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Math 4567. Homework Set # I

October 2, 2009

Chapter 7 (page 178, problems 1,2,3,5,7), (page 192, problems 2,5), Chapter 1 (page 11, problems 2,8,9), (page 17, problem 2), (page 21, problem 1), Chapter 2 (page 31, problems 1,2,6), (page 39, problems 1,2,6), (page 42, problems 1,8).

Chapter 7, page 178, Problem 1 Show that $\psi_1(x) = 1$, $\psi_2(x) = x$ are orthogonal in $C_p(-1, 1)$ with inner product $(f, g) = \int_{-1}^1 f(x)g(x)dx$. Find constants A, B such that $\psi_3(x) = 1 + Ax + Bx^2$ is orthogonal to ψ_1 and ψ_2 ,

Solution:

$$(\psi_1, \psi_2) = \int_{-1}^1 x dx = 0$$

$$(\psi_1, \psi_3) = \int_{-1}^1 (1 + Ax + Bx^2)dx = 2 + \frac{2B}{3} = 0 \quad \longrightarrow B = -3$$

$$(\psi_2, \psi_3) = \int_{-1}^1 x(1 + Ax + Bx^2)dx = \frac{2A}{3} = 0 \quad \longrightarrow A = 0$$

Chapter 7, page 178, Problem 2 Given $f, \psi_1 \in C[a, b]$ and linearly independent, find $\psi_2 = f + A\psi_1$ so that $(\psi_1, \psi_2) = 0$. Interpret geometrically.

Solution:

$$(\psi_2, \psi_1) = (f + A\psi_1, \psi_1) = (f, \psi_1) + A \|\psi_1\|^2 = 0 \quad \longrightarrow A = -\frac{(f, \psi_1)}{\|\psi_1\|^2}$$

Thus $\psi_2(x) = f(x) - \frac{(f, \psi_1)}{\|\psi_1\|^2} \psi_1(x)$. $A\psi_1$ is the orthogonal projection of f on the line spanned by ψ_1 .

Chapter 7, page 178, Problem 3 If in Problem 2, $f(x) = \cos nx + \sin nx$, $\psi_1(x) = \cos nx$, in $C[-\pi, \pi]$ for n a positive integer, show that $\psi_2(x) = \sin nx$.

Solution:

$$(f, \psi_1) = \int_{-\pi}^{\pi} \cos^2 nx \, dx + \int_{-\pi}^{\pi} \cos nx \sin nx \, dx = \pi,$$

$$\|\psi_1\|^2 = \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$$

from the orthogonality relations for $\cos nx$ and $\sin nx$. Thus $\psi_2 = f - \psi_1 = \sin nx$.

Chapter 7, page 178, Problem 5 If $f, g \in C_p(a, b)$, verify

$$\frac{1}{2} \int_a^b \int_a^b [f(x)g(y) - g(x)f(y)]^2 \, dx \, dy = \|f\|^2 \|g\|^2 - (f, g)^2$$

and establish the Schwarz inequality $|(f, g)| \leq \|f\| \|g\|$.

Solution:

$$\begin{aligned} & \frac{1}{2} \int_a^b \int_a^b [f(x)g(y) - g(x)f(y)]^2 \, dx \, dy = \\ & \frac{1}{2} \int_a^b f^2(x) \, dx \int_a^b g^2(y) \, dy - \int_a^b f(x)g(x) \, dx \int_a^b f(y)g(y) \, dy + \frac{1}{2} \int_a^b g^2(x) \, dx \int_a^b f^2(y) \, dy \\ & = \|f\|^2 \|g\|^2 - (f, g)^2 \geq 0, \end{aligned}$$

since the integrand is a perfect square. Thus $\|f\|^2 \|g\|^2 \geq (f, g)^2$. Taking the square root of both sides we have $\|f\| \|g\| \geq |(f, g)|$.

Chapter 7, page 178, Problem 7 Show that if $f, g \in C_p(a, b)$ then $\|f + g\| \leq \|f\| + \|g\|$.

Solution:

$$\begin{aligned} \|f+g\|^2 &= (f+g, f+g) = (f, f) + 2(f, g) + (g, g) \leq \|f\|^2 + 2|(f, g)| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\| \|g\| + \|g\|^2 = (\|f\| + \|g\|)^2, \end{aligned}$$

by the Schwarz inequality. Taking the square root of both sides we have $\|f + g\| \leq \|f\| + \|g\|$.

Chapter 7, page 192, Problem 2 Apply Bessel's inequality to show that

$$\sum_{n=1}^N b_n^2 \leq \frac{2}{\pi} \int_0^\pi f(x)^2 dx$$

for $f \in C_p(0, \pi)$ and b^n the Fourier Sine series coefficients.

Solution: If $\{\phi_n\}$ is an ON sequence in $C_p(0, \pi)$ then Bessel's inequality says $\|f\|^2 \geq \sum_{n=1}^N c_n^2$ for all integers N , where $c_n = (f, \phi_n)$. Choosing $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ we have

$$c_n = \sqrt{\frac{2}{\pi}} \int_0^\pi f(t) \sin nt dt = \sqrt{\frac{\pi}{2}} b_n,$$

so $c_n^2 = \frac{\pi}{2} b_n^2$. Multiplying both sides of Bessel's inequality by $\frac{2}{\pi}$ we have $\sum_{n=1}^N b_n^2 \leq \frac{2}{\pi} \int_0^\pi f(x)^2 dx$.

Chapter 7, page 192, Problem 5 For $N = 1, 2, \dots$ let

$$s_N(x) = \begin{cases} 0 & \text{when } 0 \leq x \leq \frac{1}{N}, \\ \sqrt{N} & \text{when } \frac{1}{N} < x < \frac{2}{N}, \\ 0 & \text{when } \frac{2}{N} \leq x \leq 1. \end{cases}$$

for $N = 1, 2, \dots$. Show that this sequence converges pointwise to the function $f(x) \equiv 0$ in the space $C_p(0, 1)$, but that it does not converge in the mean to f in $C_p(0, 1)$.

Solution: If $x = 0$, $s_N(0) = 0$ for all N , so $\lim_{N \rightarrow \infty} s_N(0) = 0 = f(0)$. If $0 < x \leq 1$ then $s_N(x) = 0$ for all $N \geq \frac{2}{x}$, so $\lim_{N \rightarrow \infty} s_N(x) = 0 = f(x)$. Thus $s_N(x)$ converges pointwise to $f(x)$ as $N \rightarrow \infty$ for each $x \in [0, 1]$. However, for any N we have

$$\|f - s_N\|^2 = \int_0^1 (f(x) - s_N(x))^2 dx = \int_{\frac{1}{N}}^{\frac{2}{N}} N dx = \left(\frac{2}{N} - \frac{1}{N}\right)N = 1.$$

Thus $\{s_N\}$ does not converge to f in the mean.

Chapter 1, page 11, Problem 2 Compute the Fourier cosine series and the Fourier sine series for the function $f(x) = \pi - x \in C_p(0, \pi)$.

Solution: For the cosine series

$$\begin{aligned}
 a_0 &= \frac{2}{\pi} \int_0^\pi (\pi - t) dt = \frac{2}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \pi, \\
 a_n &= \frac{2}{\pi} \int_0^\pi (\pi - t) \cos nt \, dt = \frac{2}{\pi} \left[(\pi - t) \frac{\sin nt}{n} \Big|_0^\pi + \int_0^\pi \frac{\sin nt}{n} dt \right] \\
 &= -\frac{2}{\pi} \frac{\cos nt}{n^2} \Big|_0^\pi = -\frac{2}{n^2 \pi} ((-1)^n - 1) \\
 &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n^2 \pi} & \text{if } n \text{ is odd.} \end{cases}
 \end{aligned}$$

Thus

$$\pi - x \sim \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}.$$

For the sine series

$$b_n = \frac{2}{\pi} \int_0^\pi (\pi - t) \sin nt \, dt = \frac{2}{\pi} \left(-(\pi - t) \frac{\cos nt}{n} \Big|_0^\pi + \int_0^\pi \frac{\cos nt}{n} dt \right) = \frac{2}{n}.$$

Thus

$$\pi - x \sim 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Chapter 1, page 11, Problem 8 Show that

$$\int_0^\pi \cos mx \cos nx \, dx = \begin{cases} 0 & \text{when } m \neq n \\ \pi/2 & \text{when } m = n, \end{cases}$$

where m, n are positive integers.

Solution: We use the addition formulas:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Adding we find

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B)).$$

For $m \neq n$ and m, n positive integers we set $A = mx$, $B = nx$ to get

$$\begin{aligned} \int_0^\pi \cos mx \cos nx \, dx &= \frac{1}{2} \left(\int_0^\pi \cos(m+n)x \, dx + \int_0^\pi \cos(m-n)x \, dx \right) \\ &= \frac{1}{2} \left(\frac{\sin(m+n)x}{m+n} \Big|_0^\pi + \frac{\sin(m-n)x}{m-n} \Big|_0^\pi \right) = 0. \end{aligned}$$

For $m = n$ a positive integer we set $A = mx$, $B = mx$ to get

$$\begin{aligned} \int_0^\pi \cos^2 mx \, dx &= \frac{1}{2} \left(\int_0^\pi \cos(2m)x \, dx + \int_0^\pi 1 \, dx \right) \\ &= \frac{1}{2} \left(\frac{\sin(2m)x}{2m} \Big|_0^\pi + \pi \right) = \frac{\pi}{2}. \end{aligned}$$

Chapter 1, page 11, Problem 9 Show that

$$\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} 0 & \text{when } m \neq n \\ \pi/2 & \text{when } m = n, \end{cases}$$

where m, n are positive integers.

Solution:

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

Subtracting we find

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B)).$$

For $m \neq n$ and m, n positive integers we set $A = mx$, $B = nx$ to get

$$\begin{aligned} \int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \left(\int_0^\pi \cos(m-n)x \, dx - \int_0^\pi \cos(m+n)x \, dx \right) \\ &= \frac{1}{2} \left(\frac{\sin(m-n)x}{m-n} \Big|_0^\pi - \frac{\sin(m+n)x}{m+n} \Big|_0^\pi \right) = 0. \end{aligned}$$

For $m = n$ a positive integer we set $A = mx$, $B = mx$ to get

$$\begin{aligned} \int_0^\pi \sin^2 mx \, dx &= \frac{1}{2} \left(\int_0^\pi 1 \, dx - \int_0^\pi \cos(2m)x \, dx \right) \\ &= \frac{1}{2} \left(\pi - \frac{\sin(2m)x}{2m} \Big|_0^\pi \right) = \frac{\pi}{2}. \end{aligned}$$

Chapter 1, page 17, Problem 2 Find the Fourier series for the function

$f \in C_p(-\pi, \pi)$ defined by

$$f(x) = \begin{cases} \frac{2}{\pi}x + 2 & \text{if } -\pi < x < 0 \\ 2 & \text{if } 0 \leq x \leq \pi. \end{cases}$$

Solution:

$$a_0 = \frac{1}{\pi} \left(\int_{-\pi}^0 \left(\frac{2}{\pi}t + 2 \right) dt + \int_0^{\pi} 2 dt \right) = \frac{1}{\pi} (-\pi + 2\pi + 2\pi) = 3$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left(\int_{-\pi}^0 \left(\frac{2}{\pi}t + 2 \right) \cos nt dt + \int_0^{\pi} 2 \cos nt dt \right) = \\ &= \frac{1}{\pi} \left(\frac{2}{\pi} \frac{\sin nt}{n} \Big|_{-\pi}^0 - \int_{-\pi}^0 \frac{2 \sin nt}{\pi n} dt \right) = \\ &= \frac{1}{\pi} \left(\frac{2 \cos nt}{\pi n^2} \Big|_{-\pi}^0 \right) = \frac{2}{n^2 \pi^2} (1 - (-1)^n) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left(\int_{-\pi}^0 \left(\frac{2}{\pi}t + 2 \right) \sin nt dt + \int_0^{\pi} 2 \sin nt dt \right) = \\ &= \frac{1}{\pi} \left(-\frac{2}{\pi} \frac{\cos nt}{n} \Big|_{-\pi}^0 + \int_{-\pi}^0 \frac{2 \cos nt}{\pi n} dt \right) = \\ &= \frac{1}{\pi} \left(\frac{2}{n\pi} \pi (-(-1)^n) \right) = \frac{2}{n\pi} (-1)^{n+1} \end{aligned}$$

Thus,

$$f(x) \sim \frac{3}{2} + 2 \sum_{n=1}^{\infty} \left(\frac{1}{n^2 \pi^2} (1 - (-1)^n) \cos nx + \frac{1}{n\pi} (-1)^{n+1} \sin nx \right).$$

Chapter 1, page 21, Problem 1 Use the Fourier sine series for $f(x) = x \in C_p(0, \pi)$:

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad 0 < x < \pi$$

to show that

$$u \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi u, \quad 0 < u < 1$$

Solution: Make the substitution $x = \pi u$ on both sides of the first identity and divide by π .

Chapter 2, page 31, Problem 1 Use the l'Hôpital rules to find $f(0+)$ and $f'_R(0)$ when $f(x) = \frac{e^x - 1}{x}$ for $x \neq 0$.

Solution: By l'Hôpital

$$f(0+) = \lim_{x \rightarrow 0+} \frac{e^x - 1}{x} \stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{e^x}{1} = 1$$

$$f'_R(0) = \lim_{x \rightarrow 0+} \frac{f(x) - f(0+)}{x} = \lim_{x \rightarrow 0+} \frac{\frac{e^x - 1}{x} - 1}{x} = \lim_{x \rightarrow 0+} \frac{e^x - x - 1}{x^2}$$

$$\stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{e^x - 1}{2x} \stackrel{l'H}{=} \lim_{x \rightarrow 0+} \frac{e^x}{2} = \frac{1}{2}.$$

Chapter 2, page 31, Problem 2 Show that the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$, but neither $f'_R(0)$ or $f'_L(0)$ exists.

Solution:

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

for all $x \neq 0$. Since $\lim_{x \rightarrow 0} |x| = 0$ we can apply the Squeeze Theorem of first year calculus to conclude that

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0 = f(0).$$

Thus $f(x)$ is continuous at $x = 0$. We have

$$f'_R(0) = \lim_{x \rightarrow 0+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0+} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \rightarrow 0+} \sin\left(\frac{1}{x}\right),$$

and the limit doesn't exist. Similarly

$$f'_L(0) = \lim_{x \rightarrow 0-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0-} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \rightarrow 0-} \sin\left(\frac{1}{x}\right),$$

and the limit doesn't exist.

Chapter 2, page 31, Problem 6 Derive the expression

$$D_N(u) = \frac{\sin(N + \frac{1}{2})u}{2 \sin \frac{u}{2}}, \quad u \neq 0, \pm 2\pi, \pm 4\pi, \dots$$

for the Dirichlet kernel $D_N(u) = 1/2 + \sum_{n=1}^N \cos nu$ by writing $A = u/2$, $B = un$ in the identity

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

and then summing from $n = 1$ to $n = N$.

Solution: Making the substitution we have

$$2 \sin \frac{u}{2} \cos nu = \sin(n + \frac{1}{2})u - \sin(n - \frac{1}{2})u.$$

Summing from $n = 1$ to $n = N$ we get cancellation in successive terms on the right

$$2 \sin \frac{u}{2} \sum_{n=1}^N \cos nu = \sum_{n=1}^N \left(\sin(n + \frac{1}{2})u - \sin(n - \frac{1}{2})u \right) = \sin(N + \frac{1}{2})u - \sin \frac{u}{2}.$$

Thus

$$2 \sin \frac{u}{2} \left(\frac{1}{2} + \sum_{n=1}^N \cos nu \right) = \sin(N + \frac{1}{2})u,$$

so $D_N(u) = \frac{\sin(N + \frac{1}{2})u}{2 \sin \frac{u}{2}}$ for $u \neq 0, \pm 2\pi, \pm 4\pi, \dots$.

Chapter 2, page 39, Problem 1 State why the Fourier sine series

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad 0 < x < \pi$$

is valid for all x such that $-\pi < x < \pi$.

Solution: The Fourier sine series describes an odd function on the interval $-\pi < x < \pi$. Since the function $f(x) = x$ is already odd, the sine series of x for $0 < x < \pi$ still represents $f(x)$ for $-\pi < x < \pi$. In particular it is valid for $x = 0$.

Chapter 2, page 39, Problem 2 Show why each of the following Fourier series for functions $f \in C_p(-\pi, \pi)$ is pointwise convergent for $-\pi \leq x \leq \pi$ and state the sum of the series for $x = \pi$.

(a)

$$f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi, \end{cases}$$

Solution: The Fourier series converges on $[-\pi, \pi]$ because both $f(x)$ and

$$f'(x) = \begin{cases} 0 & \text{when } -\pi < x < 0, \\ 0 & \text{when } 0 < x < \pi, \end{cases}$$

belong to $C_p(-\pi, \pi)$. At $x = \pi$ the series converges to $\frac{f(-\pi+0)+f(\pi-0)}{2} = \frac{-\pi/2+\pi/2}{2} = 0$.

(b) $f(x) = e^{ax}$.

Solution: The Fourier series converges on $[-\pi, \pi]$ because both $f(x)$ and $f'(x) = ae^{ax}$ belong to $C_p(-\pi, \pi)$. At $x = \pi$ the series converges to $\frac{f(-\pi+0)+f(\pi-0)}{2} = \frac{e^{-a\pi}+e^{a\pi}}{2} = \cosh a\pi$.

Chapter 2, page 39, Problem 6 (a) Use the correspondence

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad 0 < x < \pi,$$

to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution: Since x^2 is an even function, the Fourier cosine series for $f(x) = x^2$ on $0 < x < \pi$ extends to the Fourier series representation of x^2 on the interval $-\pi < x < \pi$. Since the extended function is continuous and $f'(x) = 2x$ is piecewise continuous for $-\pi < x < \pi$ the Fourier series converges to $f(x)$ for all x in that

interval. In particular $f(0) = 0$ and substituting $x = 0$ in the Fourier series we have

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Simplifying this equation we get the first result. Since $f(-\pi) = f(\pi) = \pi^2$, the Fourier series for $f(x)$ converges to π^2 when $x = \pi$. Thus

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Simplifying this equation we get the second result.

(b) By writing $x = \pi$ in the correspondence

$$x^4 \sim \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2 - 6}{n^4} \cos nx, \quad 0 < x < \pi$$

and referring to part (a), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution: Since $f(x) = x^4$ is an even function, the Fourier cosine series for x^4 extends to a Fourier series on $-\pi < x < \pi$. Since $f, f' \in C_p(-\pi, \pi)$, f is continuous on $-\pi < x < \pi$ and $f(-\pi) = f(\pi)$ the Fourier series converges to $f(x)$ for each x in $[-\pi, \pi]$. Setting $x = \pi$ in the Fourier series we get

$$\pi^4 = \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} \frac{(n\pi)^2 - 6}{n^4},$$

or

$$\frac{\pi^4}{10} = \pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 6 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Substituting the value of the first sum from part [a] we find

$$\frac{\pi^4}{10} = \frac{\pi^4}{6} - 6 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

or

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Chapter 2, page 42, Problem 1 Show that if

$$f(x) = \begin{cases} 0 & \text{when } -3 < x < 0, \\ 1 & \text{when } 0 < x < 3, \end{cases}$$

and $f(0) = 1/2$ then

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{3}, \quad -3 < x < 3.$$

Describe graphically the function that is represented by this series for all x , $-\infty < x < \infty$.

Solution: $f \in C_p(-3, 3)$ so we can extend it to all real x by periodicity: $f(x+6) = f(x)$. Since $f'(x) = 0$ for $-3 < x < 3$ and $x/ne0$ we have $f' \in C_p(-3, 3)$. By construction $f(x) = \frac{f(x+0)+f(x-0)}{2}$ for all x , so the Fourier series of f converges to $f(x)$ for all x . The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right) \right),$$

where

$$a_0 = \frac{1}{3} \int_{-3}^3 f(t) dt = 1,$$

and for $n \neq 0$,

$$a_n = \frac{1}{3} \int_{-3}^3 f(t) \cos\left(\frac{n\pi t}{3}\right) dt = \frac{1}{3} \int_0^3 \cos\left(\frac{n\pi t}{3}\right) dt = 3 \frac{1}{3} \frac{\sin\left(\frac{n\pi t}{3}\right)}{n\pi} \Big|_0^3 = 0,$$

$$\begin{aligned} b_n &= \frac{1}{3} \int_{-3}^3 f(t) \sin\left(\frac{n\pi t}{3}\right) dt = \frac{1}{3} \int_0^3 \sin\left(\frac{n\pi t}{3}\right) dt = -3 \frac{1}{3} \frac{\cos\left(\frac{n\pi t}{3}\right)}{n\pi} \Big|_0^3 \\ &= \frac{1}{n\pi} (1 - (-1)^n). \end{aligned}$$

Thus

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{3}, \quad -3 \leq x \leq 3.$$

Graphically this is a step function that is 0 for the strips $-3+6k < x < 6k$, 1 for the strips $6k < x < 6k+3$ and $\frac{1}{2}$ at the points $x = 6k$, where $k = 0, \pm 1, \pm 2, \dots$.

Chapter 2, page 42, Problem 8 From the Fourier series

$$f(x) = \frac{a_0}{2} + \lim_{N \rightarrow \infty} \sum_{n=1}^N \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right)$$

derive the complex series

$$f(x) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N A_n \exp\left(i \frac{n\pi x}{c}\right),$$

where $A_0 = \frac{a_0}{2}$, $A_n = \frac{a_n - ib_n}{2}$, $A_{-n} = \frac{a_n + ib_n}{2}$ for $n = 1, 2, \dots$. Derive the formula

$$A_k = \frac{1}{2c} \int_{-c}^c f(t) \exp\left(-i \frac{k\pi t}{c}\right) dt, \quad k = 0, \pm 1, \pm 2, \dots$$

Solution:

$$\begin{aligned} \sum_{n=-N}^N A_n \exp\left(i \frac{n\pi x}{c}\right) &= A_0 + \sum_{n=1}^N \left(A_n \exp\left(i \frac{n\pi x}{c}\right) + A_{-n} \exp\left(-i \frac{n\pi x}{c}\right) \right) = \frac{a_0}{2} + \\ \frac{1}{2} \sum_{n=1}^N \left[(a_n - ib_n) \left(\cos \frac{n\pi x}{c} + i \sin \frac{n\pi x}{c} \right) + (a_n + ib_n) \left(\cos \frac{n\pi x}{c} - i \sin \frac{n\pi x}{c} \right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^N \left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right) \right], \end{aligned}$$

because the cross terms cancel out in the last expansion. Furthermore,

$$A_0 = \frac{a_0}{2} = \frac{1}{2c} \int_{-c}^c f(t) dt,$$

for $k > 0$,

$$A_k = \frac{a_k - ib_k}{2} = \frac{1}{2c} \int_{-c}^c f(t) \left(\cos \frac{k\pi t}{c} - i \sin \frac{k\pi t}{c} \right) dt = \frac{1}{2c} \int_{-c}^c f(t) \exp\left(-i \frac{k\pi t}{c}\right) dt,$$

and for $k < 0$,

$$A_k = \frac{a_{-k} + ib_{-k}}{2} = \frac{1}{2c} \int_{-c}^c f(t) \left(\cos \frac{k\pi t}{c} + i \sin\left(-\frac{k\pi t}{c}\right) \right) dt = \frac{1}{2c} \int_{-c}^c f(t) \exp\left(-i \frac{k\pi t}{c}\right) dt.$$