimensional configuration space.

2 Two-center problem

We consider the one-particle, two-center quantum problem characterized by the potential $V(r_1, r_2; R)$ (see Fig.1). There are two possible (related) formulations of the eigenvalue problem: (i) spacial, with the Hamiltonian

$$\mathcal{H}^{(3)} \;=\; -\Delta^{(3)} + V(r_1, r_2; R)\ ,\ x \in \mathbf{R}^3\ , \tag{3}$$

where $\Delta^{(3)}$ is the 3D Laplacian; (ii) planar, with the Hamiltonian

$$\mathcal{H}^{(2)} \;=\; -\Delta^{(2)} + \tilde{V}(r_1, r_2; R)\ ,\ x \in \mathbf{R}^2\ , \tag{4}$$

where $\Delta^{(2)}$ is the 2D Laplacian. The problem (i): $\mathcal{H}^{(3)}\Theta = E\Theta$ admits the integral $L_\varphi \;=\; -i\,\partial_\varphi$ and dependence on the azimuthal angle variable φ can be separated out. Any eigenstate of $\mathcal{H}^{(3)}$ is characterized by azimuthal quantum number (magnetic quantum number) $m = 0, \pm 1, \pm 2 \ldots$ which is the eigenvalue of L_φ. The eigenfunctions of $\mathcal{H}^{(3)}$ have the form $\Theta = e^{im\varphi}\rho^{-1/2}\Psi(r_1, r_2)$, where $\rho^{-1/2}$ is the gauge factor and $\Psi(r_1, r_2)$ are the eigenstates of $\mathcal{H}^{(2)}$ with modified potential

$$\tilde{V}(r_1, r_2; R) \;=\; V(r_1, r_2; R) \;+\; V_\varphi(r_1, r_2; R)\ ,$$

$$V_\varphi = \frac{(m^2-\frac{1}{4})R^2}{r_1 r_2}\left[\frac{1}{(r_1+r_2+R)(r_1+r_2-R)} - \frac{1}{(r_1-r_2+R)(r_1-r_2-R)}\right] . \tag{5}$$

Hence, the $3D$ problem (i) with $\Theta \in L^2(\mathbf{R^3})$ is reduced to the $2D$ problem (ii) with a modified scalar potential $\tilde{V}(r_1,r_2)$ and $\Psi \in L^2(\tilde{\mathbf{R}}^2)$, where $\tilde{\mathbf{R}}^2 = \mathbf{R}^2_+$ is the upper half plane. In this case $\mathbf{R}^3 = \tilde{\mathbf{R}}^2 \times S^1$. Due to this reduction, it is sufficient to study the planar eigenvalue problem

$$\mathcal{H}^{(2)}\Psi \;=\; E\Psi \ , \ \Psi \in \tilde{L}^2(\tilde{\mathbf{R}}^2) \ .$$

3 Prolate spheroidal coordinates

As in §2 we consider the one-particle, two-center quantum problem - the one-electron diatomic molecular ion (Z_1, Z_2, e) in the Born-Oppenheimer approximation. If $Z_1 = Z_2 = 1$ it becomes the Hydrogen molecular ion H_2^+. The Schrödinger equation can be written in the Cartesian coordinates as, see e.g. [6]

$$\mathcal{H}^{(3)}\Theta \equiv (-\Delta^{(3)} \;-\; \frac{Z_1}{\sqrt{x^2+y^2+(z+a)^2}} - \frac{Z_2}{\sqrt{x^2+y^2+(z-a)^2}})\Theta \;=\; E\Theta, \tag{6}$$

where $\Delta^{(3)} \;=\; \partial_{xx} + \partial_{yy} + \partial_{zz}$ and $Z_{1,2}$ are charges of the fixed centers. In terms of prolate spheroidal coordinates, see e.g. [4],

$$x = a\sinh\alpha\sin\beta\cos\phi \ , \ y = a\sinh\alpha\sin\beta\sin\phi \ , \ z = a\cosh\alpha\cos\beta \ ,$$

where $R = 2a$, see Fig.1, we have

$$\Delta^{(3)} \;= \tag{7}$$

$$\frac{1}{a^2(\cosh^2\alpha - \cos^2\beta)}\left[(\partial_{\alpha\alpha} + \coth\alpha\partial_\alpha) \;+\; (\partial_{\beta\beta} + \cot\beta\partial_\beta)\right] + \frac{1}{a^2\sinh^2\alpha\sin^2\beta}\partial_{\phi\phi} \ .$$

The Schrödinger equation now takes the form

$$(\Delta^{(3)} \;+\; \frac{(Z_1+Z_2)\cosh\alpha + (Z_2-Z_1)\cos\beta}{a(\cosh^2\alpha - \cos^2\beta)})\Theta \;=\; -E\Theta. \tag{8}$$

This follows from the metric expressions

$$ds^2 = dx^2 + dy^2 + dz^2 = a^2(\cosh^2\alpha - \cos^2\beta)(d\alpha^2 + d\beta^2) + a^2\sinh^2\alpha\sin^2\beta \; d\phi^2 \ .$$

For future use we note that $\cosh^2\alpha - \cos^2\beta = \sinh^2\alpha + \sin^2\beta$. Clearly variables separate in equation (8).

Now we investigate the most general potential $V(\alpha,\beta)$ such that the equation

$$\mathcal{H}^{(3)}\Theta \equiv (-\Delta^{(3)} + V(\alpha,\beta))\Theta = E\Theta \tag{9}$$

is separable in prolate spheroidal coordinates and determine the symmetry operator that implements this variable separation. In order to find it, let we multiply the Hamiltonian $\mathcal{H}^{(3)}$ in (9) by $(\cosh^2\alpha - \cos^2\beta)$. We arrive at a generalized spectral problem for the operator

$$\tilde{\mathcal{H}} \;\equiv\; (\cosh^2\alpha - \cos^2\beta)\mathcal{H}^{(3)} \;=$$

$$-\frac{1}{a^2}\left[(\partial_{\alpha\alpha}+\coth\alpha\partial_\alpha)\ +\ (\partial_{\beta\beta}+\cot\beta\partial_\beta)+(\frac{1}{\sinh^2\alpha}+\frac{1}{\sin^2\beta})\partial_{\phi\phi}\right]\ +\ (\cosh^2\alpha-\cos^2\beta)V(\alpha,\beta)\ , \tag{10}$$

with the weight $(\cosh^2\alpha-\cos^2\beta)$. From expression (10) it follows immediately that

$$V(\alpha,\beta)=\frac{f(\alpha)+g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\ ,$$

so

$$\left(-\Delta^{(3)}+\frac{f(\alpha)+g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\right)\Theta\ =\ E\,\Theta\ , \tag{11}$$

where $f(\alpha), g(\beta)$ are arbitrary functions. For the (Z_1, Z_2, e) system those functions are

$$f(\alpha)=(Z_1+Z_2)\cosh\alpha\ ,\ g(\beta)=(Z_2-Z_1)\cos\beta\ . \tag{12}$$

This implies that the Hamiltonian of the (Z_1, Z_2, e) system in α, β variables is double-periodic, i.e., invariant under the transformation: $\alpha\to\alpha+2i\pi, \beta\to\beta+2\pi$. It is worth emphasizing if, in general,

$$f(\alpha)\ =\ \mathcal{F}(\cosh\alpha)\ ,\ g(\beta)\ =\ \mathcal{G}(\cos\beta)\ , \tag{13}$$

the Hamiltonian

$$\mathcal{H}^{(3)}\ =\ -\Delta^{(3)}+\frac{\mathcal{F}(\cosh\alpha)+\mathcal{G}(\cos\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\ ,$$

is double-periodic: $\alpha\to\alpha+2i\pi, \beta\to\beta+2\pi$.

Writing Θ in separable form $\Theta = A(\alpha)B(\beta)\Phi(\phi)$, we see that the spectral problem for the operator (10) (see also (9)) separates as

$$A''(\alpha)+\coth\alpha A'(\alpha)-af(\alpha)A(\alpha)+\left[-\frac{m^2}{\sinh^2\alpha}-a^2\lambda+a^2E\sinh^2\alpha\right]A(\alpha)\ =\ 0\ , \tag{14}$$

$$B''(\beta)+\cot\beta B'(\beta)-ag(\beta)B(\beta)+\left[-\frac{m^2}{\sin^2\beta}+a^2\lambda+a^2E\sin^2\beta\right]B(\beta)=0\ , \tag{15}$$

$$C''(\phi)+m^2C(\phi)=0\ . \tag{16}$$

Here, m, λ, E are the separation constants. It is worth noting that the separation equations similar to (14), (15) appear when the separation of variables for Riemannian spaces of constant curvature (with 0 potential) is studied [7], [8]. With m fixed, the equations (14), (15) define a bi-spectral problem with spectral parameters λ, E. Making a gauge transformation we can remove the second term in (14), (15)) and reduce, hence, (14), (15)) to one-variable Schrödinger equation form. Solving for λ in these equations we obtain

$$\lambda\Theta=\frac{1}{2}\left[\frac{1}{a^2}(\partial_{\alpha\alpha}+\coth\alpha\partial_\alpha)\ -\ \frac{f(\alpha)}{a}-\frac{1}{a^2}(\partial_{\beta\beta}+\cot\beta\partial_\beta)\ +\ \frac{g(\beta)}{a}\right.$$
$$\left.+\frac{1}{a^2}(\frac{1}{\sinh^2\alpha}-\frac{1}{\sin^2\beta})\partial_{\phi\phi}+(\sinh^2\alpha\ -\ \sin^2\beta)E\right]\Theta\ ,$$

where $\Theta = AB\Phi$. Using equations (11),(7) to solve for $E\,\Theta$ and substitute into this expression, we find

$$K\,\Theta=\lambda\,\Theta\ ,$$

where the operator

$$K \;=\; \frac{1}{a^2(\sinh^2\alpha+\sin^2\beta)}\left(\sin^2\beta\;(\partial_{\alpha\alpha}\;+\;\coth\alpha\partial_\alpha)\;-\;\sinh^2\alpha\;(\partial_{\beta\beta}+\cot\beta\partial_\beta)\right) \tag{17}$$

$$+\frac{\sin^2\beta-\sinh^2\alpha}{a^2\sinh^2\alpha\sin^2\beta}\,\partial_{\phi\phi}\;-\;\frac{\sin^2\beta\;f(\alpha)\;-\;\sinh^2\alpha\;g(\beta)}{a(\sinh^2\alpha\;+\;\sin^2\beta)}\;.$$

In the case of the (Z_1, Z_2, e) system this operator coincides with the integral found by Erikson and Hill [9].

It follows from the general theory of variable separation [4] that $[K, H] = 0$, so that K is a symmetry operator for the system. Moreover, the pure differential operator part of K can be expressed in terms of the enveloping algebra of the Euclidean Lie algebra $e(3)$. A basis for $e(3)$ is given by the 3 translation generators

$$P_1 = \partial_x,\;\; P_2 = \partial_y,\;\; P_3 = \partial_z,$$

and the 3 rotation generators

$$J_1 = y\partial_z - z\partial_y,\;\; J_2 = z\partial_x - x\partial_z,\;\; J_3 = x\partial_y - y\partial_x,$$

in Cartesian coordinates. In terms of these generators we find

$$-\mathcal{H} \;=\; P_1^2 + P_2^2 + P_3^2 - \frac{f(\alpha)+g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\;,$$

$$K = -\frac{1}{a^2}(J_1^2+J_2^2+J_3^2-a^2(P_1^2+P_2^2)) - \frac{f(\alpha)+g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)} + \frac{\cos^2\beta\;f(\alpha)+\cosh^2\alpha\;g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\;.$$

So far we studied the spacial problem (i) characterized by the Hamiltonian (3). Now we can look what would happen in planar motion formalism (ii), where the problem is described by the Hamiltonian (4), when we separate out azimuthal motion, coordinate φ, and perform the gauge transformation $\rho = (\sinh\alpha\;\sin\beta)^{-1/2}$. The $2D$ Laplacian becomes

$$\frac{1}{a^2(\cosh^2\alpha-\cos^2\beta)}\,[\partial_{\alpha\alpha}+\;\partial_{\beta\beta}]\;=\;\Delta^{(2)}\;.$$

For the case of the (Z_1, Z_2, e) system the Schrödinger equation is

$$\left\{\Delta^{(2)}\;+\;\frac{\left[\frac{\frac{1}{4}-m^2}{\cosh^2\alpha-1}+(Z_1+Z_2)\cosh\alpha\right]+\left[\frac{\frac{1}{4}-m^2}{1-\cos^2\beta}+(Z_2-Z_1)\cos\beta\right]}{a(\cosh^2\alpha-\cos^2\beta)}\right\}\Psi\;=\;-E\Psi\;. \tag{18}$$

(cf. (4)). The metric length is

$$ds^2 = dx^2 + dy^2 = a^2(\cosh^2\alpha-\cos^2\beta)(d\alpha^2+d\beta^2)\;.$$

The most general $2D$ Schrödinger equation which admits separation of variables in α, β coordinates has the form

$$\left(-\Delta^{(2)}+\frac{f(\alpha)+g(\beta)}{a(\cosh^2\alpha-\cos^2\beta)}\right)\Psi\;=\;E\Psi\;. \tag{19}$$

where $f(\alpha), g(\beta)$ are arbitrary functions. Assuming $\Psi = A(\alpha)B(\beta)$ we arrive at two equations

$$A''(\alpha) \;-\; af(\alpha)A(\alpha) \;+ \big[-a^2\lambda \;+\; a^2 E \sinh^2\alpha\,\big]\, A(\alpha) \;=\; 0 \;, \tag{20}$$

$$B''(\beta) \;-\; ag(\beta)B(\beta) \;+ \big[a^2\lambda \;+\; a^2 E \sin^2\beta\,\big]\, B(\beta) \;=\; 0 \;, \tag{21}$$

where λ is the separation parameter. These two equations represent a bi-spectral problem where E, λ are spectral parameters.

The operator

$$K^{(2)} \;=\; \frac{1}{a^2(\sinh^2\alpha + \sin^2\beta)}\left(\sin^2\beta\,\partial_{\alpha\alpha} \;-\; \sinh^2\alpha\,\partial_{\beta\beta}\right) \;-\; \frac{\sin^2\beta\; f(\alpha) - \sinh^2\alpha\; g(\beta)}{a(\sinh^2\alpha + \sin^2\beta)} \;, \tag{22}$$

(cf. (17)), commutes with the planar Hamiltonian $\mathcal{H}^{(2)}$,

$$[K^{(2)}, \mathcal{H}^{(2)}] \;=\; 0 \;.$$

In the case of the (Z_1, Z_2, e) system the operator $K^{(2)}$ coincides with the integral found by Erikson and Hill [9].

4 Elliptic coordinates

In terms of the more physical (dimensionless) elliptic coordinates, see e.g. [6],

$$\xi = \cosh\alpha \;,\;\; \eta = \cos\beta \;, \tag{23}$$

where $\xi \in [1, \infty)$, $\eta \in [-1, 1]$, which implies invariance with respect to translations $\alpha \to \alpha + 2i\pi, \beta \to \beta + 2\pi$, respectively, the planar Hamiltonian for the system (Z_1, Z_2, e) takes the form

$$\mathcal{H}^{(2)}\Psi \equiv (-\Delta^{(2)} - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} + V_\varphi)\Psi \;=\; E\Psi \;,$$

(see (5)), where $r_1 = a(\xi + \eta)$, $r_2 = a(\xi - \eta)$, see Fig.1, and

$$\Delta^{(2)} \;=\; \frac{\xi^2 - 1}{a^2(\xi^2 - \eta^2)}[\partial_{\xi\xi} + \frac{\xi}{\xi^2 - 1}\partial_\xi] + \frac{1 - \eta^2}{a^2(\xi^2 - \eta^2)}[\partial_{\eta\eta} - \frac{\eta}{1 - \eta^2}\partial_\eta] \;. \tag{24}$$

The general separable form for the Hamiltonian eigenvalue equation in ξ, η coordinates is

$$\mathcal{H}^{(2)}\Psi \equiv \left(- \Delta^{(2)} + \frac{\mathcal{F}(\xi) + \mathcal{G}(\eta)}{a^2(\xi^2 - \eta^2)}\right)\Psi \;=\; E\Psi \;, \tag{25}$$

(cf. (13)). This form naturally includes the modified potential $V_\varphi(r_1, r_2)$ (see (5)), which occurs in a transition from $3D$ case (ii) to $2D$ case (i), see Section 1,

$$V_\varphi(\xi, \eta) \;=\; \frac{m^2 - \frac{1}{4}}{a^2}\left(\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right)\frac{1}{\xi^2 - \eta^2} \;,$$

(cf. (5)). We emphasize that in the case of the (Z_1, Z_2, e) system the functions in the Hamiltonian (25) take the simple form

$$\mathcal{F}(\xi) = (Z_1 + Z_2)\xi \; + \frac{m^2 - \frac{1}{4}}{a(\xi^2 - 1)} \quad , \quad \mathcal{G}(\eta) = (Z_2 - Z_1)\eta \; + \frac{m^2 - \frac{1}{4}}{a(1 - \eta^2)} \; .$$

In general, the one-particle, two-center potential $V(r_1, r_2)$ in (3), which admits separation of variables in ellipsoidal coordinates, has the form

$$V(r_1, r_2) \;=\; \frac{f(r_1 + r_2) - g(r_1 - r_2)}{r_1} \;+\; \frac{f(r_1 + r_2) + g(r_1 - r_2)}{r_2} \;, \tag{26}$$

where f, g are arbitrary functions. In the case of (Z_1, Z_2, e) system f, g are constants. Interestingly, if f, g are linear functions, $f(x) = Ax, g(x) = Bx$, the potential (26) becomes,

$$V(r_1, r_2) \;=\; (A + B)\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right) .$$

Multiplying (25) by $(\xi^2 - \eta^2)$ and writing Ψ in separable form $\Psi = \mathcal{A}(\xi)\mathcal{B}(\eta)$, we obtain the separation equations

$$(\xi^2 - 1)\mathcal{A}'' + \xi\mathcal{A}' + a\mathcal{F}(\xi)\mathcal{A} + \left[-a^2\lambda + a^2\xi^2 E\right]\mathcal{A} \;=\; 0 \;, \tag{27}$$

$$(1 - \eta^2)\mathcal{B}'' - \eta\mathcal{B}' + a\mathcal{G}(\eta)\mathcal{B} + \left[a^2\lambda - a^2\eta^2 E\right]\mathcal{B} \;=\; 0 \;, \tag{28}$$

(cf. (14), (15)), where λ, E are the separation constants playing a role of spectral parameters in bi-spectral problem (27)-(28). Now the integral takes the form

$$K = \frac{1}{a^2(\xi^2 - \eta^2)}\Big((\xi^2 - 1)(1 - \eta^2)\partial_{\xi\xi} - \xi(\eta^2 - 1)\partial_\xi + (\xi^2 - 1)(\eta^2 - 1)\partial_{\eta\eta} + \eta(\xi^2 - 1)\partial_\eta\Big) \tag{29}$$

$$+\frac{(1 - \eta^2)\mathcal{F}(\xi) + (1 - \xi^2)\mathcal{G}(\eta)}{a(\xi^2 - \eta^2)} \;,$$

(cf. (17)) and the separation constant λ is the spectral parameter in the eigenvalue problem $K\Psi = \lambda\Psi$.

5 Solvability

The goal of the section is to describe one-particle, two-center potentials for which exact solutions can be found. In order to proceed let us take the equation (27) in the form of an eigenvalue problem

$$h_\xi\mathcal{A} \equiv \Big(-(\xi^2 - 1)\partial_\xi^2 - \xi\partial_\xi - a\mathcal{F}(\xi) - a^2E\xi^2\Big)\mathcal{A} \;=\; -a^2\lambda\mathcal{A} \;. \tag{30}$$

Written in terms of the variable α where $\xi = \cosh\alpha$, this becomes a 1D Schrödinger equation for the Hamiltonian

$$\mathcal{H}_\xi(\alpha) \;=\; -\partial_\alpha^2 + V_\xi(\xi = \cosh\alpha) \;,$$

with potential

$$V_\xi \ = \ -a\mathcal{F}(\xi) - a^2 E\xi^2 = \ -a\mathcal{F}(\cosh\alpha) - a^2 E\cosh^2\alpha \ , \tag{31}$$

and we arrive at the eigenvalue problem for the Schrödinger operator with a hyperbolic potential. (In most of the following discussion we restrict our attention to a single energy eigenspace so we can rescale matters so that $E = 0$.) Among hyperbolic potentials there is the exactly-solvable hyperbolic modified Pöschl-Teller potential (in other words, one-soliton potential for $A_s = 0$)

$$V_{PT}^{(h)} \ = \ -\frac{A_c}{\cosh^2\alpha} - \frac{A_s}{\sinh^2\alpha} \ , \tag{32}$$

with a finite number of bound states, where all of them can be found exactly(algebraically), see e.g. [6], and two three-parametric families [1] of quasi-exactly-solvable potentials:

$$V_1^{(h,qes)} \ = \ -\frac{A_c}{\cosh^2\alpha} - \frac{A_s}{\sinh^2\alpha} + A_1\cosh^2\alpha + A_2\cosh^4\alpha \ , \tag{33}$$

and

$$V_2^{(h,qes)} \ = \ -\frac{A_c}{\cosh^2\alpha} - \frac{A_s}{\sinh^2\alpha} + \frac{A_1}{\cosh^4\alpha} + \frac{A_2}{\cosh^6\alpha} \ , \tag{34}$$

where a finite number of eigenstates can be found algebraically [10]. (These potentials are all related to superintegrable systems, see, e.g., [15], eqn. (72).) Known eigenfunctions for (32) have a form

$$\cosh^\nu(\frac{\alpha}{2}) \ \sinh^\mu(\frac{\alpha}{2}) \ P_k(\cosh\alpha) \ ,$$

where ν, μ are known constants defined by $A_{c,s}$ and P_k is a polynomial of degree k, which can be found by algebraic means. Known (algebraic) eigenfunctions for (33) have a form

$$\cosh^\nu(\frac{\alpha}{2}) \ \sinh^\mu(\frac{\alpha}{2}) \ e^{-b\cosh\alpha} \ P_k(\cosh\alpha) \ ,$$

where $b^2 = A_2$. Similar form (up to a factor) appears for the potential (34).

The same analysis as for (27) can be performed for the equation (28) rewriting it like

$$h_\eta\mathcal{B} \equiv \bigg(-(1-\eta^2)\partial_\eta^2 + \eta\partial_\eta - a\mathcal{G}(\eta) + a^2 E\eta^2 \bigg)\mathcal{B} \ = \ a^2\lambda\mathcal{B} \ .$$

Indeed, we arrive at the eigenvalue problem for the Hamiltonian

$$\mathcal{H}_\eta(\beta) \ = \ -\partial_\beta^2 + V_\eta(\eta = \cos\beta) \ ,$$

with trigonometric potential,

$$V_\eta \ = \ -a\mathcal{G}(\eta) + a^2 E\eta^2 = \ -a\mathcal{G}(\cos\beta) + a^2 E\cos^2\beta \ . \tag{35}$$

Among trigonometric potentials there is the exactly-solvable modified trigonometric Pöschl-Teller potential (see e.g. [11])

$$V_{PT}^{(t)} \ = \ -\frac{B_c}{\cos^2\beta} - \frac{B_s}{\sin^2\beta} \ , \tag{36}$$

[1] The fourth parameter should take discrete values

and also two three-parametric families [2] of quasi-exactly-solvable potentials

$$V_1^{(t,qes)} \;=\; -\frac{B_c}{\cos^2\beta} - \frac{B_s}{\sin^2\beta} + B_1\cos^2\beta + B_2\cos^4\beta \ , \tag{37}$$

and

$$V_2^{(t,qes)} \;=\; -\frac{B_c}{\cos^2\beta} - \frac{B_s}{\sin^2\beta} + \frac{B_2}{\cos^4\beta} + \frac{B_3}{\cos^6\beta} \ , \tag{38}$$

where a finite number of eigenstates can be found algebraically. Known eigenfunctions for (36) have a form

$$|\cos(\frac{\beta}{2})|^{\nu}\ |\sin(\frac{\beta}{2})|^{\mu} P_k(\cos\beta)$$

where ν, μ are known constants defined by $B_{c,s}$ and P_k is a polynomial of degree k, which can be found by algebraic means. Known (algebraic) eigenfunctions for (37) have a form

$$|\cos(\frac{\beta}{2})|^{\nu}\ |\sin(\frac{\beta}{2})|^{\mu}\ e^{-b\cos\beta}\ P_k(\cos\beta)$$

where $b^2 = B_2$. Similar form (up to a factor) appears for the potential (38).

Finding $\mathcal{F}, \mathcal{G}$ from (32), (36) we construct the two-dimensional, four-parametric, exactly solvable problem with potential

$$V_{PT}(\xi,\eta) \;=\; \frac{1}{\xi^2-\eta^2}\left(\frac{A_c}{\xi^2} + \frac{A_s}{1-\xi^2} + \frac{B_c}{\eta^2} - \frac{B_s}{1-\eta^2}\right) , \tag{39}$$

see (25). If $B_c \neq 0$ the potential is singular. In (r_1, r_2) variables the potential (39) looks as follows

$$V_{PT}(r_1,r_2) \;=\; \frac{1}{r_1 r_2}\left(\frac{a_c}{(r_1+r_2)^2} + \frac{a_s}{(r_1+r_2)^2-R^2} + \frac{b_c}{(r_1-r_2)^2} - \frac{b_s}{(r_1-r_2)^2-R^2}\right) . \tag{40}$$

This model is integrable with one second-order integral (29) (with appropriate $\mathcal{F}, \mathcal{G}$) for any values of parameters $a_{c,s}, b_{c,s}$. As we will point out, for 2 linear conditions on the 4 parameters it is conformally second-order superintegrable. The question of the existence of the second, higher-order-than-two integral (thus superintegrability) for certain values of parameters is open and might be a subject of separate investigation.

This construction is similar to one which has led to TTW model (see [12]), when the separation in polar coordinates was inverted by adding 2D radial harmonic oscillator to modified trigonometric Pöschl-Teller potential in angular coordinate (36). Taking in (25) a superposition of (quasi)-exactly-solvable, hyperbolic-trigonometric potentials we will obtain (quasi)-exactly-solvable, integrable, one-particle, two-center problems.

There is single 2D Euclidean space nondegenerate superintegrable system that permits separation in elliptic coordinates, the Smorodinsky-Winternitz system (or caged oscillator) [13, 14]:

$$-\Delta^{(2)}\Psi + \left(A_1(x^2+y^2) + \frac{A_2}{x^2} + \frac{A_3}{y^2}\right)\Psi \;=\; E\Psi \ . \tag{41}$$

or, in other words, TTW model at $k=1$ [12]. In this case

$$\mathcal{F}(\xi) = a^4 A_1 \xi^2(\xi^2-1) + \frac{A_2}{\xi^2-1} - \frac{A_3}{\xi^2},\quad \mathcal{G}(\eta) = -a^4 A_1 \eta^2(\eta^2-1) + \frac{A_2}{1-\eta^2} + \frac{A_3}{\eta^2} \ . \tag{42}$$

[2] The fourth parameter should take discrete values

It has two second order integrals and the expressions are valid for all E simultaneously. If we restrict E to a fixed value, say $E = 0$, then we can consider conformal symmetries of the Schrödinger operator. The system (41) becomes now trivially second-order conformally superintegrable with 3 generators. Note that for $A_1 = 0$ the model (41) degenerates to (39). However, this is not *just* a restriction because the restricted system is now conformally second-order superintegrable with 6 linearly independent generators, see [16], eqn. (2), so the symmetry algebra is much larger. Hence, the model with (39) for a certain particular values of of parameters is conformally superintegrable. Another way that (41) leads to (39), but with different values of the parameters, is that it is conformally equivalent to a second-order superintegrable system on the 2-sheet hyperboloid, see [17], eqns. (4)-(9). Again the 1D potentials are of Pöschl-Teller type.

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