

Name: _____

Math 4567. Homework Set # II

October 16, 2009

Chapter 2 (page 54, problems 1,2,5,6,7), Chapter 3 (page 63, problem 3),
(page 71, problems 1,2,8,9), (page 76, problem 1).

Chapter 2, page 54, Problem 1 a. Show that the function

$$f(x) = \begin{cases} 0 & \text{when } -\pi \leq x \leq 0, \\ \sin x & \text{when } 0 < x \leq \pi, \end{cases}$$

satisfies all conditions for uniform convergence on $[-\pi, \pi]$.

b. Verify that the Fourier series

$$f \sim \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}, \quad -\pi < x < \pi$$

converges pointwise uniformly to f on $[-\pi, \pi]$.

c. State why the series can be differentiated on $(-\pi, \pi)$ and describe the function that is represented by the differentiated series for all x .

Solution:

- a.** f is continuously differentiable on the open intervals $0 < x < \pi$ and $-\pi < x < 0$. We have $f(-\pi) = f(0) = f(\pi) = 0$, so it is continuous on $[-\pi, \pi]$. $f'(0+0) = 1$, $f'(0-0) = 0$, $f'(\pi-0) = -1$, $f'(-\pi+0) = 0$ so f is piecewise smooth.
- b.** By part [a.] the series for f converges pointwise uniformly to f on $[-\pi, \pi]$.

- c. Since f satisfies the conditions for uniform convergence and since $f''(x)$ is piecewise continuous on $(-\pi, \pi)$ the Fourier series can be differentiated term-by-term. The differentiated series converges to 0 for $-\pi < x < 0$, to $\cos x$ for $0 < x < \pi$, to $\frac{1}{2}$ for $x = 0$ and to $-\frac{1}{2}$ for $x = \pm\pi$.

Chapter 2, page 54, Problem 2 Differentiate the series

$$f(x) = x \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \quad 0 < x < \pi$$

to get the Fourier series for $f'(x) = 1$ on the same interval and state why the procedure is reliable here.

Solution: This is the Fourier series for the function $f(x) = |x|$ on the interval $[-\pi, \pi]$. Here $f(-\pi) = f(\pi) = \pi$ and f is continuous on $[-\pi, \pi]$. Furthermore $f'(x) = 1$ for $0 < x < \pi$ and $f'(x) = -1$ for $-\pi < x < 0$ so f' is piecewise continuous on $[-\pi, \pi]$. Finally $f''(x) = 0$ for $x \neq 0, \pm\pi$. Thus differentiating the Fourier series term by term yields the sum of 1 on the interval $0 < x < \pi$.

Chapter 2, page 54, Problem 5 Integrate from $s = 0$ to $s = x$, ($-\pi \leq x \leq \pi$) the Fourier series

$$s = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin ns$$

and the Fourier series

$$f(s) = 2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)s}{2n-1}$$

for

$$f(s) = \begin{cases} -\pi/2 & \text{when } -\pi < s < 0, \\ \pi/2 & \text{when } 0 < s < \pi \end{cases}$$

In each case describe graphically the function represented by the series.

Solution:

- a. Integrating both sides of the Fourier series term-by term from 0 to x we get

$$\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (\cos nx - 1), \quad -\pi \leq x \leq \pi.$$

The series is representing the parabola $F(x) = x^2/2$ in the interval $[-\pi, \pi]$.

- b. Integrating both sides of the Fourier series term-by term from 0 to x we get

$$F(x) = \int_0^x f(s) ds = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} (\cos(2n-1)x - 1), \quad -\pi \leq x \leq \pi,$$

where

$$F(x) = \begin{cases} \frac{\pi x}{2} & \text{when } 0 \leq x \leq \pi \\ -\frac{\pi x}{2} & \text{when } -\pi \leq x < 0. \end{cases}$$

Thus $F(x) = \frac{\pi|x|}{2}$ for $-\pi \leq x \leq \pi$.

Chapter 2, page 54, Problem 6 Let p_n, q_n $n = 1, \dots, N$ be real numbers where at least one of the p_n is nonzero. By considering the quadratic equation

$$\sum_{n=1}^N (p_n x + q_n)^2 = 0,$$

derive the Cauchy inequality

$$\left(\sum_{n=1}^N p_n q_n \right)^2 \leq \left(\sum_{n=1}^N p_n^2 \right) \left(\sum_{n=1}^N q_n^2 \right).$$

Solution: Write the quadratic equation as

$$(P, P)x^2 + 2(P, Q)x + (Q, Q) = 0,$$

where

$$(P, P) = \sum_{n=1}^N p_n^2, \quad (P, Q) = \sum_{n=1}^N p_n q_n, \quad (Q, Q) = \sum_{n=1}^N q_n^2.$$

By assumption, $(P, P) > 0$. Since the original form of the quadratic equation is as a sum of squares, this equation has at most one real solution x , which would be such that $p_n x + q_n = 0$ for all n . The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$ and it has the property that $D > 0$ for the case that there are 2 distinct real roots, $D = 0$ if there is exactly one real root, and $D < 0$ when there are no real roots. In this case $D = 4(P, Q)^2 - 4(P, P)(Q, Q)$ and there is at most one real root. Hence we must have $D \leq 0$ or $(P, Q)^2 \leq (P, P)(Q, Q)$.

Chapter 2, page 54, Problem 7 Let $S_N(x)$ be the N th partial sum of the Fourier series

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

for

$$f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi \end{cases}$$

1. By writing $A = x$, $B = (2n-1)x$ in the identity,
- 2.

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

and then summing from $n = 1$ to $n = N$ derive

$$2 \sum_{n=1}^N \cos(2n-1)x = \frac{\sin 2Nx}{\sin x}, \quad x \neq 0, \pm\pi, \pm 2\pi, \dots$$

Verify that

$$S'_N(x) = \frac{\sin 2Nx}{\sin x}, \quad 0 < x < \pi.$$

3. Show that the first extremum of $S_N(x)$ in $0 < x < \pi$ is a relative maximum occurring when $x = \pi/(2N)$.
4. Show that

$$S_N\left(\frac{\pi}{2N}\right) = I_1 + I_2, \quad I_1 = \int_0^{\pi/(2N)} \frac{x - \sin x}{x \sin x} \sin 2Nx \, dx,$$

$$I_2 = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x} dx.$$

Verify that the integrands are piecewise continuous on $0 < x < \pi/(2N)$. so that the integrals converge.

5. Show that $I_1 \rightarrow 0$ as $N \rightarrow \infty$ so that

$$\lim_{N \rightarrow \infty} S_N\left(\frac{\pi}{2N}\right) = \int_0^\pi \frac{\sin t}{t} dt.$$

Solution:

1. We have

$$2 \sin x \cos(2n - 1)x = \sin 2nx - \sin 2(n - 1)x.$$

Thus by truncation

$$2 \sin x \sum_{n=1}^N \cos(2n - 1)x = \sum_{n=1}^N (\sin 2nx - \sin 2(n - 1)x) = \sin 2Nx,$$

so, dividing by $\sin x$ we have

$$2 \sum_{n=1}^N \cos(2n - 1)x = \frac{\sin 2Nx}{\sin x}, \quad x \neq 0, \pm\pi, \pm 2\pi, \dots$$

2.

$$\begin{aligned} S'_N(x) &= \frac{d}{dx} \left(2 \sum_{n=1}^N \frac{\sin(2n - 1)x}{2n - 1} \right) = 2 \sum_{n=1}^N \cos(2n - 1)x \\ &= \frac{\sin 2Nx}{\sin x}, \quad 0 < x < \pi. \end{aligned}$$

3. $S'_N(0) = 2 \sum_{n=1}^N \cos(2n - 1)0 = 2N > 0$ so $S_N(x)$ is initially increasing from $S_N(0) = 0$ for x increasing from 0. The first maximum of $S_N(x)$ is at the first positive x_N such that $S'_N(x_N) = 0$. Thus $x_N = \pi/(2N)$.

4.

$$\begin{aligned} S_N\left(\frac{\pi}{2N}\right) &= S_N(0) + \int_0^{\pi/(2N)} \frac{\sin 2Nx}{\sin x} dx = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{\sin x} dx \\ &= \int_0^{\pi/(2N)} \left[\frac{x - \sin x}{x \sin x} \sin 2Nx + \frac{\sin 2Nx}{x} \right] dx \end{aligned}$$

$$= I_1 + I_2,$$

where

$$I_1 = \int_0^{\pi/(2N)} \frac{x - \sin x}{x \sin x} \sin 2Nx \, dx, \quad I_2 = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x} dx.$$

The integrand of I_1 is piecewise continuous over $[0, \pi/(2N)]$ except perhaps at $x = 0$. However, by the l'Hôpital rules

$$\lim_{x \rightarrow 0+} \frac{x - \sin x}{x \sin x} \stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{1 - \cos x}{\sin x + x \cos x} \stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{\sin x}{2 \cos x - x \sin x} = 0$$

so the integrand goes to 0 as $x \rightarrow 0+$. Thus the integrand of I_1 is continuous over $[0, \pi/(2N)]$, hence bounded above in absolute value by a positive constant M . Similarly the integrand of I_2 is piecewise continuous over $[0, \pi/(2N)]$ except perhaps at $x = 0$. However, by the l'Hôpital rule

$$\lim_{x \rightarrow 0+} \frac{\sin 2Nx}{x} \stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{2N \cos 2Nx}{1} = 2N$$

so the integrand is piecewise continuous on $[0, \pi/(2N)]$.

5. We have

$$|I_1| \leq \int_0^{\pi/(2N)} M \, dx = \frac{M\pi}{2N} \rightarrow 0$$

as $N \rightarrow \infty$. Thus

$$\lim_{N \rightarrow \infty} S_N\left(\frac{\pi}{2N}\right) = \lim_{N \rightarrow \infty} I_2 = \lim_{N \rightarrow \infty} \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x} dx$$

$$\lim_{N \rightarrow \infty} \int_0^{\pi} \frac{\sin u}{u} du = \int_0^{\pi} \frac{\sin u}{u} du = \sigma = 1.85 \dots$$

Since $\pi/2 = 1.57 \dots$, this shows that the partial sums overshoot the function values $f(x_N)$ for $x_N \rightarrow 0+$ by the difference $\sigma - \pi/2$.

Chapter 3, page 63, problem 3 Show that the substitution $\tau = kt$ can be used to write the equation

$$u_t = k(u_{xx} + u_{yy}),$$

in the form

$$u_\tau = u_{xx} + u_{yy}.$$

Solution: Since $\tau = kt$ we have

$$\partial_t u = \frac{\partial \tau}{\partial t} u_\tau = k u_\tau.$$

Thus we can cancel the common factor k from both sides of the first equation to obtain the desired result.

Chapter 3, page 71, Problem 1 Let $u(x)$ be the steady-state temperature in a slab bounded by planes $x = 0$ $x = c$ when those faces are kept at fixed temperatures $u = 0$, $u = u_0$, respectively. Solve the boundary value problem for $u(x)$ to show that

$$u(x) = \frac{u_0}{c}x, \quad \Phi_0 = K \frac{u_0}{c},$$

where Φ_0 is the flux of heat to the left across each plane. $x = x_0$.

Solution: The boundary value problem for this system is $u = u(x)$ where u is continuous on $[0, c]$ and 2 times differentiable on $(0, c)$, with

$$u_{xx} = 0, \quad 0 < x < c \quad \text{where } u(0) = 0, \quad u(c) = u_0.$$

The general solution of the differential equation is $u(x) = ax + b$. The boundary conditions give $u(0) = b = 0$ and $u(c) = ac = u_0$ so the unique solution is $u(x) = \frac{u_0}{c}x$. The flux of heat to the left across each plane $x = x_0$ is $\Phi_0 = K \frac{du(x)}{dx} = \frac{Ku_0}{c}$.

Chapter 3, page 71, Problem 2 A slab occupies the region $0 \leq x \leq c$. There is a constant flux of heat Φ_0 into the slab through the face $x = 0$. The face $x = c$ is kept at temperature $u = 0$. Solve the boundary value problem for the steady-state temperatures $u(x)$ in the slab.

Solution: The boundary value problem for this system is $u = u(x)$ with u continuous on $[0, c]$, left differentiable at $x = 0$ and 2 times differentiable on $(0, c)$, with

$$u_{xx} = 0, \quad 0 < x < c, \quad \text{where } \Phi_0 = -Ku_x(0), \quad u(c) = u_0.$$

The general solution of the differential equation is $u(x) = ax + b$. The boundary conditions give $\Phi_0 = -Ka$ and $u(c) = ac + b = u_0$ so the unique solution is $a = -\Phi_0/K$, $b = u_0 + c\Phi_0/K$ or

$$u(x) = -\frac{\Phi_0}{K}(x - c) + u_0.$$

Chapter 3, page 71, Problem 8 Derive expressions for $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ in cylindrical coordinates.

Solution: Cylindrical coordinates are defined by relations

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z.$$

or

$$\rho = \sqrt{x^2 + y^2}, \quad \tan \phi = \frac{y}{x}, \quad z = z.$$

Thus,

$$\rho_y = \frac{y}{\sqrt{x^2 + y^2}} = \sin \phi, \quad \sec^2 \phi \phi_y = \frac{1}{x}, \quad z_y = 0.$$

Since $\sec^2 \phi = \tan^2 \phi + 1 = \frac{y^2}{x^2} + 1$ we have

$$\phi_y = \frac{x}{x^2 + y^2} = \frac{\cos \phi}{\rho}.$$

By the chain rule:

$$\partial_y = \rho_y \partial_\rho + \phi_y \partial_\phi + z_y \partial_z = \sin \phi \partial_\rho + \frac{\cos \phi}{\rho} \partial_\phi.$$

Thus

$$u_y = \sin \phi u_\rho + \frac{\cos \phi}{\rho} u_\phi.$$

and

$$\begin{aligned} u_{yy} &= \left(\sin \phi \partial_\rho + \frac{\cos \phi}{\rho} \partial_\phi \right) \left(\sin \phi u_\rho + \frac{\cos \phi}{\rho} u_\phi \right) \\ &= \sin^2 \phi u_{\rho\rho} - \frac{\sin \phi \cos \phi}{\rho^2} u_\phi + \frac{\sin \phi \cos \phi}{\rho} u_{\phi\rho} + \frac{\cos^2 \phi}{\rho} u_\rho \end{aligned}$$

$$\begin{aligned}
& + \frac{\sin \phi \cos \phi}{\rho} u_{\phi\rho} - \frac{\sin \phi \cos \phi}{\rho^2} u_{\phi} + \frac{\cos^2 \phi}{\rho^2} u_{\phi\phi} \\
= & \sin^2 \phi u_{\rho\rho} + 2 \frac{\sin \phi \cos \phi}{\rho} u_{\phi\rho} - 2 \frac{\sin \phi \cos \phi}{\rho^2} u_{\phi} + \frac{\cos^2 \phi}{\rho} u_{\rho} + \frac{\cos^2 \phi}{\rho^2} u_{\phi\phi}
\end{aligned}$$

Chapter 3, page 71, Problem 9 Suppose that temperatures u in a solid hemisphere $r \leq 1$, $0 \leq \theta \leq \pi/2$ are independent of ϕ : $u = u(r, \theta)$, and that the base of the hemisphere is insulated. Use the transformation relating spherical and cylindrical coordinates to show that

$$\frac{\partial u}{\partial \theta} = -\rho \frac{\partial u}{\partial z} + z \frac{\partial u}{\partial \rho}.$$

Thus show that u satisfies the boundary condition $u_{\theta}(r, \pi/2) = 0$.

Solution: In cylindrical coordinates ρ, ϕ, z we would have $u = u(\rho, z)$ and the insulated base condition would be $u_z(\rho, 0) = 0$ for $0 \leq \rho \leq 1$. Now we switch to spherical coordinates r, θ, ϕ , where

$$z = r \cos \theta, \quad \rho = r \sin \theta, \quad \phi = \phi.$$

or

$$r = \sqrt{\rho^2 + z^2}, \quad \tan \theta = \frac{z}{\rho}, \quad \phi = \phi.$$

From the chain rule,

$$\partial_{\theta} = \rho_{\theta} \partial_{\rho} + \phi_{\theta} \partial_{\phi} + z_{\theta} \partial_z$$

where

$$z_{\theta} = -r \sin \theta = -\rho, \quad \rho_{\theta} = r \cos \theta = z, \quad \phi_{\theta} = 0.$$

Thus

$$u_{\theta} = z u_{\rho} - \rho u_z.$$

On the base we have $u_z = 0$, $z = 0$, and $\theta = \pi/2$. Thus, the boundary condition in spherical coordinates is $u_{\theta}(r, \pi/2) = 0$ for $0 \leq r < 1$.

Chapter 3, page 76, Problem 1 A stretched string with ends fixed at $x = 0$, $x = 2c$ hangs at rest under its own weight. Show how it follows from equation

$$y_{tt}(x, t) = a^2 y_{xx} - g$$

that the static $y(x)$ must satisfy the equation $a^2 y''(x) = g$, where $a^2 = H/\delta$.

Solution: If the solution y is static then $y_t \equiv 0$, hence $y_{tt} = 0$ for all t and $y_x(x) = y'(x)$. The general solution of equation $a^2 y''(x) = g$ is $y(x) = \frac{gx^2}{2a^2} + Ax + B$ where A, B are constants. Since $y(0) = 0$, we have $B = 0$. Since $y(2c) = 0$ we have $A = -\frac{gc}{a^2}$. Thus

$$y(x) = \frac{gx}{2a^2}(x - 2c), \quad \text{and} \quad (x - c)^2 = \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right).$$

This is an inverted parabola with vertex at $x = c$ and depth $|y(c)| = \frac{gc^2}{2a^2} = \frac{g\delta c^2}{2H}$.