Wilson polynomials/functions and intertwining operators for the generic quantum superintegrable system on the 2-sphere

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Abstract. It has been known since 2007 that the Wilson and Racah polynomials can be characterized as basis functions for irreducible representations of the quadratic symmetry algebra of the quantum superintegrable system on the 2-sphere, $H\Psi = E\Psi$, with generic 3-parameter potential. Clearly, the polynomials are expansion coefficients for one eigenbasis of a symmetry operator L_1 of H in terms of an eigenbasis of another symmetry operator L_2, but the exact relationship appears not to have been made explicit. We work out the details of the expansion to show, explicitly, how the polynomials arise and how the principal properties of these functions: the measure, 3-term recurrence relation, 2nd order difference equation, duality of these relations, permutation symmetry, intertwining operators and an alternate derivation of Wilson functions – follow from the symmetry of this quantum system. There is active interest in the relation between multivariable Wilson polynomials and the quantum superintegrable system on the n-sphere with generic potential, and these results should aid in the generalization. Contracting function space realizations of irreducible representations of this quadratic algebra to the other superintegrable systems one can obtain the full Askey scheme of orthogonal hypergeometric polynomials. All of these contractions of superintegrable systems with potential are uniquely induced by Wigner Lie algebra contractions of $so(3, C)$ and $e(2, C)$. All of the polynomials produced are interpretable as quantum expansion coefficients. It is important to extend this process to higher dimensions.

1. Introduction

We define a quantum superintegrable system as an integrable Hamiltonian system on an n-dimensional pseudo-Riemannian manifold with potential: $H = \Delta_n + V$ that admits $2n - 1$ algebraically independent partial differential operators commuting with H, the maximum possible,[1]. Thus $[H, L_j] = 0, \quad n = 1, 2, \cdots, 2n-1$ where Δ_n is the Laplace-Beltrami operator on the manifold and we choose the generators L_j such that the sum of their orders is a small as possible. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically. Typically, the basis symmetries L_j generate an algebra under commutation, not usually a Lie algebra, that closes at finite order. It is this algebra that is responsible for the solvability of the quantum system.

The generic superintegrable system on the 2-sphere, system $S9$ in our listing [2], is

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \qquad a_j = \frac{1}{4} - k_j^2,$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices. Here, $s_1^2 + s_2^2 + s_3^2 = 1$. The basis symmetries can be chosen as

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2},\ L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2},\ L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

where J_1, J_2, J_3 are rotation generators: $J_3 = s_2\partial_{s_1} - s_1\partial_{s_2}, \cdots$. Note the discrete symmetry of $H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$ with respect to permutations of the indices $1, 2, 3$. With the commutator $R = [L_1, L_2]$, the algebra generated by these symmetries has the structure

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k), \tag{1}$$

where $i, j, k = 1, 2, 3$, pairwise distinct. Here, $\{A, B\} = AB + BA$, and $\{A, B, C\}$ is the symmetrizer of 3 operators. R^2 is contained in the algebra and expressed by

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \tag{2}$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+\frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.$$

This algebra is exactly $QR(3)$, the structure algebra of the Racah and Wilson polynomials, [3, 4, 5, 6, 7]. The significance of these special functions is that they are the expansion coefficients of a basis of eigenfunctions of one of the spherical coordinate generators L_i in terms of an eigenbasis of L_j, $i \neq j$. Long before, Dunkl in the remarkable paper [8], (see also [9, 10] and references contained therein) had computed these coefficients as expansions of polynomial bases (not eigenbases) and shown them to be Racah/Wilson polynomials. However, to our knowledge no one has worked out the details of this relationship between the functions and the expansion coefficients to understand how important properties of Wilson polynomials can be interpreted from a quantum mechanical viewpoint. Here we show the critical role of intertwining operators for the quantum system in determining parameter changing recurrences for Wilson/Racah polynomials. In particular, the ${}_4F_3$ expressions for Wilson polynomials/functions follow from intertwining operators alone.

We recall the definition of ${}_4F_3$ hypergeometric functions:

$${}_4F_3\left(\begin{array}{cccc} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3 & \end{array}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k k!}\, x^k$$

where $(a)_0 = 1$, $\quad (a)_k = a(a+1)(a+2)\cdots(a+k-1)$ if $k \geq 1$. If $a_1 = -n$ for n a nonnegative integer then the sum is finite with $n+1$ terms. The Wilson polynomials of order n in t^2 are

$$\Phi_n(\alpha, \beta, \gamma, \delta; t) = {}_4F_3\left(\begin{array}{cccc} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha - t, & \alpha + t \\ \alpha+\beta, & \alpha+\gamma, & & \alpha+\delta \end{array}; 1\right).$$

If $\alpha + \beta = -m$, m a nonnegative integer we have the finite set $\Phi_0, \cdots, \Phi_m$ of Racah polynomials.

2. Structure algebra and interbasis expansion coefficients

The basis simultaneous eigenfunctions of L_1 and H: take the form

$$\Psi_{N-n,n} = (s_1^2 + s_2^2)^{\frac{1}{2}(2n+k_1+k_2+1)}(1 - s_1^2 - s_2^2)^{\frac{1}{2}(k_3+\frac{1}{2})}\left(\frac{s_2^2}{s_1^2 + s_2^2}\right)^{\frac{1}{2}(k_2+\frac{1}{2})}\left(\frac{s_1^2}{s_1^2 + s_2^2}\right)^{\frac{1}{2}(k_1+\frac{1}{2})}$$

$$\times P_n^{(k_2,k_1)}(\frac{s_1^2-s_2^2}{s_1^2+s_2^2})P_{N-n}^{(2n+k_1+k_2+1,k_3)}(1-2s_1^2-2s_2^2),$$

$$L_1\Psi_{N-n,n}=\left[k_1^2+k_2^2-\frac{1}{2}-(2n+1+k_1+k_2)^2\right]\Psi_{N-n,n},\quad n=0,1,\cdots,N,$$

$$H\Psi_{N-n,n}=E_N\Psi_{N-n,n},\quad E_N=-(2N+k_1+k_2+k_3+2)^2+\frac{1}{4},\quad N=0,1,\cdots,$$

separable in spherical coordinates, $x=\cos(2\varphi), y=\cos(2\theta)$ where $s_1=\sin\theta\cos\varphi, s_2=\sin\theta\sin\varphi, s_3=\cos\varphi$, orthogonal with respect to area measure on the 1st octant of the 2-sphere. The dimension of eigenspace E_N is $N+1$.

$$P_n^{(\alpha,\beta)}(y)=\left(\begin{array}{c} n+\alpha \\ n\end{array}\right){}_2F_1\left(\begin{array}{cc} -n & \alpha+\beta+n+1 \\ \alpha+1 & \end{array};\frac{1-y}{2}\right),\quad \text{Jacobi polynomials}$$

The eigenfunctions of L_2, H are obtained by permutations $1\leftrightarrow 3$, $n\leftrightarrow q$, of the L_1 basis:

$$L_2\Lambda_{N-q,q}=\left[k_3^2+k_2^2-\frac{1}{2}-(2q+1+k_3+k_2)^2\right]\Lambda_{N-q,q},\quad q=0,1,\cdots,N,$$

$$H\Lambda_{N-q,q}=E_N\Lambda_{N-q,q},\quad E_N=-(2N+k_1+k_2+k_3+2)^2+\frac{1}{4},\quad N=0,1,\cdots.$$

They are separable in a different set of spherical coordinates X,Y expressible in terms of x,y by $X=\frac{1+x+3y-xy}{xy-x+y+3}, Y=\frac{x-y-1-xy}{2}$, and orthogonal with respect to area measure on the 1st octant of the 2-sphere. Due to the S_3 permutation symmetry there is also a basis of eigenfunctions of L_3 and H, but we will not consider it here. Using the method described in [11] we can derive the structure algebra of $S9$, just from the 1st order Gaussian differential recurrences for

$${}_2F_1\left(\begin{array}{cc} a, & b \\ c & \end{array};z\right)\to{}_2F_1\left(\begin{array}{cc} a\pm1, & b \\ c & \end{array};z\right),\quad {}_2F_1\left(\begin{array}{cc} a, & b \\ c & \end{array};z\right)\to{}_2F_1\left(\begin{array}{cc} a\pm1, & b\mp1 \\ c\mp1 & \end{array};z\right)$$

and a limiting process. One consequence from Section 5.1 of that paper, is:

$$L_2\Psi_{m,n}=A_n\Psi_{m-1,n+1}+B_n\Psi_{m,n}+C_n\Psi_{m+1,n-1}=$$

$$-\frac{4(N+k_3-n)(N+n+k_1+k_2+k_3+2)(n+1)(n+k_1+k_2+1)}{(2n+k_1+k_2+2)(2n+k_1+k_2+1)}\Psi_{m-1,n+1}$$

$$[-\frac{(k_1^2-k_2^2)\left(k_3^2-(2N+k_1+k_2+k_3+2)^2\right)}{2(2n+k_1+k_2+2)(2n+k_1+k_2)}+\frac{1}{2}(2n+k_1+k_2+1)^2+\frac{1}{4}-k_1^2-\frac{1}{2}(2N+2+k_1+k_2+k_3)^2$$

$$+\frac{1}{2}k_3^2]\Psi_{m,n}-\frac{4(N-n+1)(N+n+k_1+k_2+1)(n+k_1)(n+k_2)}{(2n+k_1+k_2)(2n+k_1+k_2+1)}\Psi_{m+1,n-1},$$

The action of L_1 on the L_2 eigenbasis follows immediately from permutation symmetry. Now we expand the L_2 eigenbasis in terms of the L_1 eigenbasis:

$$\Lambda_{N-q,q}^{(k_1,k_2,k_3)}=\sum_{n=0}^{N}R_q^n(k_1,k_2,k_3)\,\Psi_{N-n,n}^{(k_1,k_2,k_3)},\quad q=0,\cdots.N$$

Using the self-adjoint properties of L_1,L_2 we can find recurrences to compute the norm:

$$||\Psi_{N-n,n}||^2=\frac{1}{4n!\Gamma(N-n+1)}\frac{\Gamma(n+k_1+1)\Gamma(n+k_2+1)\Gamma(N-n+k_3+1)\Gamma(N+n+k_1+k_2+2)}{(2N+k_1+k_2+k_3+2)(2n+k_1+k_2+1)\Gamma(n+k_1+k_2+1)\Gamma(N+n+k_1+k_2+k_3+2)},$$

To better exploit the symmetry of our system we rescale the basis:

$$\Psi'_{N-n,n}(k_1,k_2,k_3)=\frac{(-1)^n n!\Gamma(N-n+1)}{\Gamma(N-n+k_3+1)\Gamma(n+k_2+1)}\Psi_{N-n,n}(k_1,k_2,k_3),$$

The norms of $\Lambda_{N-q,q}$ follow from the permutation $1 \leftrightarrow 3,\ n \leftrightarrow q$.

The two sets of $N+1$ basis vectors $\{\frac{\Psi'_{m,n}}{||\Psi'_{m,n}||}\}$, $\{\frac{\Lambda'_{p,q}}{||\Lambda'_{p,q}||}\}$ are each orthonormal, implying that the $(N+1)\times(N+1)$ matrix $\left(\frac{||\Psi'_{m,n}||R'^{n}_{q}}{||\Lambda'_{p,q}||}\right)$, $\quad 0\le n,q\le N$, is orthogonal. We have identities

$$\sum_{\ell=0}^{N}\frac{R'^{n_1}_{\ell}R'^{n_2}_{\ell}}{||\Lambda'_{N-\ell,\ell}||^2}=\frac{\delta_{n_1,n_2}}{||\Psi'_{m_1,n_1}||^2},\quad \sum_{\ell=0}^{N}\frac{R'^{\ell}_{q_1}R'^{\ell}_{q_2}}{||\Lambda_{p_1,q_1}||^2}=\frac{\delta_{q_1,q_2}}{||\Psi'_{N-\ell,\ell}||^2}.$$

By permutation symmetry: $\Lambda'_{p,q}(k_1,k_2,k_3)=\Psi'_{p,q}(k_3,k_2,k_1)$. We set

$$R'^{n}_{q}(k_1,k_2,k_3)_N\cdot||\Psi'_{N-n,n}(k_1,k_2,k_3)||^2\equiv\ \Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n&N&q\end{array}\right).$$

Note that Ξ' satisfies the orthogonality relations

$$\sum_{q=0}^{N}\frac{\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n_1&N&q\end{array}\right)\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n_2&N&q\end{array}\right)}{||\Lambda_{N-q,q}(k_1,k_2,k_3)||^2}=||\Psi'_{N-n_1,n_1}(k_1,k_2,k_3)||^2\delta_{n_1,n_2},$$

$$\sum_{\ell=0}^{N}\frac{\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ \ell&N&q_1\end{array}\right)\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ \ell&N&q_2\end{array}\right)}{||\Psi'_{N-\ell,\ell}(k_1,k_2,k_3)||^2}=||\Lambda'_{N-q_1,q_1}(k_1,k_2,k_3)||^2\delta_{q_1,q_2}.$$

Applying L_2 to both sides of the expansion formula we have

$$L_2\Lambda'_{q,p}=\sum_{n=0}^{N}R'^{n}_{q}\left(A'_n\Psi'_{m-1,n+1}B'_n\Psi'_{m,n}+C'_n\Psi'_{m+1,n-1}\right)=\left[-(2q+k_2+k_3+1)^2-\frac{1}{2}+k_2^2+k_3^2\right]\sum_{n=0}^{N}R'^{n}_{q}\Psi'_{m,n}.$$

Thus, equating coefficients of $\Psi'_{m,n}$, we find a 3-term recurrence formula for R'^{n}_{q}, hence for Ξ'. If we make the identifications

$$k_1=\delta+\beta-1,\ k_2=\alpha+\gamma-1,\ k_3=\alpha-\gamma,\ N=-\alpha-\beta, t=q+\frac{k_2+k_3+1}{2},$$

this formula for Ξ' agrees exactly with the 3-term recurrence formula for the Racah polynomials, so that $\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n&N&q\end{array}\right)$ is proportional to a Racah polynomial in t^2 of order n. Moreover,

$$||\Lambda'_{N-q,q}(k_1,k_2,k_3)||^2\sim\frac{\Gamma(t-\alpha+1)\Gamma(t-\beta+1)\Gamma(t-\gamma+1)\Gamma(t-\delta+1)\Gamma(t)}{\Gamma(t+\alpha)\Gamma(t+\beta)\Gamma(t+\gamma)\Gamma(t+\delta)\Gamma(t+1)},$$

so the measure defined by $||\Lambda'_{N-q,q}(k_1,k_2,k_3)||^{-2}$ is a scalar times a function of t that is symmetric with respect to all permutations of $\alpha,\beta,\gamma,\delta$. This implies that the family of orthogonal polynomials determined by this measure must admit this symmetry up to a multiplicative factor. Further, the left hand side of the orthogonality relation is proportional to

$$\sum_{q=0}^{N}\frac{(2\alpha)_q(\alpha+1)_q(\alpha+\beta)_q(\alpha+\gamma)_q(\alpha+\delta)_q}{(\alpha)_q(\alpha-\beta+1)_q(\alpha-\gamma+1)_q(\alpha-\delta+1)_q q!}\,\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n_1&N&q\end{array}\right)\Xi'\left(\begin{array}{ccc}k_1&k_2&k_3\\ n_2&N&q\end{array}\right),$$

precisely the measure for orthogonality of the Racah polynomials $\Phi_n^{(\alpha,\beta\gamma,\delta)}(t^2)$ where $t=q+\alpha$.

Duality: By making the transpositions $k_1\leftrightarrow k_3,\ n\leftrightarrow q$ we obtain the result of applying L_1 to the expansion of $\Psi'_{N-n,n}$ in an L_2 eigenbasis. This gives a 3-term recurrence relation for Ξ, defining a family of orthogonal polynomials $p'_q(n)$ in the variable n. It is a 2nd order difference equation in q, hence t, for the Racah polynomials as eigenfunctions. This action induces a model of an irreducible representation of the structure algebra of $S9$ in which the basis functions are Racah polynomials in t^2 and the symmetry operators map to difference operators.

3. Intertwining operators

Let $\mathcal{W}_{k_1,k_2,k_3}$ be the space of functions on the first octant of the 2-sphere and with Hamiltonian $H^{(k_1,k_2,k_3)}$, symmetry operators $L_j^{(k_1,k_2,k_3)}$ and inner product $<\Phi,\Psi>_{k_1,k_2,k_3}$. Let $\mathcal{W}_{k_1',k_2',k_3'}$ be another such space. An *intertwining operator* is a mapping $X^{(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1',k_2',k_3'}$ such that

$$X^{(k_1,k_2,k_3)}H^{(k_1,k_2,k_3)}=H^{(k_1',k_2',k_3')}X^{(k_1,k_2,k_3)}.$$

Note that $X^{(k_1,k_2,k_3)}$ maps eigenfunctions of $H^{(k_1,k_2,k_3)}$ to eigenfunctions of $H^{(k_1',k_2',k_3')}$ and its adjoint $X^{*(k_1,k_2,k_3)}$ reverses the action.

Such energy shifting transformations are induced by the basic differential recurrence relations obeyed by Gaussian hypergeometric functions. For example the standard recurrence

$$\left[z(1-z)\frac{d}{dz}-(b+a-1)z+c-1\right]{}_2F_1\left(\begin{array}{cc} a & b\\ c & \end{array};z\right)=(c-1){}_2F_1\left(\begin{array}{cc} a-1 & b-1\\ c-1 & \end{array};z\right),$$

induces a 1st order differential operator $T^{(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1-1,k_2-1,k_3}$ such that, in terms of the x,y variables,

$$T^{(k_1,k_2,k_3)}=\sqrt{1-x^2}\,\partial_x-\frac{1}{2}(k_2-\frac{1}{2})\sqrt{\frac{1+x}{1-x}}+\frac{1}{2}(k_1-\frac{1}{2})\sqrt{\frac{1-x}{1+x}},\ T^{(k_1,k_2,k_3)}\Psi_{m,n}^{(k_1,k_2,k_3)}=-(n+1)\Psi_{m,n+1}^{(k_1-1,k_2-1,k_3)}.$$

The adjoint is induced by $\frac{d}{dz}{}_2F_1\left(\begin{array}{cc} a & b\\ c & \end{array};z\right)=\frac{ab}{c}{}_2F_1\left(\begin{array}{cc} a+1 & b+1\\ c+1 & \end{array};z\right)$,

$$T^{*(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1+1,k_2+1,k_3},\ \Psi_{m,n}^{(k_1,k_2,k_3)}\to-(k_1+k_2+n+1)\Psi_{m,n-1}^{(k_1+1,k_2+1,k_3)},$$

$$T^{*(k_1,k_2,k_3)}=-\sqrt{1-x^2}\,\partial_x-\frac{1}{2}(k_2+\frac{1}{2})\sqrt{\frac{1+x}{1-x}}+\frac{1}{2}(k_1+\frac{1}{2})\sqrt{\frac{1-x}{1+x}},$$

Note that these intertwining operators are defined independent of basis. The action of T and T^* on the Λ-basis can again be computed from 1st order relations obeyed by Gaussian hypergeometric functions. To find these we transform to X,Y coordinates and again make use of first order hypergeometric differential recurrences. We obtain

$$T^{(k_1,k_2,k_3)}\Lambda'^{(k_1,k_2,k_3)}_{p,q}=-\frac{1}{2q+k_2+k_3+1}\Lambda'^{(k_1-1,k_2-1,k_3)}_{p+1,q}+\frac{1}{2q+k_2+k_3+1}\Lambda'^{(k_1-1,k_2-1,k_3)}_{p,q+1},$$

$$\mathcal{T}^{(\alpha-\frac12,\beta-\frac12,\gamma-\frac12,\delta-\frac12)}\,\Xi'\left(\begin{array}{ccc} k_1-1 & k_2-1 & k_3\\ n & N+1 & q\end{array}\right)=n(k_1+k_2+n-1)\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3\\ n-1 & N & q\end{array}\right),$$

$$\mathcal{T}^{(\alpha-\frac12,\beta-\frac12,\gamma-\frac12,\delta-\frac12)}f(t)=\frac{1}{2t}\left[(f(t+\frac12)-f(t-\frac12)\right],\quad t=q+\frac{k_2+k_3+1}{2}.$$

$$\mathcal{T}^{*(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}\,\Xi'\left(\begin{array}{ccc} k_1+1 & k_2+1 & k_3\\ n & N-1 & q\end{array}\right)=\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3\\ n+1 & N & q\end{array}\right),$$

$$\mathcal{T}^{*(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}f(t)=\frac{1}{2t}\left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)f(t+\frac12)-(\alpha-t)(\beta-t)(\gamma-t)(\delta-t)f(t-\frac12)\right].$$

Note that (with $t=q+\frac{k_2+k_3+1}{2}$),

$$\mathcal{T}^{*(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}\,\mathcal{T}^{(\alpha,\beta,\gamma,\delta)}\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3\\ n & N & q\end{array}\right)=n(k_1+k_2+n+1)\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3\\ n & N & q\end{array}\right)$$

a 2nd order difference equation for Ξ' as a polynomial in t^2. We will solve this equation.

3.1. Calculation of Racah polynomials

Note that $\Xi'_n(t)$ can be written as $\Xi' = G(n, N, k_1, k_2, k_3)\,\Phi_n(t)$ where $\Phi_n(t)$ is a polynomial in t^2 such that $\Phi_n(0) = 1$. Thus we can write $\Phi_n(t) = \sum_{k=0}^{n} w_k P_k(\alpha, t)$, where $P_k(\alpha, t) = (\alpha + t)_k(\alpha - t)_k$, $w_0 = 1$. Applying τ to $P_k(\alpha, t)$, we get,

$$\tau P_k(\alpha, t) = -kP_{k-1}(\alpha + \frac{1}{2}, t) \tag{3}$$

Applying τ^* to the shifted basis, we get

$$\tau^* P_k(\alpha + \frac{1}{2}, t) = -(\alpha+\beta+\gamma+\delta+k)P_{k+1}(\alpha, t) + (\alpha+\beta+k)(\alpha+\gamma+k)(\alpha+\delta+k)P_k(\alpha, t)_k. \tag{4}$$

Thus,

$$\tau^*\tau P_k(\alpha, t) = k(\alpha+\beta+\gamma+\delta+k-1)P_k(\alpha, t) - k(\alpha+\beta+k-1)(\alpha+\gamma+k-1)(\alpha+\delta+k-1)P_{k-1}(\alpha, t),$$

a 2-term recurrence relation, which implies $w_{k+1} = \frac{(-n+k)(n+\alpha+\beta+\gamma+\delta+k-1)}{(k+1)(\alpha+\beta+k)(\alpha+\gamma+k)(\alpha+\delta+k)} w_k$, $w_0 = 1$. It is easy to solve this recurrence to obtain

$$w_k = \frac{(-n)_k(n+\alpha+\beta+\gamma+\delta-1)_k}{k!(\alpha+\beta)_k(\alpha+\gamma)_k(\alpha+\delta)_k},\ k = 0, 1, \cdots. \tag{5}$$

Hence, unique up to a scalar multiple,

$$\Phi_n(t) = {}_4F_3\left(\begin{matrix} -n & n+\alpha+\beta+\gamma+\delta-1 & \alpha+t & \alpha-t \\ \alpha+\beta & \alpha+\gamma & \alpha+\delta & \end{matrix} ; 1\right). \tag{6}$$

3.2. More permutation symmetry:

Since the measure for the Racah polynomials is invariant under all permutations of $\alpha, \beta, \gamma, \delta$, and the Racah polynomials can be obtained from the measure by the Gram-Schmidt process. Each such permutation must take a Racah polynomial to a scalar multiple of itself. We find $(\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n\, \Phi_n^{(\alpha,\beta,\gamma,\delta)}(t)$ is invariant under all permutations.

3.3. More intertwining operators

The recurrence

$$\left(z\frac{d}{dz} + c - 1\right) {}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix} ; z\right) = (c-1)\,{}_2F_1\left(\begin{matrix} a & b \\ c-1 & \end{matrix} ; z\right)$$

leads to an intertwining operator $U^{(k_1,k_2,k_3)}_{(-,+,-,+)} :\ \mathcal{W}_{k_1,k_2,k_3} \longrightarrow \mathcal{W}_{k_1+1,k_2-1,k_3}$,

$$\mu^{(\beta,\delta)}\, \Phi_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma-\frac{1}{2},\delta+\frac{1}{2})}(t) = \frac{(n+\beta+\delta)(n+\alpha+\gamma-1)}{(\alpha+\gamma-1)}\, \Phi_n^{(\alpha,\beta,\gamma,\delta)}(t)$$

Its action on the Λ' basis induces the recurrence

$$\mu^{(\beta,\delta)} f(t) = \frac{1}{2t}\left[(\beta+t)(\delta+t)f(t+\frac{1}{2}) - (\beta-t)(\delta-t)f(t-\frac{1}{2})\right],$$

$$\mu^{(\beta,\delta)}\, \Xi'\left(\begin{matrix} k_1+1 & k_2-1 & k_3 \\ n & N & q \end{matrix}\right) = (n+k_1+1)(n+k_2)\, \Xi'\left(\begin{matrix} k_1 & k_2 & k_3 \\ n & N & q \end{matrix}\right).$$

The permutation invariance of the Racah polynomials leads to a family of recurrences in the μ such that any pair of $\alpha, \beta, \gamma, \delta$ can be raised by $\frac{1}{2}$ and the other pair lowered by $\frac{1}{2}$. These also follow from intertwining operators induced by Gaussian hypergeometric differential recurrences. In particular, the hypergeometric recurrence

$$\left(z\frac{d}{dz}+a\right){}_2F_1\left(\begin{array}{cc} a & b \\ & c \end{array};z\right)=a\ {}_2F_1\left(\begin{array}{cc} a+1 & b \\ c & \end{array};z\right)$$

induces the operator

$$U^{(k_1,k_2,k_3)}_{(+,+,-,-)}=\sqrt{\frac{1+y}{2}}\left[-(1-y)\partial_y-N-\frac{k_1}{2}-\frac{k_2}{2}-\frac{1}{2}+\frac{1}{2}(k_3+\frac{1}{2})\left(\frac{1-y}{1+y}\right)\right], \tag{7}$$

$$U^{(k_1,k_2,k_3)}_{(+,+,-,-)}\Psi^{(k_1,k_2,k_3)}_{m,n}=-(n+N+k_1+k_2+1)\Psi^{(k_1,k_2,k_3+1)}_{m-1,n},\quad m\geq 1, n\geq 0. \tag{8}$$

The action on the L_2 eigenbasis is

$$\mu^{(\alpha,\beta)}\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3+1 \\ n & N-1 & q \end{array}\right)=-\frac{2N+k_1+k_2+k_3+2}{2N+k_1+k_2+k_3+1}\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array}\right),$$

$$\mu^{(\alpha,\beta)}\Phi_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma-\frac{1}{2},\delta-\frac{1}{2})}=(\alpha+\beta)\Phi_n^{(\alpha,\beta,\gamma,\delta)},\quad \alpha+\beta=-N. \tag{9}$$

The recurrence $\left(z\frac{d}{dz}+b\right){}_2F_1\left(\begin{array}{cc} a & b \\ c & \end{array};z\right)=b\ {}_2F_1\left(\begin{array}{cc} a & b+1 \\ c & \end{array};z\right)$, induces the operator

$$U^{(k_1,k_2,k_3)}_{(+,-,-,+)}=\sqrt{\frac{1+y}{2}}\left[(y-1)\partial_y+N+\frac{k_1}{2}+\frac{k_2}{2}+k_3+\frac{3}{2}+\frac{1}{2}(k_3+\frac{1}{2})\left(\frac{y-1}{y+1}\right)\right], \tag{10}$$

$$U^{(k_1,k_2,k_3)}_{(+,-,-,+)}\Psi^{(k_1,k_2,k_3)}_{m,n}=(n+N+k_1+k_2+k_3+2)\Psi^{(k_1,k_2,k_3+1)}_{m,n},\quad m\geq 0, n\geq 0. \tag{11}$$

In terms of Ξ' and Φ_n the action is

$$\mu^{(\alpha,\delta)}\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3+1 \\ n & N & q \end{array}\right)=\frac{2N+k_1+k_2+k_3+2}{2N+k_1+k_2+k_3+3}\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array}\right),$$

$$\mu^{(\alpha,\delta)}\Phi_n^{(\alpha+\frac{1}{2},\beta-\frac{1}{2},\gamma-\frac{1}{2},\delta+\frac{1}{2})}=(\alpha+\delta)\Phi_n^{(\alpha,\beta,\gamma,\delta)},\qquad \alpha+\delta=N+k_1+k_2+k_3+2.$$

Note that the operators (7,10) are not basis independent, since they depend on N. However, the intertwining operator $V^{(k_1,k_2,k_3)}:\ \mathcal{W}_{k_1,k_2,k_3}\longrightarrow\mathcal{W}_{k_1,k_2,k_3+1}$,

$$V^{(k_1,k_2,k_3)}=U^{(k_1,k_2,k_3)}_{(+,-,-,+)}+U^{(k_1,k_2,k_3)}_{(+,+,-,-)}=\sqrt{\frac{1+y}{2}}\left[2(y-1)\partial_y+k_3+1\right]. \tag{12}$$

is basis independent, so defined intrinsically, as is its adjoint.

3.4. The expansion coefficients

Solving all of these recurrences for Ξ', we find

$$R'^{n}_{q}(k_1,k_2,k_3)_N\cdot||\Psi'_{N-n,n}(k_1,k_2,k_3)||^2=\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array}\right)=$$

$$\frac{4c\,(2N+k_1+k_2+k_3+2)(2n+k_1+k_2+1)\Gamma(N+1)}{\Gamma(N+k_1+k_2+k_3+2)\Gamma(k_2+1)}\Phi_n^{(\alpha,\beta,\gamma,\delta)}(t^2),\quad t=q+\frac{k_2+k_3+1}{2},$$

$$\alpha=\frac{k_2+k_3+1}{2},\ \beta=-N-\frac{k_2+k_3+1}{2},\ \gamma=\frac{k_2-k_3+1}{2},\ \delta=N+k_1+\frac{k_2+k_3+3}{2},$$

$$\Phi_n={}_4F_3\left(\begin{array}{cccc} -n, & k_1+k_2+n+1, & -q, & k_3+k_2+q+1 \\ -N, & k_2+1, & N+k_1+k_2+k_3+2 & \end{array};1\right)$$

The overall scaling factor $c(k_1,k_2,k_3)$ can be determined by evaluating the double integral for Ξ' in the simplest case $n=q=N=0$ where it factors into a product of beta integrals.

4. Extension to Wilson polynomials

Racah polynomials are $S9$ expansion coefficients for finite dimensional representations on the real 2-sphere. Wilson polynomials are expansion coefficients related to infinite dimensional representations for the Schrödinger eigenvalue equation of the generic potential on the upper sheet of the 2d hyperboloid. (Much earlier, Koornwinder [9, 10] pointed out the connection between expansion coefficients on higher dimensional hyperboloids but not specifically to the 2d case.) A Hilbert space structure is imposed on the eigenspace corresponding to a single continuous spectrum eigenvalue, where N is a negative real number, not an integer.

We expand the L_2 basis vectors in terms of the L_1 basis:

$$\Lambda_q = \sum_{n=0}^{\infty} R_q^n \Psi_n, \quad \sum_{\ell=0}^{\infty} \frac{||\Psi_{n_1}||^2}{||\Lambda_\ell||^2} R_\ell^{n_1} R_\ell^{n_2} = \delta_{n_1,n_2},$$

Applying L_2 to both sides of the expansion we can show that the R_q^n satisfy a three term recurrence relation and a difference equation as before, and the orthogonality relation can be rewritten in the form

$$\sum_{q=0}^{\infty} \frac{(2\alpha)_q(\alpha+1)_q(\alpha+\beta)_q(\alpha+\gamma)_q(\alpha+\delta)_q}{(\alpha)_q(\alpha-\beta+1)_q(\alpha-\gamma+1)_q(\alpha-\delta+1)_q\, q!} R'^{\,n_1}_{\,q} R'^{\,n_2}_{\,q} = \delta_{n_1,n_2} h_{n_1},$$

where $R'^{n}_{q} \sim \Phi_n(t)$ is a Wilson polynomial. This is equivalent to a ${}_5F_4$ identity and can all be made rigorous. Wilson recast the orthogonality into the form of a contour integral which greatly extended its domain of validity. All of the Racah-intertwining operators extend to this case.

The quantum problem on the 2d hyperboloid has mixed spectrum, both bound states and continuous spectra, [12]. An interesting task for future research is to work out the interbasis expansion associated with the spectral decomposition in this case and to relate it explicitly to partially discrete, partially continuous orthogonality relations for Wilson polynomials.

5. Extension to Wilson functions

If n is not an integer a formal calculation using (3) and (4) that ignores series convergence still gives (6) as a solution of the eigenvalue equation $\tau^*\tau\,\Phi_n = n(n+\alpha+\beta+\gamma+\delta-1)\Phi_n$. However, a careful calculation gives

$$[\tau^*\tau - n(n+\alpha+\beta+\gamma+\delta)] \sum_{k=0}^{K} \omega_k\, P(t,\alpha)_k = \frac{(-n)_{K+1}(n+\alpha+\beta+\gamma+\delta-1)_{K+1}}{K!\,(\alpha+\beta)_K(\alpha+\gamma)_K(\alpha+\delta)_K}(\alpha+t)_K(\alpha-t)_K.$$

Taking the limit as $K \to +\infty$, and making use of the Stirling formula, we obtain

$$[\tau^*\tau - n(n+\alpha+\beta+\gamma+\delta)]\,\Phi_n(t) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)}{\Gamma(-n)\Gamma(n+\alpha+\beta+\gamma+\delta-1)\Gamma(\alpha+t)\Gamma(\alpha-t)}. \quad (13)$$

Since the Γ function has a pole at the negative integers, we see that $\Phi_n(t)$ satisfies the eigenvalue equation for n a nonnegative integer, but otherwise it does not, except for isolated choices of the parameters. Now consider the functions with relations

$$Q(t,\alpha,\beta)_k = \frac{\Gamma(1-\beta+t)\Gamma(1-\beta-t)}{\Gamma(\alpha+t)\Gamma(\alpha-t)}(1-\beta+t)_k(1-\beta-t)_k,\ \tau\, Q(t,\alpha,\beta)_k = (\alpha+\beta-k-1)\, Q(t,\alpha+\frac{1}{2},\beta+\frac{1}{2})_k,$$

$$\tau^*\, Q(t,\alpha+\frac{1}{2},\beta+\frac{1}{2})_k = -(\gamma+\delta+k)\, Q(t,\alpha,\beta)_k + k(k-\beta+\delta)(k-\beta+\gamma)\, Q(t,\alpha,\beta)_{k-1}.$$

Then computing formally without regard to series convergence, we obtain the nonpolynomial solution of the $\tau^*\tau$ eigenvalue equation:

$$\Psi_n(t) = \frac{\Gamma(1-\beta+t)\Gamma(1-\beta-t)}{\Gamma(\alpha+t)\Gamma(\alpha-t)} {}_4F_3\left(\begin{matrix} 1-n-\alpha-\beta & n+\gamma+\delta & 1-\beta+t & 1-\beta-t \\ 2-\alpha-\beta & & 1-\beta+\gamma & 1-\beta+\delta \end{matrix} ;1\right). \tag{14}$$

However, a careful computation, exactly analogous to that for $\Phi_n(t)$, yields the result

$$(\tau^*\tau - n(n+\alpha+\beta+\gamma+\delta))\,\Psi_n(t) = \frac{\Gamma(2-\alpha-\beta)\Gamma(1-\beta+\gamma)\Gamma(1-\beta+\delta)}{\Gamma(1-n-\alpha-\beta)\Gamma(n+\gamma+\delta)\Gamma(\alpha+t)\Gamma(\alpha-t)}, \tag{15}$$

so $\Psi_n(t)$ doesn't satisfy the eigenvalue equation. Now, comparing (13),(15), we see that functions

$$\tilde{\Phi}_n^{(\alpha,\beta\gamma,\delta)}(t) = \Phi_n(t) - \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(1-n-\alpha-\beta)\Gamma(n+\gamma+\delta)}{\Gamma(-n)\Gamma(n+\alpha+\beta+\gamma+\delta-1)\Gamma(2-\alpha-\beta)\Gamma(1-\beta+\gamma)\Gamma(1-\beta+\delta)}\Psi_n(t) \tag{16}$$

do satisfy the eigenvalue equation for general $n = n' + c$ where n' runs over the integers and c is a fixed noninteger. Furthermore it is straightforward to verify that $\tilde{\Phi}_n^{(\alpha,\beta\gamma,\delta)}(t)$ satisfies all of the recurrence formulas induced by the intertwining operators τ, μ, τ^*, μ^* for general n, such as (9), that are satisfied by Φ_n for nonnegative integer values. The functions $\tilde{\Phi}_n(t)$ have the duality property, hence since they satisfy the 2nd order difference eigenvalue equation for general n they must also satisfy the 3-term recurrence formula. Thus for fixed c these basis functions define an infinite-dimensional irreducible representation of the quadratic algebra of $S9$ in which the eigenvalues of L_1 are indexed by an arbitrary integer n'. In the original quantum problem this representation is easy to construct: all of the recurrence relations that we have derived using hypergeometric functions remain valid for nonpolynomial hypergeometric functions. These nonpolynomial bases are no longer normalizable, but the recurrence relations remain valid. Not clear was how such representations could be realized in terms of difference operators. The solution is that the analytic continuation of the basis functions is (16), the Wilson functions which lead to associated Wilson polynomials [13, 14], and to the Wilson transform [15], which corresponds to infinite dimensional irreducible representations of the quadratic algebra of $S9$ in which the spectrum of L_1 is (partly) continuous. In the original quantum mechanical system the analytic continuation is evident, but the integral over the sphere, i.e. in x, y, giving the interbasis expansion coefficients is deformed into a Pochhammer contour integral on a Riemann surface over the x-plane with branch points at ± 1 and a similar surface over the y-plane. All of the intertwining operator recurrences can be verified by integration by parts.

6. Discussion and Conclusions

- We showed explicitly how Racah and Wilson polynomials, and the Wilson functions, arise as expansion coefficients for the generic superintegrable system on the complex 2-sphere, relating two different sets of spherical coordinate bases.
- We showed how the principal properties of these functions: the measure, 3-term recurrence relation, 2nd order difference equation, duality of these relations, permutation symmetry, and intertwining operators – follow from the symmetry of this quantum system.
- The duality between the 3-term recurrence formula and the 2nd order difference equation is a consequence of the permutation symmetry of the quantum Hamiltonian.
- The parameter changing difference relations for the polynomials follow from intertwining operators for the quantum system. All of the properties of this system are induced by the fundamental differential recurrence relations of the Gaussian hypergeometric functions.
- The orthogonality measure for the polynomials and its symmetry follow from the symmetry of the norms of the quantum basis functions.

- There is active interest in the relation between multivariable Wilson polynomials and the quantum superintegrable system on the n-sphere with generic potential, and these results should aid in the generalization.
- By contracting function space realizations of irreducible representations of the $S9$ quadratic algebra to the other superintegrable systems one obtains the full Askey scheme of orthogonal hypergeometric polynomials, uniquely induced by Lie algebra contractions of $so(3, C)$ and $e(2, C)$, [16, 17]. This work should be extended to multivariable orthogonal polynomials.

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