# A GRADUALLY UNMASKING METHOD FOR LIMITED DATA TOMOGRAPHY

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## ABSTRACT

In limited data tomography, with applications such as electron microscopy, medical imaging, industrial non-destructive testing, etc., the scanning views are within an angular range that is either limited (i.e., less than the full $180°$) or sparsely sampled. In these situations, standard reconstruction algorithms produce reconstructions with notorious intrinsic artifacts. We propose a novel technique that gradually recovers (or "unmasks") the densities in the image, and whose implementation is based on the algebraic reconstruction techniques (ART). Using our method, we show that the artifacts are thus significantly reduced.

*Keywords: Limited angle; few projections; ART; image reconstruction*

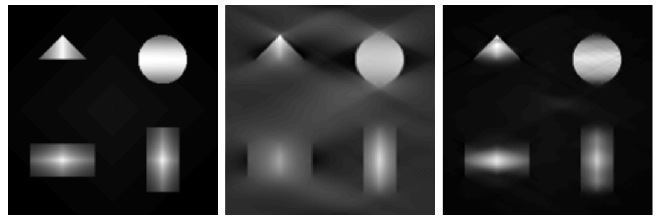

**Fig. 1**. From left to right: true image with smoothly varying background and foreground regions; reconstruction by ART from $120°$ range; reconstruction by our gradually unmasking method from the same angular range. Note the significant reduction of "butterfly" artifacts in the right image. The mean squared errors (MSE) of the reconstructions are, respectively, $7.1 \cdot 10^{-3}$ and $1.1 \cdot 10^{-3}$. All the images are $192 \times 192$.

## 1. INTRODUCTION

It is well known that the the recovery of the density distribution of an object from its (usually noisy) projections, also termed *tomography*, is ill-posed. Therefore, many algorithms (see, e.g., [3, 6]) have been proposed, which typically work reasonably well when the data are densely collected over a complete angular range of 180 degrees. In many cases, however, data can only be collected over an angular range that is either limited (such as due to physical constraints) or sparsely sampled (such as due to cost savings). The former case is commonly referred to as *limited angle tomography*; however, in this paper we term both as *limited data tomography*.

In limited data tomography, the use of standard reconstruction algorithms produces reconstructions with notorious intrinsic artifacts, having either a butterfly pattern when the scanning angular range is limited or a star pattern when the data are sparsely sampled. The center image of, respectively, Fig. 1 and Fig. 5 contains examples of these.

In dealing with the limited angle tomography, researchers have attempted several approaches such as Fourier methods, sinogram methods, regularization methods, Bayesian method, wavelet techniques, etc. (see, e.g., references cited in [2] and more recently [7, 8]). In terms of artifacts, methods that apply regularization in the image domain show higher degrees of success. Nevertheless, they need to assume piecewise smoothness of the unknown image and are still vulnerable to the artifacts, unless a "reference image" is used, which is not always available. For the sparse angular sampling problem, methods like the total variation [9] have been shown to be very promising [1]. However, in addition to the piecewise constantness assumption, little is known about the effect of measurement noise in the reconstructions.

We propose a very simple yet effective approach that targets those artifacts. Our method is based on the algebraic reconstruction techniques (ART) [3], which starts with a uniform image. Interleaved with the iterative steps, the reconstructed values are then set to be greater than or equal to a threshold $t > 0$, which in turn is slowly decreased during the run. This strategy applies to images with isolated high density regions over a low density background. If instead the regions have lower densities, we proceed the opposite and set the reconstructed values to be smaller than or equal to $t$, which is now slowly increased. Our algorithm does not require the piecewise smoothness nor sharp boundary assumption, and furthermore, it shows degrees of robustness to noise. Fig. 1 compares standard ART with the proposed approach.

The application of a transformation of the image in between two iterative steps has been referred to as *trick* in the

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literature [4]. Thus, our approach is an ART algorithm with a trick that aims at reducing the artifacts. In this paper we only show experimentally the usefulness of this framework; its theoretical explanations are currently under investigation.

In Section 2 we explain and give a heuristic justification of our approach, and we show in Section 3 the advantages and limitations of the proposed method on a variety of images and noise conditions. We provide conclusions and discussions in the last section.

## 2. THE PROPOSED METHOD

### 2.1. Algebraic Reconstruction Techniques

To simplify the discussions, we consider 2D images. Generalizations to 3D and other scanning geometries are straightforward. In ART the image solution $C(x, y)$ is approximated by a weighted sum of basis functions, each of which is a shifted basic basis function $b(x, y)$:

$$C(x,y) = \sum_{i,j} c_{ij} b\left(x - x_i, y - y_j\right). \tag{1}$$

The most common basic basis function is a pixel that has a square support and is valued one inside the support and zero otherwise. The weights $c_{ij}$ of the summation are the coefficients of the resulting approximated image vector.

In real applications, we can only measure indirectly a finite number of line integrals. Let $w_l$ be a measured integrals along a line $l$, then in the absence of noise we have for an image in the form (1) that

$$w_l = \sum_{i,j} c_{ij} r_{ij}(l), \tag{2}$$

where $r_{ij}(l)$ is the integral of $b\left(x - x_i, y - y_j\right)$ along $l$, which can be calculated from the geometry of data collection. When noise is present, as it is the case in real applications, the equality in (2) becomes an approximation.

In ART, a system of equalities, each of which in the form (2) is formed by considering all the lines along which data have been collected. An ART-type algorithm is essentially based on the following relaxation method for solving a consistent system of linear equalities [3]. Let the system be

$$Rc = w \tag{3}$$

with $S$ unknowns and $K$ equations. For $1 \leq k \leq K$, let $R_k$ be the transpose of the $k$-th row of $R$ and $w_k$ be the $k$-th component of $w$; then

$$c^{(0)} \text{ is the } S\text{-dimensional column vector of zeros}, \tag{4}$$

$$c^{(m+1)} = c^{(m)} + \lambda v^{(m)} R_{k_m}, \tag{5}$$

with

$$v^{(m)} = \frac{\left(w_{k_m} - R_{k_m}^t c^{(m)}\right)}{R_{k_m}^t R_{k_m}}, \tag{6}$$

where $k_m = m \,(\mathrm{mod}\, K) + 1$ and $\lambda \in [0, 2)$ is a relaxation parameter that controls how well the current equation is satisfied. The sequence $c^{(m)}$ converges to the unique minimum Euclidean norm solution of the system (3).

In the inconsistent case, there is no convergence proof. However, ART-type algorithms are widespread, due to its relatively superior performance for various reconstruction tasks [10]. In practice $\lambda$ is kept low (about 0.1-0.01) to reduce noise fitting and/or to improve convergence speed. Another important parameter is the order in which the data is accessed. It has been observed that faster convergence can be achieved if a linear (i.e., increasing or decreasing) order is given up in favor of a non-sequential (e.g., directions that are as orthogonal as possible to the previous ones) order.

Another reason for its popularity is the possibility of incorporating prior knowledge in the reconstruction process. For example, if the image is known to be non-negative, then after each iteration one can set the negative values to zero prior to the next iterative step. Such adjustment has been shown to improve the speed of convergence to a desirable reconstruction (see next subsection 2.2). The application of a transformation of the image in between two iterative steps has been referred to as *trick* [4]. To be mathematically precise, in an ART-type algorithm with trick, the step in (5) is replaced by

$$\tilde{c}^{(m+1)} = c^{(m)} + \lambda v^{(m)} R_{k_m}, \tag{7}$$

$$c^{(m+1)} = T\left(\tilde{c}^{(m+1)}\right), \tag{8}$$

where $T$ is the transformation that defines the trick.

### 2.2. Cause of the artifacts

Here we give a heuristic explanation of the cause of the artifacts, at least within an ART-type algorithm. In the case when the true image consists of high (low) density regions, we think of the artifacts as a result of "allowing" some surrounding background to take values that are lower (higher)

| | | |
|---|---|---|
| | | |
| | 1 | |
| | | |

(a)

| | | |
|---|---|---|
| −1/9 | 2/9 | −1/9 |
| 2/9 | 5/9 | 2/9 |
| −1/9 | 2/9 | −1/9 |

(b)

| | | |
|---|---|---|
| | | |
| | ~1 | |
| | | |

(c)

**Fig. 2**. Effect of the non-negativity trick. (a) phantom, (b) solution by a standard ART, and (c) solution with the non-negativity constraint.

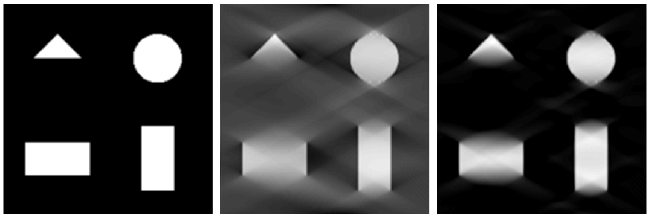

**Fig. 3**. From left to right: true image; reconstruction by ART from 120° range; reconstruction by our gradually unmasking method from the same angular range. Note the significant reduction of "butterfly" artifacts in the right image. The MSE of the reconstructions are, respectively, $2.2 \cdot 10^{-2}$ and $6.9 \cdot 10^{-3}$.

than the true one, during the iterations. To see this, take for example a 3x3 binary image with 1 in the middle and 0 in the rest (Fig. 2-a). Suppose that we wish to reconstruct this image from its three horizontal and three vertical noiseless line integrals (with value 0 or 3). If we apply standard ART, as defined in (4) and (5) (i.e., with no trick) and with $\lambda = 1$, the solution converges to the image in Fig. 2-b, which is quite different from the original one. On a gray scale display, the true image is all black except at the center pixel, where it is white. On the other hand, the values 2/9 in the reconstruction would appear gray, which would form artifacts. In this simple example, we can see that indeed the values 2/9 are consequence of "allowing" negative values, which are -1/9, during the iterations. If instead we use a non-negativity trick by setting the function $T(c)$ to be $\max\{c, 0\}$ (component-wise), then a more desirable solution is found (Fig. 2-c). It is easy to verify that, due to the linearity of the algorithm, one can extend this idea to a more general case of non-zero uniform background: $T(c) = \max\{c, t\}$, where $t$ is the background value.

### 2.3. The gradually unmasking method

The observations in the previous subsection motivate the following approach. In the case when the image consists of high density regions, we set the function $T(c)$ to be a function of iterative step $m$

$$T_m(c) = \max\{c, t_m\} \tag{9}$$

where $t_m$ is a threshold that decreases with $m$. (If instead the regions have lower intensities than the background, then $T_m(c) = \min\{c, t_m\}$, and $t_m$ is increased.) In this paper we consider $t_m$ to be linear in $m$: $t_m = t_0 - dm$, where $t_0$ is the starting threshold and $d$ the "unmasking rate," which is fixed and measures how fast the threshold is changed. Since $d$ is a constant, it is convenient to define $\widetilde{d} = K \cdot d$, where $K$ is the number of views.

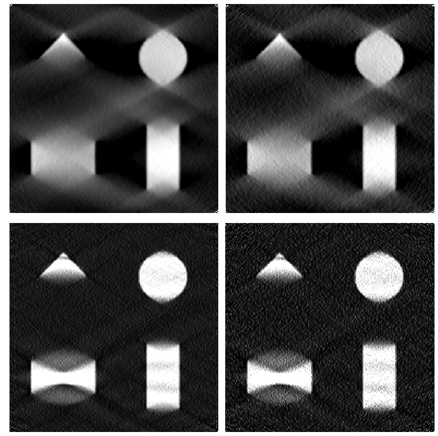

**Fig. 4**. Effect of Gaussian noise with standard deviation 0.5 (left) and 1 (right). Top row corresponds to standard ART reconstructions, and the bottom row reconstructions are produced by our proposed method. The phantom used is the left image of Fig. 3. The MSE of the reconstructions are, respectively, $2.9 \cdot 10^{-2}$ and $1.8 \cdot 10^{-2}$ (for the left column); and $3.1 \cdot 10^{-2}$ and $2.4 \cdot 10^{-2}$ (for the right column).

## 3. EXPERIMENTS

Unless otherwise stated, the following are the settings. All the images are 192×192, and the projection data consist of 120 views that correspond to 120 consecutive degrees (i.e., 2/3 of the full range of 180 degrees). Within ART, the relaxation parameter $\lambda$ is set to 0.01, and the data access order is random (with no repetition). The basic basis function is the pixel, though we precalculate and store the projection matrix $R$ using MATLAB's function called *radon*. In our method, we start with the zero image, $\widetilde{d}$ is set to 0.0002, $t_0 = 0.5$, and the algorithm stops when $t_m = 0$.

We first reconstructed an image consisting of four uniform foreground regions from limited angle data, whose results are shown in Fig. 3.

### 3.1. Non-uniform densities

Next, we made the regions and the background non-uniform with a Gaussian profile. The results, which are already displayed in Fig. 1, show that the butterfly artifacts in the background are significantly reduced by our gradually unmasking method.

### 3.2. Noisy data

The effectiveness of our approach is also revealed when data are noisy. Fig. 4 shows the reconstructions from projections contaminated by independent Gaussian noise with zero mean and standard deviations 0.5 and 1. The artifacts in the background are once again reduced using our approach, which also

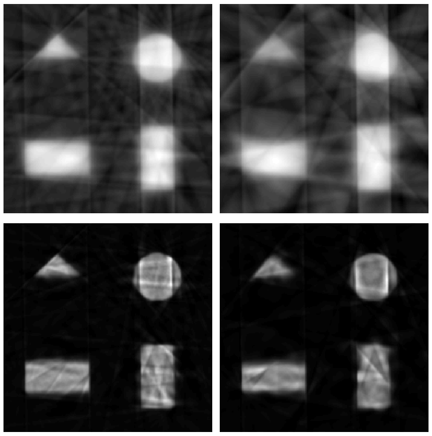

**Fig. 5**. Effect of limited number of equally spaced views. Top row: standard ART using 15 and 9 views. Bottom row: the same by our approach. The phantom used is the left image of Fig. 3. The MSE of the reconstructions are, respectively, $1.5 \cdot 10^{-2}$ and $6.9 \cdot 10^{-3}$ (for the left column); and $2.7 \cdot 10^{-2}$ and $10^{-2}$ (for the right column).

improves the contrast and the shape of the circles.

### 3.3. Limited number of views

Here we test our approach when there are a very few (9 and 15) views but distributed uniformly over the full 180 degrees. One can see that in the reconstructions by our method (Fig. 5), although the contrast is lower and the density of the foreground regions is noticeably non-uniform, their boundaries are better defined and there are less artifacts.

## 4. CONCLUSIONS AND DISCUSSIONS

We have proposed a novel approach that effectively reduces the artifacts caused by limited angle. We have also showed its usefulness in the case of limited number of views. Unlike many existing methods, it does not require a "good" starting image or a "reference image" or any type of information that is either unavailable or too restrictive to the particular task.

The possibility of inserting a transformation in between two iterations of ART constitutes a powerful advantage over some iterative methods, such as the *simultaneous iterative reconstruction techniques* (see e.g., [6]). We applied a similar strategy to SIRT, but it did not work: after one iteration, artifacts are already formed and thus cannot be reduced.

Our method is unique in that it gradually recovers the densities in the image, by starting from the highest values to the lowest ones (or vice-versa), without keeping a “residual image, ” as done in the CLEAN algorithm [5]. The use of CLEAN did not produce satisfactory results in our settings.

Preliminary experiments indicate that the reduction of artifacts by our approach in reconstructions from real data is also effective.


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