Superintegrability as an organizing principle for special function theory

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Abstract 1

An n-dimensional system $H = \Delta_n + V(\mathbf{x})$, where Δ_n is the Laplace operator, is integrable if it admits n algebraically independent commuting symmetry operators. It is superintegrable if it is integrable and admits 2n-1 algebraically independent symmetry operators (the maximum possible, but of course not all commuting). If the independent symmetries can all be chosen of order k or less as differential operators the system is kth order superintegrable. Superintegrability is much more restrictive than integrability.

Abstract 2

The operators of a superintegrable system typically close under commutation to form an algebra, not usually a Lie algebra, and the representations of this algebra lead to new special functions beyond those which rise by solving the quantum eigenvalue problem $H\Psi = E\Psi$. First order superintegrable systems such as Helmholtz, Laplace or wave equations $(V = 0)$ have Lie symmetry algebras and most of the special functions of mathematical physics arise by separation of variables from such equations, characterized by commuting sets of 2nd order symmetries.

Absract 3

Eigenvalue problems for higher order superintegrable systems are associated with additional special functions, such as Painlevé transcendents. Irreducible representations of symmetry algebras lead to many new classes of special functions, including discrete orthogonal polynomials, such as multivariable Wilson polynomials. We give examples and argue for superintegrability as a useful organizing principle for the theory of special functions.

Desirable properties of a theory

- Should lead naturally to at least the special functions of mathematical physics, the basic orthogonal polynomials, and other special functions.
- Should yield directly important basic properties of these functions: recurrences, generating functions, addition theorems, orthogonality, etc.

Superintegrability

Hamiltonian: $H = \Delta_n + V(\mathbf{x})$ where Δ_n is the Laplacian on a Riemannian manifold, expressed in local coordinates x_j.

Superintegrable if there are $2n-1$ algebraically independent differential symmetry operators

$$S_j, \quad j = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_j] \equiv HS_j - S_jH = 0$.

This is a very restrictive condition!

Order of a System

- The commutator, linear combinaton and product of two symmetries is again a symmetry, so they generate a nonabelian symmetry algebra.
- One of the miracles of superintergrability is that this algebra typically closes under commutation at finite order.
- A superintegrable system is of order ℓ if ℓ is the maximum order of the generating symmetries other than the Hamiltonian.

Integrability and Superintegrability

- An integrable system has n independent commuting symmetry operators.
- A superintegrable system has $2n - 1$ algebraically independent symmetry operators (not all commuting). In many cases a superintegrable system is multiply integrable.

Note that if Ψ satisfies the eigenvalue equation $H\Psi = E\Psi$ and S is a symmetry operator, then $S\Psi$ also satisfies the eigenvalue equation, i.e. symmetries map solutions to solutions.

Symmetries as Recurrences

- If the superintegrable system admits n commuting 2nd order symmetry operators $S_1, \cdots, S_n$ then typically it is separable and the special function solutions are the common eigenfunctions $S_j \Psi_\lambda = \lambda_j \Psi_\lambda$ where $\lambda_1 = E, \lambda_2, \cdots, \lambda_n$ are the separation constants.
- Then the remaining generating symmetries S_k, $k > n$, define special function recurrence relations $S_k \Psi_\lambda = \sum_\mu c_{\lambda,\mu} \Psi_\mu$.

Conformal Superintegrability 1

There is a similar definition of conformal superintegrability for Laplace-type equations $H\Psi = 0$ where Hamiltonian: $H = \Delta_n + V(\mathbf{x})$.

Conformally Superintegrable if there are $2n-1$ algebraically independent differential symmetry operators

$$S_j, \quad j = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_j] \equiv HS_j - S_jH = L_jH$, for some differential operator L_j.

Conformally Superintegrable 2

Note that if $H\Psi = 0$, then $H(S_j\Psi) = 0$, so conformal symmetries map solutions of the Laplace equation to solutions.

Again, conformal symmetries lead to recurrence relations for R-separable special function solutions of the Laplace equation.

Helmholz Equation on the 2-Sphere

- 2-sphere: $s_1^2 + s_2^2 + s_3^2 = 1$
- $\Delta_2 \Psi = E\Psi$ or $(J_1^2 + J_2^2 + J_3^2)\Psi = E\Psi$
- $J_3 = x_1\partial_{x_2} - x_2\partial_{x_1}$, J_2, J_1, angular momentum operators
- $n = 2$, $2n - 1 = 3$. 1st order superintegrable with generators J_1, J_2, J_3
- Get finite dimensional eigenspaces for $E_\ell = -\ell(\ell + 1)$, dimension $2\ell + 1$

Spherical Harmonics 2

- Eigenfunctions of J_3 define spherical coordinate separation. Basis $\{Y_\ell^m\}$.
- Action of J_2, J_3 on the basis yields the m-recurrence relations for spherical harmonics.
- The symmetry algebra is a Lie algebra which exponentiates to the rotation group and gives the addition formula for spherical harmonics.
- On the n-sphere get polyspherical harmonics.
- Get different special function bases for different commuting sets of n 2nd order symmetries.

Hypergeometric Functions 1

- With change of variables can write flat space Laplace equation $\Delta_4\Psi = 0$ as $(\partial_{u_1}\partial_{u_2} - \partial_{u_3}\partial_{u_4})\Psi = 0$.
- $n = 4$, $2n - 1 = 7$, Can find 15 linearly independent 1st order conformal symmetries with 7 of them algebraically independent. Simplest symmetries are dilation generators, such as $u_1\partial_{u_1} + u_3\partial_{u_3}$.
- Can find 4 commuting dilation generators whose common eigenfunctions take the form $\Psi = {}_2F_1\left(\alpha, \beta; \gamma; \frac{u_3u_4}{u_1u_2}\right) u_1^{-\alpha}u_2^{-\beta}u_3^{\gamma-1}$

Hypergeometric Functions 2

- The 11 remaining symmetries determine the differential recurrence relations for hypergeometric functions.
- The 15 1st order conformal symmetries generate the symmetry algebra $s\ell(4,C)$ This exponentiates to the group $SL(4,C)$ and gives the addition formulas as well as other properties of ${}_2F_1$s,
- Generalizes in various ways for larger n. (Appell functions, Lauricella functions, Horn functions,$\cdots$)

Generic Potential on the 2-Sphere 1

- Potential $V = \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$ where $s_1^2 + s_2^2 + s_3^2 = 1$.
- Equation $H\Psi \equiv (J_1^2 + J_2^2 + J_3^2 + V)\Psi = E\Psi$ where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$, etc.
- Generating symmetries S_1, S_2, S_3 where $S_1 = J_3^2 + \frac{a_1 s_2^2}{s_1^2} + \frac{a_2 s_1^2}{s_2^2}$, plus cyclic permutations. Here $H = S_1 + S_2 + S_3 + a_1 + a_2 + a_3$.
- 2nd order superintegrable system

Generic Potential on the 2-Sphere 2

- Separates in spherical coords. Bound state eigenfunctions are Karlin-McGregor polynomials.
- Separates in ellipsoidal coordinates. Bound states are products of ellipsoidal wave fuctions.
- Action of symmetry generators S_1, S_2, S_3 on these bases leads to recurrences.
- The symmetries generate a quadratic algebra (not a Lie algebra).

Quadratic Algebra Structure

- $R = [S_1, S_2]$
- $[S_i, R] = 4\{S_i, S_k\} - 4\{S_i, S_j\} - (8 + 16a_j)S_j + (8 + 16a_k)S_k + 8(a_j - a_k)$,
- $R^2 = \frac{8}{3}\{S_1, S_2, S_3\} - (16a_1 + 12)S_1^2 - (16a_2 + 12)S_2^2 - (16a_3 + 12)S_3^2 + \frac{52}{3}(\{S_1, S_2\} + \{S_2, S_3\} + \{S_3, S_1\}) + \frac{1}{3}(16 + 176a_1)S_1 + \frac{1}{3}(16 + 176a_2)S_2 + \frac{1}{3}(16 + 176a_3)S_3 + \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.$
- i, j, k chosen so $\epsilon_{ijk} = 1$; $\{S_i, S_j\} = S_iS_j + S_jS_i$

Reps of the Quadratic Algebra 1

- The generators S_1, S_2, S_3 map the energy eigenspaces into themselves
- To understand multiplicities and structure of the bound states, must classify the irreducible representations of the quadratic algebra.
- Since H commutes with all S_j, it is constant in an irreducible representation.
- Classify the structure of the irreducible representations in terms of an S_1 eigenbasis.

Reps of the Quadratic Algebra 2

- Surprise! For an irreducible representation the action of S_2 on the S_1 basis yields the general three-term recurrence relation for the Wilson polynomials $p_n(t^2, \alpha, \beta, \gamma, \delta) =$
$${}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & & \alpha+\delta \end{matrix}\right.$$
with $\alpha = -\frac{a_1+a_3+1}{2} + \mu$, $\beta = \frac{a_1+a_3+1}{2}$, $\gamma = \frac{a_1-a_3+1}{2}$, $\delta = \frac{a_1+a_3-1}{2} + a_2 - \mu + 2$.
- If the representation is finite dimensional, we obtain the Racah polynomials.

Reps of the Quadratic Algebra 3

- In this model S_2 is multiplication by t^2 and S_1 is the second order divided difference operator for the Wilson polynomials.
- The structure relations give information about the Wilson polynomials.
- Thus the quantum mechanical generic potential on the 2-sphere is intimately related to the Wilson polynomials, even though the 1st system involves differential equations and the 2nd involves difference equations.

Generic Potential on the 3-Sphere 1

- $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1, \quad 2n - 1 = 5$
- $H\Psi = E\Psi$ where

$$H = \sum_{1\le i<j\le 4} (s_i\partial_j - s_j\partial_i)^2 + \sum_{k=1}^{4} \frac{a_k}{s_k^2}, \qquad \partial_i \equiv \partial_{s_i}.$$

- 2nd order superintegrable with generators

$$S_{ij} \equiv S_{ji} = (s_i\partial_j - s_j\partial_i)^2 + \frac{a_i s_j^2}{s_i^2} + \frac{a_j s_i^2}{s_j^2},$$

for $1 \le i < j \le 4. \quad 5 \Longrightarrow 6$ Theorem

Generic Potential on the 3-Sphere 2

- The generators determine a quadratic algebra. The structure is far more complicated than the 2D case and one identity is of order 5 in the generators.
- The irreducible representations of the quadratic algebra lead exactly to the defining relations for the 2-variable Tratnik polynomials, a generalization of Wilson and Racah polynomials,
- Structure equations for Tratnik polynomials were previously unknown.

Generic Potential on the 3-Sphere 3

Simultaneous eigenfuntions of commuting pair $S_{12}, S_{12} + S_{14} + S_{24}$ are Tratnik polynomials

$$p_{n_1}\left(t^2, \alpha, \beta, \gamma, \delta\right) p_{n_2}\left(s^2, n_1 + \beta + \frac{\alpha+\delta}{2}, \gamma + \frac{\alpha+\delta}{2},\right.$$

$$\left. M + 1 + b_4 + \beta + \frac{\alpha+\delta}{2}, M + 1 + \beta + \frac{\alpha+\delta}{2}\right),$$

where α, δ depend linearly on s and t so w_{n_1} is a polynomial in both s and t. Other commuting pairs give other bases.

Generic Potential on the 3-Sphere 4

- Result with E.G. Kalnins and Sarah Post.
- Conjecture that generic potential on N-sphere corresponds to $N-1$-variable Tratnik polynomials.
- Similarly multivariable Hahn polynomials related to the oscillator, and so it goes.
- In 2nd order superintegrability theory it appears that all such systems are limits (using the Bôcher procedure) of generic systems on spheres. Conjecture that this is in close correspondence to the Askey scheme for orthogonal polynomials.

Painlevè transcendents

$$(\partial_x^2 + \partial_y^2 + \hbar^2\omega_1^2 P_1(\omega_1 x) + \hbar^2\omega_2^2 P_1(\omega_2 y))\Psi = E\Psi$$

Here, P_1 is the first Painlevè transcendent. 3rd order superintegrable. Other superintegrable systems with the transcendents P_2, P_3, P_4.

- My claim: Superintegrability organizes the theory of solutions of differential and difference equations that are interesting enough to be considered “special".
- Extension to q-difference equations?