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The Coulomb–Oscillator Relation on n-Dimensional Spheres and Hyperboloids*

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Abstract—We establish a relation between Coulomb and oscillator systems on n-dimensional spheres and hyperboloids for $n \geq 2$. We show that, as in Euclidean space, the quasiradial equation for the $(n+1)$-dimensional Coulomb problem coincides with the $2n$-dimensional quasiradial oscillator equation on spheres and hyperboloids. Using the solution of the Schrödinger equation for the oscillator system, we construct the energy spectrum and wave functions for the Coulomb problem. © *2002 MAIK "Nauka/Interperiodica"*.

1. INTRODUCTION

It has long been known that the Coulomb and oscillator potentials are two paradigms in quantum mechanics that possess dynamical or hidden symmetries: $O(n+1)$ for motion in a Coulomb field [1] and $SU(n)$ for an oscillator. On the other hand, the connections with these two Lie groups of dynamical symmetries provide relations between the Coulomb and oscillator systems. In particular, the $(n+1)$-dimensional radial Schrödinger equation for the Coulomb system is identical to the oscillator equation for $2n$ dimensions by the duality transformation [2]. It is also known that the complete relation (not only for the radial part) is possible only for special dimensions of $(2,2)$, $(3,4)$, and $(5,8)$. The dual mappings in these cases are so-called Levi-Civita, Kustaanheimo–Stiefel, and Hurwitz transformations [3–5].

The generalization of the Coulomb problem to a three-sphere was performed in the famous article of Schrödinger [6]; for the n-dimensional hyperboloid, this problem was solved in [7]. Later, the Coulomb and the oscillator problem on spheres and pseudospheres were discussed from many points of view in [8–19].

In [20], we constructed a series of complex mappings $S_{2C} \to S_2$, $S_{4C} \to S_3$ and $S_{8C} \to S_5$, which extend to spherical geometry the Levi-Civita, Kustaanheimo–Stiefel, and Hurwitz transformations, which are well known for Euclidean space. We showed that these transformations establish a correspondence between Coulomb and oscillator problems in classical and quantum mechanics for dimensions of (2, 2), (3, 4), and (5, 8) on the spheres. A detailed analysis of the real mapping on a curved space was performed in [21]. It was shown that, in the stereographic projection (see also [22]), the relation between Coulomb and oscillator problems functionally coincide with the flat-space Levi-Civita and Kustaanheimo–Stiefel relations.

In the present paper, we find the relation between the quasiradial Schrödinger equations for Coulomb and oscillator problems on an n-dimensional sphere and two-sheeted hyperboloids for $n \geq 2$.

2. COULOMB–OSCILLATOR RELATION ON n-SPHERE

The Schrödinger equation describing a nonrelativistic quantum motion on the n-dimensional sphere $s_0^2 + s_1^2 + \cdots + s_n^2 = R^2$, where s_i are Cartesian coordinates in the ambient $(n+1)$-dimensional Euclidean space, has the form ($\hbar = \mu = 1$)

$$\mathcal{H}\Psi = \left[-\frac{1}{2}\Delta_{\text{LB}} + V(\mathbf{s})\right]\Psi = E\Psi, \qquad (1)$$

where the Laplace–Beltrami operator in arbitrary curvilinear coordinates ξ_μ is

$$\Delta_{\text{LB}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi_\mu} g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial \xi_\nu}, \qquad (2)$$
$$g = \det||g_{\mu\nu}||, \quad g_{\alpha\mu}g^{\mu\nu} = \delta_\alpha^\nu.$$

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For any central potential $V(\chi)$, the Schrödinger equation admits separation of variables in the hyperspherical coordinates that are specified as

$$
\begin{aligned}
s_0 &= R\cos\chi, \\
s_1 &= R\sin\chi\cos\vartheta_1, \\
s_2 &= R\sin\chi\sin\vartheta_1\cos\vartheta_2, \\
&\dots \\
s_{n-1} &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{n-2}\cos\varphi, \\
s_n &= R\sin\chi\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{n-2}\sin\varphi,
\end{aligned}
$$

where $\chi, \vartheta_1, \dots, \vartheta_{n-2} \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. We can separate the angular part of the wave function using the ansatz

$$
\begin{aligned}
&\Psi(\chi, \vartheta_1, \dots, \vartheta_{n-2}, \varphi) \\
&= \mathcal{R}(\chi)\, Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \dots, \vartheta_{n-2}, \varphi),
\end{aligned}
\tag{3}
$$

where l_i are the angular hypermomenta, L is the total angular momentum, and the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \dots, \vartheta_{n-2}, \varphi)$ is a solution of the Laplace–Beltrami eigenvalue equation on an $(n-1)$-dimensional sphere. After the separation of variables in (1), we obtain the quasiradial equation

$$
\begin{aligned}
&\frac{1}{\sin^{n-1}\chi}\frac{d}{d\chi}\sin^{n-1}\chi\frac{d\mathcal{R}(\chi)}{d\chi} \\
&+ \left[2R^2E - \frac{L(L+n-2)}{\sin^2\chi} - 2R^2V(\chi)\right]\mathcal{R}(\chi) = 0.
\end{aligned}
\tag{4}
$$

Using the substitution

$$
Z(\chi) = (\sin\chi)^{(n-1)/2}\mathcal{R}(\chi), \tag{5}
$$

we find

$$
\begin{aligned}
&\frac{d^2Z}{d\chi^2} + \Big[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sin^2\chi} \\
&- 2R^2V(\chi)\Big]Z = 0,
\end{aligned}
\tag{6}
$$

where $\tilde{E} = 2R^2E + (n-1)^2/4$ and the quasiradial wave function $Z(\chi)$ satisfies the normalization condition

$$
\int\limits_0^{\pi} Z(\chi)Z^*(\chi)\, R^n\, d\chi = 1. \tag{7}
$$

(i) Let us now consider the n-dimensional oscillator potential [8, 9]

$$
\begin{aligned}
V(\chi) &= \frac{\omega^2R^2}{2}\frac{s_1^2 + s_2^2 + \dots + s_n^2}{s_0^2} \\
&= \frac{\omega^2R^2}{2}\tan^2\chi.
\end{aligned}
\tag{8}
$$

Substituting the oscillator potential into Eq. (6), we obtain the Pöschl–Teller-type equation

$$
\begin{aligned}
&\frac{d^2Z}{d\chi^2} + \Big[\epsilon - \frac{\nu^2 - 1/4}{\cos^2\chi} \\
&- \frac{(L + (n-2)/2)^2 - 1/4}{\sin^2\chi}\Big]Z = 0,
\end{aligned}
\tag{9}
$$

where $\nu = \sqrt{\omega^2R^4 + 1/4}$ and $\epsilon = \tilde{E} + \omega^2R^4$. The solution of the above equation that is regular for $\chi \in [0, \pi/2]$ and which is expressed in terms of the hypergeometric function is [23]

$$
\begin{aligned}
Z(\chi) \equiv Z^n_{n_rL\nu}(\chi) &= \sqrt{\frac{2(2n_r + L + \nu + n/2)\Gamma(n_r + L + \nu + n/2)\Gamma(n_r + L + n/2)}{R^n[\Gamma(L + n/2)]^2\Gamma(n_r + \nu + 1)(n_r)!}}, \\
&(\sin\chi)^{L+(n-1)/2}(\cos\chi)^{\nu+1/2}\,{}_2F_1\Big(-n_r, n_r + L + \nu + \frac{n}{2}; L + \frac{n}{2}; \sin^2\chi\Big),
\end{aligned}
\tag{10}
$$

and the ϵ is quantized as

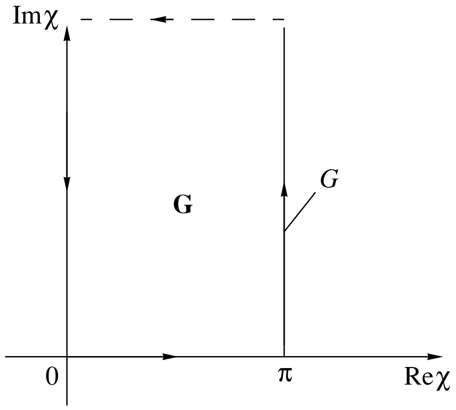
$$
\epsilon = \left(2n_r + L + \nu + \frac{n}{2}\right)^2, \tag{11}
$$

where $n_r + L = 0, 1, 2, \dots$ is a "quasiradial" quantum number. The energy spectrum of the n-dimensional oscillator is given by

$$
\begin{aligned}
E^n_N(R) = \frac{1}{2R^2}\Big[&(N+1)(N+n) \\
&+ (2\nu - 1)(N + \frac{n}{2})\Big],
\end{aligned}
\tag{12}
$$

where $N = 2n_r + L = 0, 1, \dots$ is principal quantum number. In the contraction limit where $R \to \infty$, $\chi \to 0$, and $R\chi \sim r$ is fixed and for $\nu \sim \omega R^2$, we see that

$$
\lim_{R\to\infty} E^n_N(R) = \omega\left(N + \frac{n}{2}\right) \tag{13}
$$



Domain $\mathbf{G} = \{0 \le \mathrm{Re}\ \chi \le \pi;\quad 0 \le \mathrm{Im}\ \chi < \infty\}$ in the complex plane of χ.

and

$$\lim_{R\to\infty} (R)^{(n-1)/2}\, Z^n_{NL\nu}(\chi) \tag{14}$$

$$= \frac{(\omega)^{L/2+n/4}}{\Gamma(L+n/2)} \sqrt{2\Gamma\left(\frac{N+L+n}{2}\right)\Big/\left(\frac{N-L}{2}\right)!}$$

$$\times\, r^{L+(n-1)/2} e^{-\omega r^2/2}\, {}_1F_1\left(-\frac{N-L}{2},\, L+\frac{n}{2};\, \omega r^2\right).$$

Formula (14) coincides with the known formula for n-dimensional flat radial wave functions [24].

(ii) The potential that is the analog of the Coulomb potential on the n-dimensional sphere has the form [6, 8, 9]

$$V(\chi) = -\frac{\alpha}{R}\frac{s_0}{\sqrt{s_1^2+s_2^2+\ldots+s_n^2}} = -\frac{\alpha}{R}\cot\chi. \tag{15}$$

The Schrödinger equation (6) for this potential is

$$\frac{d^2Z}{d\chi^2} + \left[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sin^2\chi} + 2\alpha R\cot\chi\right] Z = 0. \tag{16}$$

We now go over to the new variable $\theta \in [0, \pi/2]$ defined as

$$e^{i\chi} = \cos\theta. \tag{17}$$

This is possible if we continue the variable χ in the complex domain $\mathbf{G}$: Re $\chi = 0$, $0 \le$ Im $\chi < \infty$ (see figure). We also complexify the coupling constant α by setting $k = i\alpha$ with

$$\alpha\cot\chi = k(1 - 2\sin^{-2}\theta). \tag{18}$$

As a result, we obtain the equation

$$\frac{d^2W}{d\theta^2} + \left[\epsilon - \frac{\nu^2 - 1/4}{\cos^2\theta} - \frac{(2L+n-2)^2 - 1/4}{\sin^2\theta}\right] W = 0, \tag{19}$$

where $W(\theta) = (\cot\theta)^{1/2} Z(\theta)$ and

$$\epsilon = \tilde{E} + 2kR, \quad \nu^2 = \tilde{E} - 2kR. \tag{20}$$

From the above equation, we see that, apart from the substitution in (20) and the transformation $L \to 2L$, the quasiradial equation (19) for the $n^{\mathrm{Coul}} = (d+1)$-dimensional Coulomb problem coincides with the $n^{\mathrm{osc}} = 2d$-dimensional quasiradial oscillator Eq. (9). This means that relations between these two systems are possible only for oscillators in even dimensions: $n^{\mathrm{osc}} = 2, 4, 6, 8 \ldots.$

Thus, Eq. (19) describes the $2(n-1)$-dimensional oscillator quasiradial functions with even angular momentum $2L$. According to (10), the regular (for $\theta \in [0, \pi/2]$ and $\nu \le 1/4$) solution of this equation has the form

$$Z(\theta) = \frac{W(\theta)}{\sqrt{\cot\theta}} \equiv Z_{n_rL}(\theta) \tag{21}$$

$$= C^n_{n_rL}(\nu)\,(\sin\theta)^{2L+n-1}\,(\cos\theta)^{\nu} \times {}_2F_1(-n_r, n_r + 2L + \nu + n - 1;\, 2L + n - 1;\, \sin^2\theta),$$

where $C^n_{n_rL}(\nu)$ is the normalization constant. To compute the constant $C^n_{n_rL}(\nu)$ for the corresponding Coulomb quasiradial function, we require that the wave function (21) satisfy the normalization condition

$$R^n \int\limits_0^{\pi} Z_{n_rL}\, Z^{\diamond}_{n_rL}\, d\chi = 1, \tag{22}$$

where the symbol "$\diamond$" means the complex conjugate together with the inversion $\chi \to -\chi$; i.e., $Z^{\diamond}(\chi) = Z^*(-\chi)$. [We choose the scalar product as $Z^{\diamond}$ because, for $\chi \in \mathbf{G}$ and real α and $\tilde{E}$, the function $Z^{\diamond}(\chi)$ also belongs to the solution space of (16).] By analogy with what was done in [20], we consider the integral along the contour G in the complex plane of variable χ (see figure):

$$\oint Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi \tag{23}$$

$$= \int_0^{\pi} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi + \int\limits_{\pi}^{\pi+i\infty} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi + \int\limits_{\pi+i\infty}^{i\infty} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi$$

$$+\int\limits_{i\infty}^{0} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi.$$

Considering that the integrand vanishes in proportion to $e^{2i\nu\chi}$ and that $Z_{n_rL}(\chi)$ is regular in the domain **G** (see figure), we then find, with the aid of the Cauchy theorem, that

$$\int\limits_{0}^{\pi} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi \tag{24}$$
$$= \left(1 - e^{2i\pi\nu}\right) \int\limits_{0}^{i\infty} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi.$$

Making the change of variable according to (17) in the integral on the right-hand side of Eq. (24), we obtain

$$\int\limits_{0}^{\pi} Z_{n_rL}(\chi)\, Z^{\diamond}_{n_rL}(\chi) d\chi \tag{25}$$
$$= i\left(1 - e^{2i\pi\nu}\right) \int\limits_{0}^{\pi/2} [Z_{n_rL}] \tan\theta\, d\theta.$$

After integration with respect to the angle θ, we finally get [24]

$$C^{n}_{n_rm}(\nu) = \sqrt{\frac{(-2i\nu)(\nu + 2n_r + 2L + n - 1)\,(n_r)!\Gamma(2L + n_r + \nu + n - 1)}{R^n[1 - e^{2i\pi\nu}](2n_r + 2L + n - 1)\,(n_r + 2L + n - 2)!\Gamma(n_r + \nu + 1)}}. \tag{26}$$

Comparing now Eq. (11) with (20) and setting $k = i\alpha$, we get

$$\nu = -\left(n_r + L + \frac{n-1}{2}\right) + i\sigma, \tag{27}$$
$$\sigma = \frac{\alpha R}{n_r + L + (n-1)/2},$$

and obtain the energy spectrum for the Coulomb problem,

$$E_n = \frac{N(N + n - 1)}{2R^2} - \frac{\alpha^2}{2(N + (n-1)/2)^2}, \tag{28}$$
$$N = n_r + L = 0, 1, 2, \ldots$$

Returning to the variable χ, we see that the Coulomb quasiradial wave function has the form

$$Z_{NL}(\chi) = C_{NL}(\sigma)\,(\sin\chi)^{L+(n-1)/2} \tag{29}$$
$$\times \exp[-i\chi(N - L - i\sigma)]$$
$$\times {}_2F_1\Big(-N + L, L + \frac{n-1}{2}$$
$$+ i\sigma; 2L + n - 1; 1 - e^{2i\chi}\Big),$$

where the normalization constant $C_{NL}(\sigma)$ is

$$C^{n}_{NL}(\sigma) = 2^{L+(n-1)/2}\, e^{\pi\sigma/2} \tag{30}$$
$$\times \frac{|\,\Gamma(L + (n-1)/2 - i\sigma)\,|}{\Gamma(2L + n - 1)}$$
$$\times \sqrt{\frac{[(N + (n-1)/2)^2 + \sigma^2](N + L + n - 2)!}{2R^n\pi(N + (n-1)/2)(N - L)!}}.$$

Thus, by using the relation between Coulomb and oscillator systems, we have constructed the quasiradial wave functions and the energy spectrum for a Coulomb system on an n-dimensional sphere.

Finally, we note that, in the contraction limit $R \to \infty$ (for details, see [14]), it is easy to recover the well-known formulas for the flat-space n-dimensional Coulomb problem both for the discrete and for the continuous spectrum [1].

3. COULOMB–OSCILLATOR RELATION ON AN n-DIMENSIONAL TWO-SHEETED HYPERBOLOID

The pseudospherical coordinates on the n-dimensional two-sheeted hyperboloid $s_0^2 - s_1^2 - s_2^2 - \ldots - s_n^2 = R^2$, $s_0 \geq R$, are

$$s_0 = R\cosh\tau,$$
$$s_1 = R\sinh\tau\cos\vartheta_1,$$
$$s_2 = R\sinh\tau\sin\vartheta_1\cos\vartheta_2,$$
$$\ldots$$
$$s_{n-1} = R\sinh\tau\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{n-2}\cos\varphi,$$
$$s_n = R\sinh\tau\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{n-2}\sin\varphi,$$

where $\tau \in [0, \infty)$. The variables in the Schrödinger Eq. (1) may be separated for any central potential $V(\tau)$ by the ansatz

$$\Psi(\tau, \vartheta_1, \ldots, \vartheta_{n-2}, \varphi) \tag{31}$$
$$= \mathcal{R}(\tau)\, Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1, \ldots, \vartheta_{n-2}, \varphi),$$

where, as in the preceding case, l_i are the angular hypermomenta and L is the total angular momentum; the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1,\ldots,\vartheta_{n-2},\varphi)$ is a solution of the Laplace–Beltrami equation on the $(n-1)$-dimensional sphere. After the separation of variables, we obtain the quasiradial equation

$$\frac{1}{\sinh^{n-1}\tau}\frac{d}{d\tau}\sinh^{n-1}\tau\frac{d\mathcal{R}}{d\tau} + \left[2R^2E - \frac{L(L+n-2)}{\sinh^2\tau} - 2R^2V(\tau)\right]\mathcal{R} = 0. \quad (32)$$

Using now the substitution

$$Z(\tau) = (\sinh\tau)^{(n-1)/2}\mathcal{R}(\tau), \quad (33)$$

we arrive at the equation

$$\frac{d^2Z}{d\tau^2} + \Big[\tilde{E} - \frac{(2L+n-1)(2L+n-3)}{4\sinh^2\tau} - 2R^2V(\tau)\Big]Z = 0, \quad (34)$$

where $\tilde{E} = 2R^2E - (n-1)^2/4$ and the quasiradial wave function $Z(\tau)$ satisfies the normalization condition

$$\int_0^\infty Z(\tau)Z^*(\tau)\,R^n\,d\tau = 1. \quad (35)$$

(i) The oscillator potential on the two-sheeted n-dimensional hyperboloid is given by

$$V(\tau) = \frac{\omega^2R^2}{2}\frac{s_1^2+s_2^2+\ldots+s_n^2}{s_0^2} = \frac{\omega^2R^2}{2}\tanh^2\tau. \quad (36)$$

From Eq. (34), we obtain

$$\frac{d^2Z}{d\tau^2} + \Big[\epsilon + \frac{\nu^2-1/4}{\cosh^2\tau} - \frac{(L+(n-2)/2)^2-1/4}{\sinh^2\tau}\Big]Z = 0, \quad (37)$$

where $\nu = \sqrt{\omega^2R^4+1/4}$ and $\epsilon = \tilde{E} - \omega^2R^4$. Thus, the oscillator problem on a hyperboloid is described by the modified Pöschl–Teller equation; in contrast to the oscillator equation on a sphere which has only bound states, Eq. (37) possesses both bound and unbound states.

The discrete-spectrum wave functions regular on the line $\tau \in [0,\infty)$ have the form [16, 19, 25]

$$Z(\tau) \equiv Z_{n_rL}(\tau) = \frac{1}{\Gamma(L+n/2)} \times\sqrt{\frac{2(\nu-L-2n_r-n/2)\Gamma(\nu-n_r)\Gamma(n_r+L+n/2)}{R^n(n_r)!\Gamma(\nu-L-n_r-n/2+1)}} \times(\sinh\tau)^{L+\frac{n-1}{2}}(\cosh\tau)^{2n_r-\nu+\frac{1}{2}} \times {}_2F_1(-n_r,-n_r+\nu;\,L+\frac{n}{2};\,\tanh^2\tau), \quad (38)$$

with $n_r = 0,1,\ldots,\ n_r^{\max} = [(\nu-L-n/2)/2]$. The quantity ϵ is quantized as

$$\epsilon = -(2n_r+L-\nu+n/2)^2, \quad (39)$$

and the energy spectrum for a quantum oscillator on an n-dimensional two-sheeted hyperboloid is

$$E_N^n(R) = \frac{1}{2R^2}\Big[-N(N+n-1) + (2\nu-1)(N+\frac{n}{2})\Big]. \quad (40)$$

Here, $N = 2n_r + L$ is the principal quantum number, and the bound-state solution is possible only for

$$0 \le N \le \Big[\nu - \frac{n}{2}\Big]. \quad (41)$$

In the contraction limit where $R\to\infty$, $\tau\sim r/R$, and $\nu\sim\omega R^2$, we see that the continuous spectrum vanishes while the discrete spectrum is infinite, and it is easy to reproduce the oscillator energy spectrum (13) and wave function (14).

(ii) The Coulomb potential on the two-sheeted n-dimensional hyperboloid has the form [7, 12]

$$V(\tau) = -\frac{\alpha}{R}\left(\frac{s_0}{\sqrt{s_1^2+s_2^2+\ldots+s_N^2}} - 1\right) = -\frac{\alpha}{R}(\coth\tau - 1). \quad (42)$$

Substituting the potential (42) into the Schrödinger Eq. (34), we arrive at the equation

$$\frac{d^2Z}{d\tau^2} + \Big[(\tilde{E} - 2\alpha R) - \frac{(2L+n-1)(2L+n-3)}{4\sinh^2\tau} + 2\alpha R\coth\tau\Big]Z = 0, \quad (43)$$

which is known to represent the problem of the Manning–Rosen potential [26].

Making the transformation from variable τ $(0\le\tau<\infty)$ to the new variable $\mu\in[0,\infty)$,

$$e^\tau = \cosh\mu, \quad (44)$$

and setting $Z(\mu) = W(\mu)/\sqrt{\coth\mu}$, we arrive at the modified Pöschl–Teller equation

$$\frac{d^2W}{d\mu^2} + \Big[\tilde{E} + \frac{(-\tilde{E}+4\alpha R)-1/4}{\cosh^2\mu} \quad (45)$$

$$- \frac{(2L+n-2)^2 - 1/4}{\sinh^2\mu}\Big]W = 0.$$

As can be seen from Eq. (45) with the substitution

$$\epsilon = \tilde{E}, \quad \nu^2 = -\tilde{E} + 4\alpha R \tag{46}$$

and the transformation $L \to 2L$, the quasiradial equation (19) for $n^{\rm Coul} = 2(d+1)$-dimensional Coulomb problem coincides with the $n^{\rm osc} = 2d$-dimensional quasiradial oscillator Eq, (37).

Thus, the regular {for $\mu \in [0,\infty)$} solution of (43) or (45) has the form

$$Z(\mu) = \frac{W(\mu)}{\sqrt{\coth\mu}} \equiv Z^n_{n_rL}(\mu) \tag{47}$$

$$= A^n_{n_rL}(\nu)\,(\sinh\mu)^{L+n/2}\,(\cosh\mu)^{2n_r-\nu}$$

$$\times\, {}_2F_1\left(-n_r, -n_r+\nu;\, L+\frac{n}{2};\, \tanh^2\mu\right),$$

where $A^n_{n_rL}(\nu)$ is the normalization constant. The constant $A^n_{n_rL}(\nu)$ is computed from the requirement that the wave function (47) satisfy the normalization condition

$$R^n \int_0^\infty |Z^n_{n_rL}(\tau)|^2\, d\tau \tag{48}$$

$$= R^n \int_0^\infty |Z^n_{n_rL}(\mu)|^2\, \tanh\mu\, d\mu = 1$$

and has the form

$$A^n_{n_rL}(\nu) = \frac{1}{\Gamma(L+n/2)} \tag{49}$$

$$\times \sqrt{\frac{2\nu(\nu-L-2n_r-n/2)\Gamma(\nu-n_r)\Gamma(n_r+L+n/2)}{R^n(L+2n_r+n/2)(n_r)!\Gamma(\nu-L-n_r-n/2+1)}}.$$

Comparing now Eq. (46) with (39) and passing from the oscillator to the Coulomb angular quantum number, $L \to 2L$, with the substitution $n \to 2(n-1)$ of dimensions, we get

$$\nu = (n_r + L + \sigma + \frac{n-1}{2}), \tag{50}$$

$$\sigma = \frac{\alpha R}{n_r + L + (n-1)/2}.$$

Thus, the discrete energy spectrum of the Coulomb problem on an n-dimensional two-sheeted hyperboloid is described by the formula

$$E^n_N(R) = -\frac{N(N+n-1)}{2R^2} \tag{51}$$

$$- \frac{\alpha^2}{2(N+(n-1)/2)^2} + \frac{\alpha}{R},$$

where $N = n_r + L$ is the principal quantum number, and the bound states occur for

$$0 \le N \le \left[\sigma - \frac{n-1}{2}\right]. \tag{52}$$

The discrete-spectrum wave function has the form

$$Z^n_{NL}(\tau) = A^n_{NL}(\sigma)\,(\sinh\tau)^{L+(n-1)/2} \times \tag{53}$$

$$\times e^{\tau(N-L-\sigma)}{}_2F_1\Big(-N+L, L+\frac{n-1}{2}$$

$$+\sigma;\, 2L+n-1;\, 1-e^{-2\tau}\Big),$$

where the normalization constant $A^n_{NL}(\sigma)$ is

$$A^n_{NL}(\sigma) = \frac{2^{L+(n-1)/2}}{\Gamma(2L+n-1)}\sqrt{\frac{[\sigma^2-(N+(n-1)/2)^2]\Gamma(N+L+n-1)\Gamma(\sigma+L+(n-1)/2)}{R^n(N+(n-1)/2)(N-L)!\Gamma(\sigma-L-(n-1)/2+1)}}. \tag{54}$$

The solution for the Coulomb quasiradial equation, for both the energy spectrum and the wave functions, is identical to that obtained in [12] by applying the path-integral approach. We do not consider here the contraction limit $R \to \infty$ to a flat Euclidean space E_n for the Coulomb problem because this was done in [12].

It should be noted that, instead of substitution (44), it is possible to use the trigonometric transformation

$$e^{-\tau} = \cos\varphi, \quad \varphi \in [0, \pi/2]. \tag{55}$$

It is easy to see that, apart from the permutation

$$\epsilon = -\tilde{E} + 4\alpha R, \quad \nu^2 = -\tilde{E} \tag{56}$$

and the transformation $L \to 2L$, the quasiradial Eq. (43) for the $n^{\rm Coul} = (d+1)$-dimensional Coulomb problem goes over to the $n^{\rm osc} = 2d$-dimensional quasiradial oscillator Eq. (9). Thus, the Coulomb problem on a two-sheeted hyperboloid is related to the oscillator problem on a sphere or a two-sheeted hyperboloid.

4. COULOMB–OSCILLATOR RELATION ON AN n-DIMENSIONAL ONE-SHEETED HYPERBOLOID

The pseudospherical coordinates on the n-dimensional one-sheeted hyperboloid $s_0^2 - s_1^2 - s_2^2 - \ldots$

$-\, s_n^2 = -R^2$ are

$$s_0 = R\sinh\tau,$$
$$s_1 = R\cosh\tau\cos\vartheta_1,$$
$$s_2 = R\cosh\tau\sin\vartheta_1\cos\vartheta_2,$$
$$\dots$$
$$s_{n-1} = R\cosh\tau\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{n-2}\cos\varphi,$$
$$s_n = R\cosh\tau\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{n-2}\sin\varphi,$$

where $\tau\in(-\infty,\infty)$. The variables in the Schrödinger Eq. (1) may be separated by using the ansatz (31)

$$\Psi(\tau,\vartheta_1,\dots,\vartheta_{n-2},\varphi) = \mathcal{R}(\tau)\,Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1,\dots,\vartheta_{n-2},\varphi),$$

where, as in the preceding case l_i are the angular hypermomenta, L is total angular momentum, and the hyperspherical function $Y_{L,l_1,l_2,l_{n-2}}(\vartheta_1,\dots,\vartheta_{n-2},\varphi)$ is a solution of the Laplace—Beltrami equation on the $(n-1)$-dimensional sphere. After the separation of variables, we obtain the quasiradial equation

$$\frac{1}{\cosh^{n-1}\tau}\frac{d}{d\tau}\cosh^{n-1}\tau\frac{d\mathcal{R}}{d\tau} + \left[2R^2E + \frac{L(L+n-2)}{\cosh^2\tau} - 2R^2V(\tau)\right]\mathcal{R} = 0. \tag{57}$$

Using now the substitution

$$Z(\tau) = (\cosh\tau)^{(n-1)/2}\mathcal{R}(\tau), \tag{58}$$

we arrive at the equation

$$\frac{d^2Z}{d\tau^2} + \left[\tilde{E} + \frac{(2L+n-1)(2L+n-3)}{4\cosh^2\tau} - 2R^2V(\tau)\right]Z = 0, \tag{59}$$

where $\tilde{E} = 2R^2E - (n-1)^2/4$ and the quasiradial wave function $Z(\tau)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty} Z(\tau)Z^*(\tau)\,R^n\,d\tau = 1. \tag{60}$$

(i) The oscillator potential on an n-dimensional one-sheeted hyperboloid is given by

$$V(\tau) = \frac{\omega^2R^2}{2}\,\frac{s_1^2+s_2^2+\ldots+s_n^2}{s_0^2} = \frac{\omega^2R^2}{2}\coth^2\tau. \tag{61}$$

For Eq. (59), we then have

$$\frac{d^2Z}{d\tau^2} + \left[\epsilon + \frac{(L+(n-2)/2)^2 - 1/4}{\cosh^2\tau} - \frac{\nu^2 - 1/4}{\sinh^2\tau}\right]Z = 0, \tag{62}$$

where $\nu = \sqrt{\omega^2R^4 + 1/4}$ and $\epsilon = \tilde{E} - \omega^2R^4$. As in the preceding case, the oscillator system is described by the modified Pöschl—Teller equation and possesses a discrete and a continuous spectrum. Since, however, the situation here is different from that in the case of motion on a two-sheeted hyperboloid, the number of bound states depends on the total angular momentum. The discrete-state wave functions regular on the line $\tau\in(-\infty,\infty)$ are

$$Z(\tau) \equiv Z_{n_rL}(\tau) = \sqrt{\frac{(L-\nu-2n_r+n/2-2)\Gamma(L-n_r+n/2-1)\Gamma(n_r+\nu+1)}{R^n(n_r)![\Gamma(\nu+1)]^2\Gamma(L-\nu-n_r+n/2-1)}} \times (\sinh\tau)^{\nu+\frac{1}{2}}(\cosh\tau)^{2n_r-L-n/2+3/2}{}_2F_1\left(-n_r, -n_r+L+\frac{n}{2}-1;\,\nu+1;\,\tanh^2\tau\right), \tag{63}$$

and

$$\epsilon = -(2n_r - L + \nu - \frac{n}{2} + 2)^2, \tag{64}$$

where the bound states occur for $n_r = 0,1,\dots,$ $n_r^{\max} = [(L-\nu+n/2-2)/2]$. The last formula means that the discrete spectrum depends on the quantum number L, and the energy spectrum of the oscillator system takes the form

$$E_{n_rL}(R) = -\frac{1}{2R^2}\Big[(2n_r - L + 2)(2n_r - L - n + 3) + (2\nu-1)(2n_r - L - \frac{n}{2} + 2)\Big]. \tag{65}$$

(ii) The Coulomb potential on the n-dimensional hyperboloid has the form [7, 12]

$$V(\tau) = -\frac{\alpha}{R}\left(\frac{s_0}{\sqrt{s_1^2+s_2^2+\ldots+s_n^2}} + 1\right) = -\frac{\alpha}{R}(\tanh\tau + 1). \tag{66}$$

The Schrödinger equation for this potential is

$$\frac{d^2Z}{d\tau^2} + \Big[(\tilde{E} + 2\alpha R) \tag{67}$$

$$+\frac{(2L+n-1)(2L+n-3)}{4\cosh^2\tau}+2\alpha R\tanh\tau\Big]Z=0,$$

which coincides with the Rosen–Morse equation [25].

Making the transformation from the variable τ $(-\infty<\tau<\infty)$ to the new variable $\mu\in[0,\infty)$,

$$e^\tau=\sinh\mu, \tag{68}$$

we arrive at the equation

$$\frac{d^2W}{d\mu^2}+\Big[(\tilde{E}+4\alpha R) \tag{69}$$

$$+\frac{(2L+n-2)^2-1/4}{\cosh^2\mu}-\frac{(-\tilde{E})-1/4}{\sinh^2\mu}\Big]W=0,$$

where $W(\mu)=(\tanh\mu)^{1/2}Z(\mu)$. From this equation, we see that, apart from the substitution

$$\tilde{E}\to\tilde{E}+4\alpha R,\quad \nu^2=-\tilde{E}, \tag{70}$$

and the simultaneous transformation $L\to 2L$ for total angular momentum, the quasiradial Eq. (69) for the Coulomb problem on a $n^{\rm Coul}=(d+1)$-dimensional one-sheeted hyperboloid coincides with the $n^{\rm osc}=2d$-dimensional quasiradial oscillator Eq. (62).

Comparing now Eq. (69) with (62) and taking into account Eqs. (64) and (70), we see that the discrete-spectrum wave function satisfying the normalization condition

$$R^n\int\limits_{-\infty}^{\infty}|Z^n_{n_rL}(\tau)|^2\,d\tau \tag{71}$$

$$=R^n\int\limits_{-\infty}^{\infty}|Z^n_{n_rL}(\mu)|^2\,\coth\mu\,d\mu=1$$

has the form

$$Z^n_{n_rL}(\tau)=\frac{2^{n_r-L-\frac{n}{2}}}{\Gamma(L-n_r+\frac{n-1}{2})}\sqrt{\frac{[(L-n_r+(n-3)/2)^2-\sigma^2]\Gamma(2L-n_r+n-2)\Gamma(L+(n-1)/2)}{R^n(L-n_r+(n-3)/2)(n_r)!\Gamma(L-\sigma+1/2)}} \tag{72}$$

$$\times(\cosh\tau)^{n_r-L-(n-1)/2}\,(e)^{\tau(\sigma-1)}\times{}_2F_1\left(-n_r,-n_r+L+n-2;\,L-n_r+\frac{n-3}{2}+\sigma;\,\frac{1}{1+e^{-2\tau}}\right),$$

with the discrete energy spectrum of the Coulomb problem being described by the formula

$$E_n=-\frac{(L-n_r-1)(L-n_r+n-2)}{2R^2} \tag{73}$$

$$-\frac{\alpha^2}{2(L-n_r+(n-3)/2)^2}-\frac{\alpha}{R}.$$

Bound states occur for $n_r=0,1,\ldots,n_r^{\max}=[(L+(n-3)/2+\sigma)]$.

Finally, we note that, in contrast to a sphere and a two-sheeted hyperboloid, the contraction limit $R\to\infty$ on one-sheeted hyperboloids is meaningless for the oscillator and Coulomb problems.

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