Nonexistence of class $\mathcal{O}_{\mathcal{H}^0}(4)$ Calogero-like 2nd order superintegrable systems on the complex 3-sphere

B. K. Berntson
KTH Royal Institute of Technology, Stockholm, Sweden
E. G. Kalnins
Department of Mathematics, University of Waikato,
Hamilton, New Zealand
and W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, U.S.A.

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Abstract

2nd order superintegrable systems with a 2-parameter potentials and 5 symmetry operators that are functionally linearly dependent have never been classified. The best known such example is the Calogero system with 3 bodies on a line. In the paper we have worked out the structure theory for such systems in conformally flat spaces and shown that they always admit a 1st order symmetry. We have given a complete classification for all such systems in 3-dimensional flat space that are of the class $\mathcal{O}_{\mathcal{H}^0}(4)$. In this note we prove the nonexistence of Calogero-like systems in class $\mathcal{O}_{\mathcal{H}^0}(4)$ on the complex 3-sphere.

1 Introduction

In the paper [1] we have derived structure results for all 2nd order superintegrable FLD systems on conformally flat real or complex spaces that have potentials that depend on 2 functionally independent variables (the maximum possible) and have determined all such systems on 3-dimensional complex flat space that are of class $\mathcal{O}_{\mathcal{H}^0}(4)$. In this note we carry out the analogous computations for the complex 3-sphere and show that no such systems exist in class $\mathcal{O}_{\mathcal{H}^0}(4)$. The notation and method of classification are taken from paper [1].

2 The complex 3-sphere

We choose a standardized Cartesian-like coordinate system $\{x, y, z\}$ on the 3-sphere such that the Hamiltonian is

$$\mathcal{H} = (1 + \frac{r^2}{4})^2(p_x^2 + p_y^2 + p_z^2) + V, \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$. These coordinates can be related to the standard realization of the sphere using complex coordinates $\mathbf{s} = (s_1, s_2, s_3, s_4)$ such that $\sum_{j=1}^4 s_j^2 = 1$ and $ds^2 = \sum_{j=1}^4 ds_j^2$ via

$$s_1 = \frac{4x}{4 + r^2}, \quad s_2 = \frac{4y}{4 + r^2}, \quad s_3 = \frac{4z}{4 + r^2}, \quad s_4 = \frac{4 - r^2}{4 + r^2} \tag{2}$$

with inverse $x = 2s_1/(1 + s_4)$, $y = 2s_2/(1 + s_4)$, $z = 2s_3/(1 + s_4)$. A basis of Killing vectors for the zero-potential system is $J_{jh}, K_h,\ j, h = 1, 2, 3,\ j < h$, where

$$J_{23} = yp_z - zp_y, \quad J_{31} = zp_x - xp_z, \quad J_{12} = xp_y - yp_x, \tag{3}$$

$$K_1 = (1+\frac{x^2 - y^2 - z^2}{4})p_x+\frac{xy}{2}p_y+\frac{xz}{2}p_z,\ K_2 = (1+\frac{y^2 - x^2 - z^2}{4})p_y+\frac{xy}{2}p_x+\frac{yz}{2}p_z,$$

$$K_3 = (1 + \frac{z^2 - x^2 - y^2}{4})p_z + \frac{xz}{2}p_x + \frac{yz}{2}p_y.$$

The nonzero commutation relations are

$$\{J_{23}, J_{31}\} = J_{12}, \quad \{K_1, K_2\} = J_{12}, \quad \{K_1, J_{31}\} = K_3 \tag{4}$$

and their cyclic permutations. The relation between this basis and the standard basis of rotation generators on the sphere $I_{\ell m} = s_\ell p_m - s_m p_\ell = -I_{m\ell}$ is

$$J_{23} = I_{23},\ J_{31} = I_{31},\ J_{12} = I_{12},\ K_1 = I_{41},\ K_2 = I_{42},\ K_3 = I_{43}. \tag{5}$$

By relabeling, we can express one of the quadratic parts of the constants of the motion $\hat{\mathcal{S}}_{(0)}$ for a FLD system as a linear combination of the quadratic parts of the remaining 4 generators $\hat{\mathcal{S}}_{(1)}, \ldots, \hat{\mathcal{S}}_{(4)}$:

$$\hat{\mathcal{S}}_{(0)} = \sum_{\ell=1}^{4} c^{(\ell)}(\mathbf{x})\hat{\mathcal{S}}_{(\ell)}. \tag{6}$$

Again we limit ourselves to the maximal case where the expansion (6) is unique. The generators $\hat{\mathcal{S}}_{(0)}, \hat{\mathcal{S}}_{(1)}, \hat{\mathcal{S}}_{(2)}, \hat{\mathcal{S}}_{(3)}, \hat{\mathcal{S}}_{(4)}$ are polynomial in x, y, z of order at most 4 and are linearly independent. We can solve for the expansion coefficients in the form $c^{(\ell)}(x,y,z) = s^{(\ell)}(x,y,z)/s^{(0)}(x,y,z)$, $\ell = 1, \ldots, 4$ where $s^{(0)}, s^{(1)}, \ldots, s^{(4)}$ are polynomials in x, y, z of order at most 4. It follows that

$$\sum_{a_1,a_2,a_3} A(a_1,a_2,a_3) x^{a_1} y^{a_2} z^{a_3} \equiv s^{(0)} \hat{\mathcal{S}}_{(0)} - \sum_{r=1}^{4} s^{(r)} \hat{\mathcal{S}}_{(r)} = 0, \tag{7}$$

where each coefficient $A(a_1,a_2,a_3)$ must vanish. In particular, the sum of all terms homogeneous of degree n must vanish for each $n = 0, \ldots, 4$:

$$B(n) \equiv \sum_{a_1+a_2+a_3=n} A(a_1,a_2,a_3) x^{a_1} y^{a_2} z^{a_3} = 0.$$

Each of the generators $\hat{\mathcal{S}}_{(r)}$ is a linear combination of terms $K_i K_j$, (maximal order 4), $J_i K_j$, (maximal order 3) and $J_i J_j$, (order 2).

Since the free part of the Hamiltonian $\mathcal{H}$ is not homogeneous, it is not true that the generators must be homogeneous polynomials. However, once the highest order terms of a generator $\mathcal{S}_{(0)}$ are fixed are fixed, the necessary and sufficient conditions on the lower order terms for $\mathcal{S}_{(0)}$ to be a symmetry are uniquely determined from the relation $\{\mathcal{H}, \mathcal{S}_{(0)}\} = 0$ and the requirement that the lower order terms cannot by themselves be a first order symmetry.

From Corollary 1 of [1] applied to the 3-sphere we see that, up to conjugacy, there are just 2 cases to consider: $\mathcal{J} = J_{12}$ and $\mathcal{J} = J_{12} + iJ_{23}$.

2.1 $\mathcal{J} = J_{12}$

Here the centralizer of $\mathcal{J}$ is the group generated by rotations about the z-axis, and transformations $\exp(\alpha K_3)$. We can use this freedom to simplify the computation. Since J_{12} is a symmetry the potential must be of the form $V(x^2+y^2, z)$. Writing a 2nd order symmetry in the form

$$\mathcal{S} = F_{11}(x,y,z)\ p_1^2 + F_{22}(x,y,z)\ p_2^2 + F_{33}(x,y,z)\ p_3^2 + F_{12}(x,y,z)\ p_1 p_2 +$$
$$F_{13}(x,y,z)\ p_1 p_3 + F_{23}(x,y,z)\ p_2 p_3 + F_0(x,y,z)$$

and requiring that $\{\mathcal{S}, \mathcal{H}\} = 0$, we can solve for the F_{jk} to get

$$\begin{aligned} F_{11} &= \frac{1}{48}(48c_{20} + 3c_{14} - c_3)y^4 + \frac{1}{48}(8c_2 x + 24c_{13} + 4c_4)y^3 + \\ &\quad \frac{1}{48}((6c_{14} + 96c_{20} - 2c_3)z^2 + (-8c_6 x - 4c_5 + 24c_9)z + \end{aligned} \tag{8}$$

$$
\begin{aligned}
& (-6c_{14}+96c_{20}-6c_3)x^2+(-32c_{15}-16c_{17})x-24c_{18}+384c_{20} \\
& -16c_3)y^2+\frac{1}{48}((8c_2x+24c_{13}+4c_4)z^2+(-8*c_1x^2+96c_{10}x-16c_1 \\
& +48c_{11})z-(24(\frac{1}{3}c_2x+c_{13}+\frac{1}{6}c_4))(x^2+4))y+\frac{1}{48}(48c_{20}+3c_{14}-c_3)z^4+ \\
& \frac{1}{48}(-8c_6x-4c_5+24c_9)z^3+\frac{1}{48}((-6c_{14}+96c_{20}+2c_3)x^2 \\
& +(-16c_{15}+16c_{17})x-24c_{19}+384c_{20})z^2+\frac{1}{12}(x^2+4)(2c_6x+c_5-6c_9)z \\
& +\frac{1}{16}(c_{14}+16c_{20}-\frac{1}{3}c_3)(x^2+4)^2, \\
F_{12} \;=\; & \frac{1}{12}c_2x^4+\frac{1}{12}((c_3+3c_{14})y+c_1z+6c_{13}+c_4)x^3+ \\
& \frac{1}{12}(-6c_2y^2+(6c_6z+12c_{15}+6c_{17})y-18z(c_{10}-\frac{1}{6}c_7))x^2+ \\
& \frac{1}{12}((-c_3-3c_{14})y^3+(-3c_1z-18c_{13}-3c_4)y^2+((-3c_{14}+3c_3)z^2+ \\
& (-12c_9+6c_5)z+12c_{18})y+c_1z^3+(3c_4-6c_{13})z^2-12c_{11}z+24c_{13}+4c_4)x+ \\
& \frac{1}{12}c_2y^4+\frac{1}{12}(-2c_6z-4c_{15}-2c_{17})y^3+3z(c_{10}-\frac{1}{6}c_7)y^2\frac{1}{2}+ \\
& \frac{1}{12}(-2c_6z^3+6c_{17}z^2+12c_8z-16c_{15}-8c_{17})y-\frac{1}{12}c_2z^4+\frac{1}{12}(-3c_7-6c_{10})z^3 \\
& -c_{12}z^2+\frac{1}{12}(12c_7+24c_{10})z-4c_2\frac{1}{3}, \\
F_{13} \;=\; & -\frac{1}{12}c_6x^4+\frac{1}{12}((3c_{14}-c_3)z+yc_1+6c_9-c_5)x^3+ \\
& \frac{1}{12}(6c_6z^2+(-6c_2y+6c_{15}-6c_{17})z-18y(c_{10}+\frac{1}{6}c_7))x^2+ \\
& \frac{1}{12}((c_3-3c_{14})z^3+(-3c_1y+3c_5-18c_9)z^2+ \\
& ((-3c_{14}-3c_3)y^2+(-12c_{13}-6c_4)y+12c_{19})z+c_1y^3+(-6c_9-3c_5)y^2 \\
& -12yc_{11}+24c_9-4c_5)x-\frac{1}{12}c_6z^4+\frac{1}{12}(2c_2y-2c_{15}+2c_{17})z^3+ \\
& 3y(c_{10}+\frac{1}{6}c_7)z^2\frac{1}{2}+\frac{1}{12}(2y^3c_2+(-6c_{15}-6c_{17})y^2+12c_{12}y-8c_{15}+8c_{17})z \\
& +\frac{1}{12}c_6y^4+\frac{1}{12}(3c_7-6c_{10})y^3-c_8y^2+\frac{1}{12}(-12c_7+24c_{10})y+4c_6\frac{1}{3}, \\
F_{22} \;=\; & \frac{1}{24}(24c_{20}-c_3)x^4+\frac{1}{24}(4c_2y-8c_{15}-4c_{17})x^3+\frac{1}{24}((6c_{14}+48c_{20})y^2 \\
& +(2c_1z+24c_{13}+4c_4)y+(48c_{20}-2c_3)z^2-4zc_5+12c_{14}-12c_{18}+192c_{20}
\end{aligned}
$$

$$
\begin{aligned}
& -4c_3)x^2 + \frac{1}{24}(-4y^3c_2 + (8c_6z + 8c_{15} + 4c_{17})y^2 + \\
& (4c_2z^2 + (-24c_{10} + 12c_7)z - 16c_2)y + (-8c_{15} - 4c_{17})z^2 + \\
& (16c_6 - 24c_8)z + 32c_{15} + 16c_{17})x + \frac{1}{24}(24c_{20} - c_3)y^4 - \frac{1}{12}c_1y^3z \\
& + \frac{1}{24}((48c_{20} + 2c_3)z^2 + 4zc_5 + 192c_{20} - 8c_3)y^2 + \\
& \frac{1}{12}z(c_1z^2 + 4c_4z - 4c_1)y + \frac{1}{24}(24c_{20} - c_3)z^4 - \frac{1}{6}z^3c_5 + \\
& \frac{1}{24}(-12c_{16} + 192c_{20})z^2 + 2zc_5\frac{1}{3} + 16c_{20} - 2c_3\frac{1}{3}, \\
F_{23} \;=\; & -\frac{1}{24}c_1x^4 + \frac{1}{24}(4c_2z - 4c_6y + 24c_{10})x^3 + \frac{1}{24}((12c_{14}z + 24c_9)y + 24c_{13}z \\
& + 24c_{11})x^2 + \frac{1}{24}(-4c_6y^3 + (-12c_2z - 12c_7)y^2 + (12c_6z^2 + 24c_{15}z \\
& + 24c_8)y + 4c_2z^3 + 12c_7z^2 + 24c_{12}z - 96c_{10})x + \frac{1}{24}c_1y^4 + \\
& \frac{1}{24}(-4c_3z - 4c_5)y^3 - \frac{1}{4}z(c_1z + 2c_4)y^2 + \frac{1}{24}(4c_3z^3 + 12c_5z^2 + \\
& 24c_{16}z - 16c_5)y + \frac{1}{24}(z^2 + 4)(c_1z^2 + 4c_4z - 4c_1), \\
F_{33} \;=\; & x^4c_{20} + \frac{1}{12}(-2c_6z - 2c_{15} + 2c_{17})x^3 + \frac{1}{12}(24c_{20}y^2 + (c_1z + 2c_4)y + \\
& (3c_{14} + 24c_{20} - c_3)z^2 + (-2c_5 + 12c_9)z + 6c_{14} - 6c_{19} + 96c_{20} - 2c_3)x^2 \\
& + \frac{1}{12}((-2c_6z - 2c_{15} + 2c_{17})y^2 + (-4c_2z^2 + (-12c_{10} - 6c_7)z \\
& -12c_{12} - 8c_2)y + (2(z^2 + 4))(c_6z + c_{15} - c_{17}))x + y^4c_{20} + \\
& \frac{1}{12}(c_1z + 2c_4)y^3 + \frac{1}{12}((24c_{20} - 2c_3)z^2 - 4zc_5 - 6c_{16} + 96c_{20} - 4c_3)y^2 \\
& -\frac{1}{12}(z^2 + 4)(c_1z + 2c_4)y + c_{20}(z^2 + 4)^2,
\end{aligned}
$$

where the c_j are constants to be determined. In addition we obtain a series of equations for the first derivatives $\partial_x F_0, \partial_y F_0, \partial_z F_0$, which lead to Bertrand-Darboux equations for $V(x^2 + y^2, z)$. At the end we have to find 5 linearly independent solutions for $\mathcal{S}$ and verify that they are functionally linearly dependent.

We can get a basis $\{L_j,\, j = 1, \ldots, 20\}$ for the 20-dimensional space of symmetries of the zero-potential system by defining the symmetry L_j as that for which $c_j = 1$ and $c_k = 0$ for all $k \neq j$. However a more convenient basis is that of eigenvectors of $\mathrm{Ad}_{\mathcal{J}}$. The result is:

Order 2 basis:

1.
$$S_{22+} = \frac{i}{2}L_{19} - \frac{i}{2}L_{16} + L_{12}, \quad \text{e.v.} = 2i,$$

2.
$$S_{22-} = -\frac{i}{2}L_{19} + \frac{i}{2}L_{16} + L_{12}, \quad \text{e.v.} = -2i,$$

3.
$$S_{20} = L_{18}, \quad \text{e.v.} = 0,$$

4.
$$S_{200} = L_{19}, \quad \text{e.v.} = 0,$$

5.
$$S_{21+} = -iL_8 + L_{11}, \quad \text{e.v.} = i,$$

6.
$$S_{21-} = iL_8 + L_{11}, \quad \text{e.v.} = -i,$$

Order 3 basis:

1.
$$S_{32+} = iL_9 - 4iL_5 + \frac{2}{3}L_{10} + L_7, \quad \text{e.v.} = 2i,$$

2.
$$S_{32-} = -iL_9 + 4iL_5 + \frac{2}{3}L_{10} + L_7, \quad \text{e.v.} = -2i,$$

3.
$$S_{31+a} = -\frac{i}{2}L_{13} - iL_4 + L_{17}, \quad \text{e.v.} = i,$$

4.
$$S_{31-a} = \frac{i}{2}L_{13} + iL_4 + L_{17}, \quad \text{e.v.} = -i,$$

5.
$$S_{31+b} = -\frac{i}{2}L_{13} + iL_4 + L_{15}, \quad \text{e.v.} = i,$$

6.
$$S_{31-b} = \frac{i}{2}L_{13} - iL_4 + L_{15}, \quad \text{e.v.} = -i,$$

7.
$$S_{30} = -2L_{10} + L_7, \qquad \text{e.v.} = 0,$$

8.
$$S_{300} = \frac{1}{12}L_9 + L_5, \qquad \text{e.v.} = 0,$$

Order 4 basis:

1.
$$S_{42+} = L_{14} + iL_2 + L_3, \quad \text{e.v.} = 2i,$$

2.
$$S_{42-} = L_{14} - iL_2 + L_3, \quad \text{e.v.} = -2i,$$

3.
$$S_{41+} = 2iL_1 + L_6, \quad \text{e.v.} = i,$$

4.
$$S_{41-} = -2iL_1 + L_6, \quad \text{e.v.} = -i,$$

5.
$$S_{40} = L_{20}, \qquad \text{e.v.} = 0,$$

6.
$$S_{400} = \frac{1}{3}L_{14} + L_3, \qquad \text{e.v.} = 0,$$

Thus the possible actions of $\mathrm{Ad}_{\mathcal{J}}$ on an eigenbasis are described by the canonical forms
$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$
$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

where $\lambda_j = \pm i,\ \pm 2i$.

2.1.1 Form 9

Since the eigenvalues for the real 3-sphere must occur in complex-conjugate pairs, a system of this form is only possible for the hyperboloid. There are numerous FLD systems with this form, but none admit a 2-parameter potential.

2.1.2 Form 10

There are several FLD systems with this form, but none admit a 2-parameter potential.

2.1.3 Form 11

Since the eigenvalues for the real 3-sphere must occur in complex-conjugate pairs, a system of this form is only possible for the hyperboloid. Checking over all possibilities for systems with this eigenvalue form we find only one system that is FLD and, for it, V depends on only a single function.

2.1.4 Form 12

Checking over all possibilities for systems with this eigenvalue form, we find that none are FLD.

2.2 $\mathcal{J} = J_{12} + iJ_{23}$

In this case the potential must be of the form, $V = V(z+ix, y^2 - 2ix(z+ix))$. This suggests the change of variables

$$x = -\rho\left[e^{-\theta} + e^{\theta}\left(1/4 - r^2\right)\right], \quad y = -\rho r e^{\theta}, \tag{13}$$

$$z = \mathrm{i}\rho\left[e^{-\theta} - e^{\theta}\left(1/4 + r^2\right)\right],$$

so that in the new coordinates we can write $\mathcal{J} = \frac{1}{2}p_r$ and the Hamiltonian is

$$\mathcal{H} = (\rho^2+4)^2\left(\frac{e^{-2\theta}p_r^2}{\rho^2} + p_\rho^2 - \frac{p_\theta^2}{\rho^2}\right) + V(\rho, e^\theta). \tag{14}$$

As in section (2.1) we can get a basis $\{L_j,\, j = 1, \ldots, 20\}$ for the 20-dimensional space of symmetries of the zero-potential system by defining the symmetry L_j as that for which $c_j = 1$ and $c_k = 0$ for all $k \neq j$. However a more convenient basis is in terms of generalized eigenvectors of $\mathrm{Ad}_{\mathcal{J}}$. The result is:

Order 2 basis:

1. $$J_4 = -\frac{1}{3}L_{16} - \frac{1}{6}L_{19},$$
2. $$J_3 = \frac{i}{3}L_{11} + \frac{1}{3}L_{12},$$
3. $$J_2 = \frac{1}{3}L_{18} - \frac{2}{3}L_{19} + \frac{1}{3}L_{16},$$
4. $$J_1 = 2iL_{11} - 2L_{12},$$
5. $$J_0 = 2L_{18} + 4iL_8 - 2L_{16} = 2\mathcal{J}^2,$$
6. $$J_{00} = L_{16} + L_{18} + L_{19}.$$

The elements of the order 2 basis satisfy

$$\mathrm{Ad}_{\mathcal{J}}(J_j) = J_{j-1}, \quad j = 1, \ldots, 4,$$

and

$$\mathrm{Ad}_{\mathcal{J}}(J_0) = \mathrm{Ad}_{\mathcal{J}}(J_{00}) = 0.$$

The subscript j on the operator J_j indicates that this basis function is a polynomial of order j in the variable r.

Order 3 basis:

1.
$$M_2 = -\frac{i}{8}L_9 - L_{15} + \frac{1}{2}L_{17} - \frac{3i}{2}L_5,$$

2.
$$M_1 = -\frac{1}{2}L_{13} + 3L_4,$$

3.
$$M_0 = -\frac{i}{4}L_9 + 2L_{15} - L_{17} - 3iL_5,$$

The subscript j on the operator M_j indicates that this basis function is a polynomial of order j in the variable r.

4.
$$N_4 = \frac{1}{16}L_{13} + \frac{i}{48}L_{10} + \frac{1}{8}L_4 - \frac{i}{32}L_7,$$

5.
$$N_3 = \frac{i}{16}L_9 - \frac{1}{4}L_{17} - \frac{i}{4}L_5,$$

6.
$$N_2 = \frac{i}{4}L_{10} + \frac{i}{8}L_7,$$

7.
$$N_1 = -\frac{3i}{8}L_9 - \frac{3}{2}L_{17} + \frac{3i}{2}L_5,$$

8.
$$N_0 = -\frac{3}{2}L_{13} + \frac{i}{2}L_{10} - 3L_4 - \frac{3i}{4}L_7.$$

The subscript j on the operator N_j indicates that this basis function is a polynomial of order j in the variable r. The elements of the order 3 basis satisfy

$$\mathrm{Ad}_{\mathcal{J}}(M_j) = M_{j-1}, \quad j = 1, 2, \qquad \mathrm{Ad}_{\mathcal{J}}(M_0) = 0.$$

and

$$\mathrm{Ad}_{\mathcal{J}}(N_j) = N_{j-1}, \quad j = 1, \ldots, 4, \qquad \mathrm{Ad}_{\mathcal{J}}(N_0) = 0.$$

.

Order 4 basis:

1.
$$K_4 = L_3,$$

2.
$$K_3 = 2iL_1 - \frac{1}{2}L_2,$$

3.
$$K_2 = -L_{14} + 3L_3 + \frac{3i}{2}L_6 + \frac{1}{12}L_{20},$$

4.
$$K_1 = 3iL_1 - \frac{3}{2}L_2,$$

5.
$$K_0 = -3L_{14} + 3L_3 + 3iL_6 + \frac{1}{8}L_{20},$$

6.
$$K_{00} = L_{20} = \mathcal{H}_0.$$

The subscript j on the operator K_j indicates that this basis function is a polynomial of order j in the variable r. The elements of the order 4 basis satisfy
$$\mathrm{Ad}_{\mathcal{J}}(K_j) = K_{j-1}, \quad j = 1, \dots, 4,$$
and
$$\mathrm{Ad}_{\mathcal{J}}(K_0) = \mathrm{Ad}_{\mathcal{J}}(K_{00}) = 0.$$

Each canonical form must correspond to bases that are invariant under the action of $\mathrm{Ad}_{\mathcal{J}}$ and contain both the symmetries J_0 and K_{00}. There are 5 canonical forms to consider:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

2.3 Form (15)

There is only one case corresponding to this form, and it is FLD. However, it does not admit a 2-parameter potential depending on 2 variables.

2.4 Form (16)

There are no 2-parameter FLD systems for this form.

2.5 Form (17)

There are no FLD systems for this form.

2.6 Form (18)

There is only 1 FLD system for this form and it admits only a 1-parameter solution.

2.7 Form (19)

There is only 1 FLD system for this form and it admits only a 1-parameter solution.

3 Conclusions

This note is part of a program to classify all 2nd order superintegrable classical and quantum systems on 3-dimensional conformally flat complex manifolds. In the paper [1] we have worked out the basic structure theory for Calogero-like superintegrable systems on these manifolds and classified all such systems in class $\mathcal{O}_{\mathcal{H}^0}(4)$ on flat spaces. Here we have shown that there are no such systems in class $\mathcal{O}_{\mathcal{H}^0}(4)$ on the complex 3-sphere.

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