Coupling constant metamorphosis and nth order symmetries for superintegrable systems

with a survey of recent structure theory and classification results for superintegrable systems

Willard Miller, Jr., University of Minnesota

Many results joint with E. G. Kalnins, J. M. Kress, G. Pogosyan and S. Post

Superintegrable Systems

In n dimensions, a classical or quantum Hamiltonian

$$\mathcal{H} = \sum_{i,j=0}^{n} g^{ij} p_i p_j + V(x_i), \qquad H = \hbar^2 \Delta + V(x_i, \hbar)$$

is (maximally) Superintegrable if it admits $2n-1$ symmetry operators, i.e.

$$\{\mathcal{L}_i, \mathcal{H}\} = 0, \quad \text{or} \quad [L_i, H] = 0, \quad \forall i = 0, \ldots, 2n-1$$

which are polynomial in the momenta or finite order differential operators. Symmetries chosen lowest order possible.

We also require that these operators be independent in a sense based on the functional independence of the classical symmetries. Maximum order of generating symmetries $=$ order of superintegrable system.

Issue 1

Most researchers assume that a superintegrable system in n dimensions is also integrable, i.e., n of the generating symmetries are in involution:

$$\{\mathcal{L}_i, \mathcal{L}_j\} = 0, \quad 1 \le i < j \le n.$$

Are there superintegrable systems that are not integrable?

Associated Polynomial Algebras

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Often, if we include these new symmetries in our algebra, it will close, i.e.

$$\{\mathcal{L}_i, R_{ij}\} = P(\mathcal{L}_i), \qquad P \text{ is a polynomial.}$$

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These relations together determine the symmetry algebra.

The Algebras

We are interested in these polynomial algebras, particularly their representations and classifications.

Why?

- To understand classes of superintegrable systems with equivalent symmetry algebra.
- To get information about the spectrum of the symmetry operators and quantization of the energy values.
- The representations are related to special function theory-Ex. Wilson, dual Hahn, Jacobi Polynomials, Airy functions, etc.

Quadratic algebra example

The Euclidean space operator in 2D Cartesian coordinates:

$$H = \Delta - \omega^2(x^2+y^2) + \frac{\frac{1}{4}-a^2}{x^2} + \frac{\frac{1}{4}-b^2}{y^2}$$

The 2nd order symmetries L_1, L_2 generate symmetry algebra,

$$[L_1, L_2] = R, \qquad [R, L_1] = 8L_1^2 - 8HL_1 + 16\omega^2 L_2 - 8\omega^2,$$

$$[R, L_2] = 8HL_2 - 8\{L_1, L_2\} + (12-16a^2)H + (16a^2+16b^2-24)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_2\} - 8H\{L_1, L_2\} + (16b^2+16a^2-\frac{200}{3})L_1^2 - 16\omega^2 L_2^2 + (32a^2-\frac{200}{3})HL_1$$

$$+(16-16a^2)H^2 + \frac{176}{3}\omega^2 L_2 = -64\omega^2(b^2+a^2-b^2a^2-\frac{29}{48}).$$

In an irreducible representation, H, ω^2, a, b are all parameters of our representation.

2nd order superintegrability

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1) The generating symmetries are functionally linearly dependent:

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Then the potential must satisfy a linear 1st order PDE $XV = 0$.
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2) The generating symmetries are functionally linearly independent. Then the potentials must satisfy a system of $(n+2)(n-1)/2$ 2nd order linear PDEs.

2nd order superintegrability 2

Example for $n = 2$ in Cartesian-like coordinates

$$V_{yy} - V_{xx} = A^{22}V_x + B^{22}V_y, \; V_{xy} = A^{12}V_x + B^{12}V_y.$$

The solutions of these potential equations form a vector space of order $\leq n + 2$. One dimension corresponds to the trivial addition of a constant to the potential.

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If the integrability conditions for these equations are satisfied identically then the potential is $n + 1$-parameter (nondegenerate). Potentials depending on fewer parameters are degenerate.

2nd order superintegrability 3

For $n = 2$ all spaces and potentials are known.

0,1 (1st order symmetry),and 3-parameter potentials occur.

For 1-parameter potentials, quadratic algebra closes at order 4. For nondegenerate (3-parameter potentials) quadratic algebra closes at order 6.

All systems are Stäckel equivalent to constant curvature space systems.

2nd order superintegrability 4

Closure of the quadratic algebra is due to dimensional considerations.

For $n = 2$ all quadratic algebras are known and their physically important irreducible representations have been worked out (thesis of S. Post).

Function space models of the irreducible representations have been worked out. (differential or difference operators in 1 variable).

2nd order superintegrability 5

Close connection with special functions. Generic nondegenerate potential on 2-sphere has quadratic algebra generated by the recurrences of the most general Wilson polynomials.

For $n = 3$ probably all nondegenerate (4-parameter) potentials are known

There are 6 linearly independent 2nd order symmetries for all 4-parameter potentials.

The quadratic algebra generated by the 6 symmetries closes at order 6. There is an order 8 functional relationship between the 6 generators.

All 4-parameter systems are Stäckel equivalent

2nd order superintegrability 6

Many examples but only modest structure theory for degenerate potentials with $n = 3$. It is known that true 3-parameter potentials (not extendable to 4 parameters) are multiseparable but their symmetry algebras do not close at order 6. (rational closure?)

Many examples for $n > 3$ and some theory for the nondegenerate cases.

superintegrability of order ≥ 3

Some classification of 3rd order 2D Euclidean space systems where there is a 2nd order constant associated with separation in either Cartesian or polar coordinates (Gravel, Winternitz) Examination of the representation theory of the associated cubic algebras (Marquette, Winternitz).

The Dracht list of 2D 3rd order systems is incomplete.

superintegrability of order ≥ 3 2

Virtually no structure theory known. A major complication for 3rd and higher order superintegrability is that the differential equations for the potentials are nonlinear.

There has been a recent explosion in the number of examples of higher order superintegrable systems on a variety of spaces, (some not conformally flat) so it is to be expected that structure theory results will folllow

Stäckel Transform

Consider the Hamiltonian-Jacobi equation,

$$\mathcal{H} = \mathcal{H}_0 + V + \alpha U = E$$

we make the transform

$$\tilde{\mathcal{H}} \equiv \frac{1}{U}(H_0 + V - E) = \alpha$$

This gives a new Hamiltonian on a (possibly) new manifold.

Q: Does the new Hamiltonian have the same symmetry structure?

That is, can we define a transform that will take the symmetries of H to (well defined) symmetries of $\tilde{H}$?

Previous Results

For second order superintegrable systems we already have a transform which accomplishes our goals, the Stäckel transform. It allows us to:

- Determine the metrics which admit multiple symmetry operators for the potential free case. (Koenigs)
- Transform all superintegrable systems to ones on spaces of constant curvature.

Second Order Stäckel transform

We separate the Hamiltonian as,

$$\mathcal{H} = \mathcal{H}_0 + V + \alpha U \qquad \mathcal{L} = \mathcal{L}_0 + W + \alpha W_U$$

with, $\mathcal{H}_0$ and $\mathcal{L}_0$ the free parts and

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with, $\mathcal{H}_0$ and $\mathcal{L}_0$ the free parts and

$$\{\mathcal{H}_0 + V, \mathcal{L}_0 + W\} = \{\mathcal{H}, \mathcal{L}\} = 0$$

then we have new constants of the motion, $\{\tilde{\mathcal{H}}, \tilde{\mathcal{L}}\} = 0$

$$\tilde{\mathcal{H}} = \frac{\mathcal{H}_0 + V}{U} \qquad \tilde{\mathcal{L}} = \mathcal{L}_0 + W - W_U \tilde{\mathcal{H}}.$$

Here $\tilde{H}$ is the new Hamiltonian, possibly on a new manifold.

Symmetry Algebra

This gives an equivalence of superintegrable systems with similar symmetry algebras.

$$\mathcal{H} = \mathcal{H}_0 + V + \alpha U = E \iff \tilde{\mathcal{H}} = \frac{\mathcal{H}_0 + V - E}{U} = \alpha$$

The symmetry algebras of second order superintegrable systems are unchanged by the Stäckel transform except for interchanging α and E.

Example

The harmonic oscillator and the Kepler system are related by an example of a Stäckel transform.

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$$\partial_x^2 + \partial_y^2 + \alpha(x^2 + y^2) = E$$

Solving for α

$$\frac{1}{x^2+y^2}\left(\partial_x^2 + \partial_y^2 - E\right) = -\alpha$$

$\tilde{H} = \frac{H-E}{U}$

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The new energy

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Using the change of variables $\nu = \frac{1}{4}(x^2 - y^2), \qquad \zeta = xy$ we obtain

$$\tilde{H} = \partial_\nu^2 + \partial_\zeta^2 - \frac{E}{2\sqrt{\nu^2 + \zeta^2}}.$$

Example cont.

The symmetry operators for the SHO are

$$X = M = x\partial_y - y\partial_x$$

$$L_1 = \frac{1}{2}(M\partial_x + \partial_x M) - \frac{\alpha y}{2\sqrt{x^2+y^2}}, \quad L_2 = \frac{1}{2}(M\partial_y + \partial_y M) - \frac{\alpha x}{2\sqrt{x^2+y^2}}.$$

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The transformed constants are

$$\tilde{X} = M = x\partial_y - y\partial_x$$

$$\tilde{L_1} = \frac{1}{2}(M\partial_x + \partial_x M) - \frac{\tilde{H} y}{2\sqrt{x^2+y^2}}, \quad \tilde{L_2} = \frac{1}{2}(M\partial_y + \partial_y M) - \frac{\tilde{H} x}{2\sqrt{x^2+y^2}}.$$

Example cont.

The symmetry algebra for the SHO is

$$[L_1, X] = L_2 \qquad [L_2, X] = -L_1 \qquad [L_1, L_2] = -\alpha X$$

$$L_1^2 + L_2^2 + \alpha X^2 - \frac{\alpha}{4} - \frac{E^2}{16} = 0.$$

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The symmetry algebra for the Coulomb system

$$[\tilde{L}_1, \tilde{X}] = \tilde{L}_2 \qquad [\tilde{L}_2, \tilde{X}] = -\tilde{L}_1 \qquad [\tilde{L}_1, \tilde{L}_2] = \tilde{E}\tilde{X}$$

$$\tilde{L}_1^2 + \tilde{L}_2^2 - \tilde{E}\tilde{X}^2 + \frac{\tilde{E}}{4} - \frac{\tilde{\alpha}^2}{4} = 0.$$

Example cont.

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$$\tilde{L}_1^2 + \tilde{L}_2^2 - \tilde{E}\tilde{X}^2 + \frac{\tilde{E}}{4} - \frac{\tilde{\alpha}^2}{4} = 0.$$

These are identical under the identifications
$\tilde{L}_1 = L_1, \quad \tilde{L}_2 = L_2, \quad \tilde{E} = -\alpha, \quad \tilde{\alpha} = -\frac{1}{2}E.$

Example cont.

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$$[\tilde{L}_1, \tilde{X}] = \tilde{L}_2 \qquad [\tilde{L}_2, \tilde{X}] = -\tilde{L}_1 \qquad [\tilde{L}_1, \tilde{L}_2] = \tilde{E}\tilde{X}$$

$$\tilde{L}_1^2 + \tilde{L}_2^2 - \tilde{E}\tilde{X}^2 + \frac{\tilde{E}}{4} - \frac{\tilde{\alpha}^2}{4} = 0.$$

We can use the Stäckel transfrom to understand symmetry algebras that differ by a permutation of the structure constants!

CCM

How can we generalize this to Nth order superintegrable systems?

Answer 1: Coupling constant metamorphosis (CCM) J. Hietarinta, B. Grammaticos, B. Dorizzi and A. Ramani (1984).

Let $\mathcal{H}(\mathbf{x}, \mathbf{p}) + \alpha U(\mathbf{x}) = E$ define a Hamiltonian system in $2n$ dimensional phase space. Assume that for every value of the parameter α the system admits a constant of the motion $\mathcal{K}(\alpha)$, analytic in α.

Theorem 1 *The Hamiltonian* $\mathcal{H}' = (\mathcal{H} - E)/U$ *admits the constant of the motion* $\mathcal{K}' = \mathcal{K}(-\mathcal{H}')$, *where now* E *is a parameter.*

Drawback

In general, the constant of the motion need not be polynomial in α and so the transformed constant might not be polynomial in the momenta.

Example-Gravel (2004)

$$\mathcal{H} = p_1^2 + p_2^2 + b_1\sqrt{x_1} + b_2 x_2$$

admits

$$\mathcal{K}^{(2)} = p_2^2 + b_2 x_2 \qquad \mathcal{K}^{(3)} = p_1^3 + \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3b_1^2}{4b_2} p_2$$

The choice $\alpha = b_2$ leads to symmetry a symmetry operator which is a rational function of the momenta.

Attempt 2: Jacobi Transform

Suppose the system $\mathcal{H} = \mathcal{H}_0 + \alpha U$ admits an Nth order constant of the motion $\mathcal{K}(\alpha)$ analytic in α.

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$\mathcal{H}_0$ is the free Hamiltonian-Here $V \equiv 0$. This is a strong restriction!

Attempt 2: Jacobi Transform

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Then $\mathcal{K}(\alpha)$ may be taken to be off the form

$$\mathcal{K} = \sum_{j=0}^{[N/2]} (\alpha)^j \, \mathcal{K}_{N-2j}.$$

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and CCM gives us the transformed operator

$$\hat{\mathcal{K}} = \sum_{j=0}^{[N/2]} \left(-\frac{\mathcal{H}_0 - E}{U}\right)^j \mathcal{K}_{N-2j}$$

which is an Nth order constant of the motion for the system

$$\tilde{\mathcal{H}} = (\mathcal{H}_0 - E)/U.$$

Example

Lets return to the example before: $\mathcal{H} = p_1^2 + p_2^2 + b_1\sqrt{x_1} + b_2 x_2$, and let $U = b_1\sqrt{x_1} + b_2 x_2 + b_3$ for some fixed b_1, b_2, b_3 with $b_1 b_2 \neq 0$.

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$$\mathcal{K}^{(3)} = p_1^3 + \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3b_1^2}{4b_2} p_2$$

So the coefficients of p_1 and p_2 are linear in the scaling constant α.

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The new Hamiltonian is

$$\hat{\mathcal{H}} = \frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2 x_2 + b_3}.$$

and the Jacobi transforms of $\mathcal{K}^{(2)}, \mathcal{K}^{(3)}$ are

$$\hat{\mathcal{K}}^{(2)} = p_2^2 - b_2 x_2 \left(\frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2 x_2 + b_3} \right),$$

$$\hat{\mathcal{K}}^{(3)} = p_1^3 - (\frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3b_1^2}{4b_2} p_2) \left(\frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2 x_2 + b_3} \right).$$

Jacobi Transform

Pro: The Jacobi transform will work for all Superintegrable systems with a scalable potential

Jacobi Transform

Pro: The Jacobi transform will work for all Superintegrable systems with a scalable potential

Con: It is not the most general transform since we must divide by the whole potential instead of an arbitrary piece

Stäckel Transform generalized

What if we want instead to just split off a piece of the potential?

If we assume that $V(\mathbf{x}, \mathbf{b})$ is linear in $\mathbf{b} = (b_0, b_1, \cdots b_M)$, U is of the form $U(\mathbf{x}) = V(\mathbf{x}, \mathbf{b}^0)$ and the potentials $V(\mathbf{x}, \mathbf{b})$ span a space of dimension $M + 1$. In particular,

$$V(\mathbf{x}, \mathbf{b}) = b_0 + \sum_{i=1}^{M} U^{(i)}(\mathbf{x}) b_i$$

where the set of functions $\{1, U^{(1)}(\mathbf{x}), \cdots, U^{(M)}(\mathbf{x})\}$ is linearly independent.

Note: this is always the case for second order symmetry operators.

Stäckel Transform generalized

We consider only those higher order constants of the motion of order N of the form

$$\mathcal{K} = \sum_{j=0}^{[N/2]} \mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b})$$

where $\mathcal{K}_{N-2j}(a\mathbf{p}, \mathbf{b}) = a^{N-2j}\mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b})$ and $\mathcal{K}_{N-2j}(\mathbf{p}, a\mathbf{b}) = a^{j}\mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b})$ for any parameter a.

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Then

$$\mathcal{K}(\alpha) \equiv \mathcal{K}(\mathbf{p}, \mathbf{b} + \alpha\mathbf{b}^{(0)})$$

is an Nth order constant of the motion for the system with Hamiltonian

$$\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b}) + \alpha U(\mathbf{x})$$

Stäckel Transform generalized

Now, we are ready to perform the Stäckel transform.

Theorem 2 *Let $\mathcal{K}$ be an Nth order constant of the motion for the system $\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b})$ of the form above, Then*

$$\tilde{\mathcal{K}} = \mathcal{K}\left(-\frac{\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b})}{U(\mathbf{x})}\right) = \sum_{j=0}^{[N/2]} \tilde{\mathcal{K}}_{N-2j}(\mathbf{p}, \mathbf{b})$$

is an Nth order constant of the motion for the system

$$\tilde{\mathcal{H}} = (\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b}))/U(\mathbf{x})$$

Ex. Tremblay, Turbiner, Winternitz (20

The Hamiltonian and the constants are,

$$\mathcal{H} = p_1^2 + p_2^2 + a(x_1^2 + x_2^2) + b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2}$$

$$\mathcal{K}_2 = (x_1 p_2 - x_2 p_1)^2 + 4b\frac{x_1^2 x_2^2}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^4 + x_2^4)}{x_1^2 x_2^2}$$

$$\mathcal{K}_4 = (p_1^2 - p_2^2)^2 + [2ax_1^2 + 2b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} - 2c\frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2}]p_1^2$$

$$+[-4ax_1x_2 + 8b\frac{x_1 x_2}{(x_1^2 - x_2^2)^2}]p_1p_2 + [2ax_2^2 + 2b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + 2c\frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2}]p_2^2$$

$$+a^2(x_1^2 - x_2^2)^2 + \frac{b^2}{(x_1^2 - x_2^2)^2} + c^2\frac{(x_1^2 - x_2^2)^2}{x_1^4 x_2^4} + 8ab\frac{x_1^2 x_2^2}{(x_1^2 - x_2^2)} + 2\frac{bc}{x_1^2 x_2^2}$$

Notice it has the required scaling relations

TTW cont.

These constants of the motion generate a closed Poisson algebra. Let $\mathcal{R} = \{\mathcal{K}_2, \mathcal{K}_4\}$. The relations are

$$\{\mathcal{K}_2, \mathcal{R}\} = 32(\mathcal{H}^2 - 2\mathcal{K}_4)\mathcal{K}_2 - 64(b+2c)\mathcal{K}_4 + 64(b-c)\mathcal{H}^2 - 128ab\mathcal{K}_1 - 128ab(b+2c),$$

$$\{\mathcal{K}_4, \mathcal{R}\} = 32\mathcal{K}_4(\mathcal{K}_4 - \mathcal{H}^2) + 128a\mathcal{K}_2\mathcal{H}^2 - 384a^2\mathcal{K}_2^2 + 128ab\mathcal{K}_4 - 64(b+4c)a\mathcal{H}^2$$
$$+256a^2(2c-b)\mathcal{K}_2 + 128a^2(b^2 + 40c^2 + 20bc)$$

There is a Casimir constraint

$$\mathcal{R}^2 = 64\mathcal{K}_2\mathcal{K}_4(\mathcal{H}^2 - \mathcal{K}_4) - 64b\mathcal{H}^4 + 128(b-c)\mathcal{K}_4\mathcal{H}^2 - 64(b+2c)\mathcal{K}_4^2 - 128a\mathcal{K}_2^2\mathcal{H}^2 +$$
$$256a^2\mathcal{K}_2^3 - 256ab\mathcal{K}_2\mathcal{K}_4 + 128a(b+4c)\mathcal{H}^2\mathcal{K}_2 + 256a^2(b-c)\mathcal{K}_2^2 - 256ab(b+2c)\mathcal{K}_4$$
$$+256a(7bc + b^2 - 2c^2)\mathcal{H}^2 - 256a^2(b^2 + 4c^2 + 20bc)\mathcal{K}_2 - 256a^2(2c+b)(b^2 + 16bc - 4c^2).$$

Ex.-TTW cont.

The Stäckel transformed Hamiltonian is

$$\tilde{\mathcal{H}} = \frac{p_1^2 + p_2^2 + a(x_1^2 + x_2^2) + b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + c\frac{(x_1^2+x_2^2)}{x_1^2x_2^2} + d}{(x_1^2 + x_2^2) + B\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + C\frac{(x_1^2+x_2^2)}{x_1^2x_2^2} + D}$$

is also superintegrable with 4th and 2nd order generating constants of the motion.

Ex.-TTW cont.

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This system is on a new space of non-constant curvature.

Ex.-TTW cont.

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is also superintegrable with 4th and 2nd order generating constants of the motion.

This system is on a new space of non-constant curvature.
The symmetry algebra is also preserved but the constants and energy change places again.

Though it seems as though we have split off all of the potential, we can make some choices of A, B, C, D so that we are effectively only splitting off a piece.

For example, the choice

$$U = (x_1^2 + x_2^2) + 4$$

gives us a superintegrable system on a Darboux space of type 3.

Quantum Stäckel transform

... is harder because the equation $[H, K] = 0$ doesn't naturally split as before

Consider the system

$$H = H_0 + V + \alpha U \qquad K(\alpha) = K_3' + K_1'$$

The requirement $[H, K] = 0$ is complicated, but we can use the fact that V and H_0 are independent of α to write

$$K(\alpha) = K_3 + \alpha K_1$$

which gives the equations

$$[K_3, H_0] = 0, \quad [K_3, U] + [K_1, H_0] = 0, \quad [K_1, U] = 0.$$

Quantum Stäckel transform cont.

The Stäckel transformed system is

$$\tilde{H} = U^{-1}(H_0 + b) \qquad \tilde{K} = K_3 - K_1 U^{-1}(H_0 + b),$$

where the operator order is important and b is a constant. A straightforward computation using the identities above yields $[\tilde{K}, U^{-1}(H_0 + b)] = 0.$

Furthermore, the transformed symmetry operator remains skew-adjoint.

This requires that there be a part of the potential which is scalable. For 3rd order quantum superintegrable systems, there are potentials which have non-scalable parts.

Example-Gravel

Consider the superintegrable system

$$H(0) = \partial_{11} + \partial_{22} + a(9x_1^2 + x_2^2) - 2/x_2^2$$

With a scalable part $U = (9x_1^2 + x_2^2) + c$. The symmetry operators are,

$$L = \partial_{22} + ax_2^2, \quad K = \{x_1\partial_2 - x_2\partial_1, \partial_{22}\} + \{\frac{a}{3}x_2^3 + \frac{1}{x_2}, \partial_1\} - \{3x_1(ax_2^2 + \frac{1}{x_2^2}), \partial_2\}.$$

The transformed Hamiltonian is,

$$\tilde{H} = \frac{1}{(9x_1^2 + x_2^2) + c}\left(\partial_{11} + \partial_{22} + a(9x_1^2 + x_2^2) - \frac{2}{x_2^2} + b\right).$$

The relation with the classical system is more evident by taking the normalization $H(0) = -(\hbar^2/2)(\partial_{11} + \partial_{22}) + a(9x_1^2 + x_2^2) + \hbar^2/x_2^2$

Nth order quantum

For the most general Nth order symmetries, that is

$$K(\alpha) = K'_N + \sum_{j=1}^{[n/2]} K'_{N-2j}(\alpha)$$

where each K'_{N-2j} is self-adjoint or skew-adjoint, depending on the parity of N. We require the K'_{N-2j} can be written in the form

$$K'_{N-2j} = \sum_{i=0}^{j} A^{(i)}_{N-2j}\alpha^i, \quad j = 0, 1, \cdots, [N/2]$$

where the $A^{(i)}_{N-2j}$ are independent of α and modulo terms of the form $f^{(i)}(\alpha)L_{N-2j}$ where L_{n-2j} is an α-independent symmetry of $H + \alpha U$ for $j > 0$. It is not clear that this covers all possible symmetries

Nth order quantum cont.

Theorem 3 *Let* $H(\alpha) = H_0 + V + \alpha U \equiv H + \alpha U$ *and* $N \geq 2$.

Let $K(\alpha)$ *be a nonzero* N*th order operator symmetry of* $H(\alpha)$ *analytic at* $\alpha = 0$, *self-adjoint for even* N *and skew-adjoint for odd* N.

Suppose further there are operators A_{N-2j} *such that the identities above are satisfied.*

Then, we can define K_{N-2h} *so that the operator* $\tilde{K} = \sum_{h=0}^{[N/2]}(-1)^h K_{N-2h}(U^{-1}(H+\alpha))^h$ *is an* N*th order symmetry for the system* $\tilde{H} = U^{-1}(H+\alpha)$ *with the appropriate adjoint properties.*

Example: TTW1 (k=2)

Let

$$H = \partial_{11} + \partial_{22} + a(x_1^2 + x_2^2) + b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2}$$

There are two basic self-adjoint symmetry operators, one of 2nd order,

$$K_2 = (x_1\partial_2 - x_2\partial_1)^2 + 4b\frac{x_1^2 x_2^2}{(x_1^2 - x - 2^2)^2} + c\frac{(x_1^4 + x_2^4)}{x_1^2 x_2^2}$$

and one of 4th order,

$$K_4 = (\partial_{11} - \partial_{22})^2 + [2ax_1^2 + 2b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} - 2c\frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2}]\partial_{11}$$

Example:: TTW2 (k=2)

These operators generate a closed symmetry algebra. Let $R = [K_2, K_4]$.The relations are

$$[K_2, R] = 32H^2K_2 - 32\{K_4, K_2\} - 64(b+2c+4)K_4 +$$
$$64(b-c+2)H^2$$
$$-128a(b+1)K_1 - 128a(b^2+2bc+4b+6c+4),$$
$$[K_4, R] = 32K_4^2 - 32H^2K_4 + 128aK_2H^2 - 384a^2K_2^2$$
$$+128a(b+1)K_4 - 64a(b+4c+6)H^2$$
$$+256a^2(2c-b+14)K_2 + 128a^2(b^2+4c^2+20bc+18b+8c-8),$$

$R^2 = \cdots$ at order 10.

Example: TTW4 (k=2)

Then the Stäckel transformed system

$$\tilde{H} = \frac{1}{(x_1^2 + x_2^2) + B\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + C\frac{(x_1^2+x_2^2)}{x_1^2 y_2^2} + D} \Big(\partial_{11} + \partial_{22} + a(x_1^2 + x_2^2) + b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2} + d \Big)$$

is also superintegrable with 4th and 2nd order generating self-adjoint symmetries.

Higher order systems: TTW (k)

In polar coordinates for 2D Euclidean space,

$$V = \alpha r^2 + \frac{\beta}{r^2\cos^2(k\theta)} + \frac{\gamma}{r^2\cos^2(k\theta)}.$$

In terms of the new variable $r = e^R$ the Hamiltonian is

$$H = e^{-2R}(p_R^2 + p_\theta^2 + \alpha e^{4R} + \frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)}).$$

Higher order systems 2: TTW (k)

We construct functions $M(R, p_R)$, $N(\theta, p_\theta)$ satisfying $\{M, H\} = e^{-2R}$ and $\{N, H\} = e^{-2R}$.

The separation equations are

$$L_2 = p_\theta^2 + \frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)}$$

$$p_R^2 + L_2 + \alpha e^{4R} - e^{2R} H = 0.$$

$M - N$ is a constant of the motion.

Higher order systems 3: TTW (k)

These equations have solutions

$$M = \frac{i}{4\sqrt{L_2}}B, \quad N = -\frac{i}{4\sqrt{L_2}k}A,$$

where

$$\sinh B = i\frac{(2L_2 e^{-2R} - H)}{\sqrt{H^2 - 4\alpha L_2}}, \ \cosh B = \frac{2\sqrt{L_2}e^{-2R}p_R}{\sqrt{H^2 - 4\alpha L_2}},$$

$$\sinh A = i\frac{-\gamma + \beta - L_2\cos(2k\theta)}{\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}},$$

$$\cosh A = \frac{\sqrt{L_2}\sin(2k\theta)p_\theta}{\sqrt{(L_2 - \beta - \gamma)^2 - 4\beta\gamma}}.$$

Higher order systems 4: TTW (k)

If k is rational, $k = \frac{p}{q}$ then

$$\sinh(-4ip\sqrt{L_2}[N - M])$$

gives rise to a a classical constant of the motion which is polynomial in the momenta, as does

$$\cosh(-4ip\sqrt{L_2}[N - M]).$$

Higher order systems 4: TTW (k=1/2)

For example, if $p = 1,\ q = 2$ we have (with $\mathcal{L} = -\gamma + \beta - L_2 \cos\theta$),

$$\cosh(2A + B) =$$

$$\frac{2\sqrt{L_2}\left[(e^{-2R} p_R (L_2 \sin^2\theta p_\theta^2 - \mathcal{L}^2) - \sin\theta(2L_2 e^{-2R} - H)\mathcal{L}\right]}{[(L_2 - \beta - \gamma)^2 - 4\beta\gamma]\sqrt{H^2 - 4\alpha L_2}},$$

Higher order systems 4.5: TTW

$$\sinh(2A+B) = \frac{\left[(2L_2e^{-2R} - H)(L_2\sin^2\theta p_\theta^2 - \mathcal{L}^2) + 4L_2\sin\theta e^{-2R}p_\theta p_R\mathcal{L}\right]}{[(L_2-\beta-\gamma)^2 - 4\beta\gamma]\sqrt{H^2-4\alpha L_2}}.$$

The bracketed quantities in the numerators are 5th and 6th order constants of the motion, respectively.

Higher order systems 5

The method works for a great variety of cases, in all dimensions, particularly when an angular separable coordinate occurs. Relies on addition formulas for trig, hyperbolic and elliptic functions.

HOS 6: Not conformally flat

In 4 dimensions, consider

$$\begin{aligned}
L_4 &= p_{\theta_3}^2 + \frac{\beta_3}{\cos^2(\theta_3)} + \frac{\beta_4}{\sin^2\theta_3)} \\
L_3 &= p_{\theta_2}^2 + \frac{\beta_2}{\cos^2(\theta_2)} + \frac{L_4}{\sin^2(\theta_2)} \\
L_2 &= p_{\theta_1}^2 + \frac{\beta_1}{\cos^2(2\theta_1)} + \frac{L_3}{\sin^2(2\theta_1)} \\
H = L_1 &= p_r^2 + \alpha r^2 + \frac{L_2}{r^2}.
\end{aligned}$$

HOS 7: Conformally curved

$$L_1'' = \left(H - \frac{2L_2}{r^2}\right)\frac{\sin(2\theta_1)}{r}p_{\theta_1}p_r +$$

$$\frac{2(L_2\cos(2\theta_1) + L_3 - \beta_1)}{r^2}p_r^2 - \frac{1}{4}(H^2 - \alpha L_2)\cos(4\theta_1)$$

is an additional constant that is quartic in the momenta, and

$$L_2'' = \sin^2(2\theta_2)(2L_3\csc^2(2\theta_1) + \beta_1 - L_2 - L_3)p_{\theta_2}^2$$

$$-2(L_3\cos(2\theta_2) + L_4 - \beta_2)\cot(2\theta_1)\sin(2\theta_2)p_{\theta_1}p_{\theta_2}$$

$$-((\beta_2 - L_3 - L_4)^2 - 4L_3L_4)\csc^2(2\theta_1)$$

is an additional constant that is quartic in the momenta.

HOS 8: Conformally curved

$$L_3'' = (2L_4 \csc^2(\theta_2) + \beta_2 - L_3 - L_4) \sin(2\theta_3) p_{\theta_3}$$
$$-2(L_4 \cos(2\theta_3) + \beta_4 - \beta_3) \cot(\theta_2) p_{\theta_2}$$

is an additional constant that is cubic in the momenta. This space is NOT conformally flat