Euler's eqns for body ang mom of a free rigid body

Given a symm, pos def 3×3 matrix $\mathbb{I}$, def the 1st order ODE

$$\dot{\underset{\sim}{M}} = \underset{\sim}{M} \times (\mathbb{I}^{-1} \underset{\sim}{M}) \qquad \text{on } \mathbb{R}^3$$

Rel to rigid body? Later.

Def $E : \mathbb{R}^3 \to \mathbb{R}$ by

$$E(\underset{\sim}{M}) := \tfrac{1}{2} \langle \underset{\sim}{M}, \mathbb{I}^{-1} \underset{\sim}{M} \rangle$$

Properties: ~~i)~~ If $\underset{\sim}{M}(t)$ is a sol curve,

ii) $\|\underset{\sim}{M}(t)\| = \text{const}$

iii) $E(\underset{\sim}{M}(t)) = \text{const}$

~~Pf~~ i) $\dot{\underset{\sim}{M}} = \underset{\sim}{M} \times \nabla E(\underset{\sim}{M})$

Pf: i) Obvious

ii, iii) Prove for $\dot{\underset{\sim}{M}} = \underset{\sim}{M} \times \nabla E(\underset{\sim}{M})$, E and diff fcn on $\mathbb{R}^3$ (Ham wrt Poisson structure on $\mathbb{R}^3$)

$$\frac{d}{dt} \|\underset{\sim}{M}(t)\|^2 = 2 \langle \underset{\sim}{M}(t), \dot{\underset{\sim}{M}}(t) \rangle$$
$$= \langle \underset{\sim}{M}(t), \underset{\sim}{M} \times \nabla E(\underset{\sim}{M}) \rangle = 0$$

$$\frac{d}{dt} E(\underset{\sim}{M}(t)) = \langle \nabla E(\underset{\sim}{M}(t)), \dot{\underset{\sim}{M}}(t) \rangle$$
$$= \langle \nabla E, \underset{\sim}{M} \times \nabla E \rangle = 0.$$

Consequence? Traj's lie in intersections of spheres w/ level sets of E.

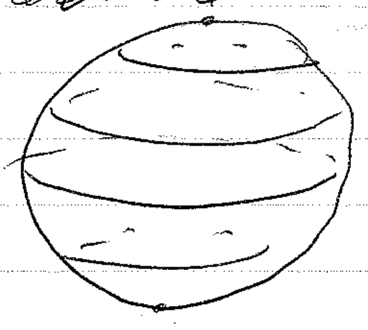
Case 1: $\mathbb{I} = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}^+$.

$\dot{\underset{\sim}{M}} = \underset{\sim}{M} \times (\frac{1}{\lambda} \underset{\sim}{M}) = \underset{\sim}{0}$

$\Rightarrow$ all pts are eq.

Case 2: $\mathbb{I} = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_3)$, $\lambda_1 \neq \lambda_3$ mod change of basis.

Level sets of E are 'upright' spheroids passing through 0 w/ vertical axes of symm. Intersections w/ spheres are horiz circles, plus poles.



Equator + poles are eq.
all pts in N/S hemisphere rotate in same direction

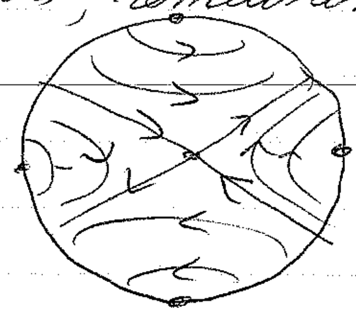
Explicit calc: $\mathbb{I} = \lambda_1 \mathbb{1} + (\lambda_3 - \lambda_1) \underset{\sim}{e}_3 \underset{\sim}{e}_3^T$

$\mathbb{I}^{-1} = \frac{1}{\lambda_1} \mathbb{1} + (\frac{1}{\lambda_3} - \frac{1}{\lambda_1}) \underset{\sim}{e}_3 \underset{\sim}{e}_3^T$

$$\Rightarrow \dot{\underset{\sim}{M}} = \underbrace{\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right)}_{\text{sign dets dir of rotation in each hemisphere.}} \langle \underset{\sim}{M}, \underset{\sim}{e}_3 \rangle \underset{\sim}{e}_3 \times \underset{\sim}{M}$$

$\underset{\sim}{M}$ rotates w/ const speed, since $\langle \underset{\sim}{M}, \underset{\sim}{e}_3 \rangle$ const along traj.

Case 3: $\mathbb{I} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > \lambda_2 > \lambda_3$
mod change of basis.
Level sets of E are triaxial ellipsoids.
Fixing a sphere & varying ellipsoids, find six pts of intersection, remaining intersections are curves:



Pts are eq - eigenvectors of $\mathbb{I}$.
~~Can show 2 saddles (ev's of λ_2), four centers~~: lin stab?

Let $X(\underline{M}) := \underline{M} \times (\mathbb{I}^{-1}\underline{M})$.
If $\mathbb{I}\underline{M}_e = \lambda_j \underline{M}_e$, then $X(\underline{M}_e) = \underline{0}$.
Let $\hat{\ }: \mathbb{R}^3 \to$ skew-sym 3×3 matrices:

$$\hat{y} z = y \times z \quad \forall y, z \in \mathbb{R}^3.$$

$$X(\underline{M}) = \hat{\underline{M}}\,\mathbb{I}^{-1}\underline{M} = -\widehat{\mathbb{I}^{-1}\underline{M}}\,\underline{M}, \text{ so}$$

$$\nabla X(\underline{M}_e) = \hat{\underline{M}}_e \mathbb{I}^{-1} - \widehat{\mathbb{I}^{-1}\underline{M}_e}$$

$$= \hat{\underline{M}}_e \mathbb{I}^{-1} - \frac{1}{\lambda_j}\hat{\underline{M}}_e$$

$$= \hat{\underline{M}}_e \left(\mathbb{I}^{-1} - \frac{1}{\lambda_j}\mathbb{1}\right).$$

$$= \hat{\underline{M}}_e \,\mathrm{diag}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_j}, \frac{1}{\lambda_2}-\frac{1}{\lambda_j}, \frac{1}{\lambda_3}-\frac{1}{\lambda_j}\right).$$

0 is an evalue w/ evector $\underline{M}_e$ — there's no motion off the sphere!

If $j=1$ or 3, get conj pair of imag evalues; if $j=2$, get real pair summing to 0.

<u>Restriction of Euler eqs to sphere</u>

We can regard $\dot{\underline{M}} = \underline{M} \times \nabla E(\underline{M})$ as an ODE on a sphere, since $\|\underline{M}(t)\| = \text{const}$.

Advantages: i) lower dim

ii) 'boring' direction is eliminated (for $E(\underline{M}) = \frac{1}{2}\langle \underline{M}, \mathbb{I}^{-1}\underline{M}\rangle$, $\mathbb{I}$ w/ distinct evalues, zero evalues of lin at eq are eliminated).

~~Disadvantages~~ iii) 2-spheres are symplectic manifolds.

Dis-adv: i) nonlinear manifold

ii) signif of $\mathbb{R}^3 \sim so(3)$ is possibly obscured (next example).

Full free rigid body eqs

Quick review of rotation group $SO(3)$:

$SO(3) = \{A \in \mathbb{R}^{3\times 3} : A^T A = \mathbb{1},\ \det A = 1\}$,

i.e. orth + or.-pres.

$SO(3)$ is a non-lin. manifold, + matrix group.

Tangent space?

Given a curve $A(\varepsilon)$ in $SO(3)$,

$\mathbb{1} \equiv A(\varepsilon)^T A(\varepsilon)$

$\Rightarrow \mathbb{0} = (A'(0))^T A(0) + A(0)^T A'(0)$

Let $A_0 = A(0)$, $B = A_0^{-1} A'(0) = A_0^T A'(0)$.

$$\begin{aligned}\mathbb{0} &= (A_0 B)^T A_0 + A_0^T (A_0 B)\\ &= B^T (A_0^T A_0) + (A_0^T A_0) B\\ &= B^T + B\end{aligned}$$

$\Rightarrow B$ is skew-symmetric.

On the other hand, given B skew-symm, def. $A(\varepsilon) := A_0 \exp(\varepsilon B)$.

$$\begin{aligned}A(\varepsilon)^T A(\varepsilon) &= (\exp(\varepsilon B))^T A_0^T A_0 \exp(\varepsilon B)\\ &= \exp(\varepsilon B^T) \exp(\varepsilon B)\end{aligned}$$

$$= \exp(-\varepsilon B)\exp(\varepsilon B)$$
$$= \mathbb{1}$$

$A(\varepsilon)$ orth $\Rightarrow |\det A(\varepsilon)| = 1$;
$\det A(0) = \det A_0 = 1$, so $A(\varepsilon) \in SO(3)$.

$$\Rightarrow T_{A_0} SO(3) = \{A_0 B : B \text{ skew-sym}\}$$
$$= \{A_0 \hat{\underline{x}} : \underline{x} \in \mathbb{R}^3\}$$
$$\simeq \mathbb{R}^3.$$

Since ~~Same~~ $\cancel{A_0 \hat{x} A_0}$ $\widehat{A_0^T \underline{x}} = A_0^T \hat{\underline{x}} A_0$,
we also have $T_{A_0} SO(3) = \{A_0 (\widehat{A_0^T x}) : \underline{x} \in \mathbb{R}^3\}$
$$= \{\hat{\underline{x}} A_0 : \underline{x} \in \mathbb{R}^3\}.$$

Def. $\tau_{L,R} : SO(3) \times \mathbb{R}^3 \to TSO(3)$
$$\tau_L(A, \underline{x}) := A\hat{\underline{x}} \qquad \tau_R(A, \underline{x}) := \hat{\underline{x}} A.$$

τ_L^{-1} is left triv of $TSO(3)$ (diffeo)
τ_R^{-1} " right " " " .

Define ODE on $SO(3) \times \mathbb{R}^3$:
$$\dot{A} = A\hat{\underline{\Omega}}$$
$$\mathbb{I}\dot{\underline{\Omega}} = (\mathbb{I}\underline{\Omega}) \times \underline{\Omega}$$

Alt: $$\dot{A} = A\,(\widehat{\mathbb{I}^{-1}\underline{M}})$$
$$\dot{\underline{M}} = \underline{M} \times (\mathbb{I}^{-1}\underline{M}) = -(\widehat{\mathbb{I}^{-1}\underline{M}})\underline{M}$$

Left triv full free rigid body eqs.
Triv of 2nd order ODE on SO(3).

Important pt: eq for $\underset{\sim}{\Omega}$ (or $\underset{\sim}{M}$) doesn't depend on A!

~~Note: Both A & M have inst ang vel~~
~~$\underset{\sim}{\Omega} = \mathbb{I}^{-1}\underset{\sim}{M}$.~~

Note: A has inst "ang vel" $\underset{\sim}{\Omega} = \mathbb{I}^{-1}\underset{\sim}{M}$;
$\underset{\sim}{M}$ has inst ang vel. $-\underset{\sim}{\Omega}$.

What happens if you rotate $\underset{\sim}{M}$ by $\underset{\sim}{A}$?
~~Define $\underset{\sim}{\mu} :=$~~ Given traj $(A(t), \underset{\sim}{M}(t))$,
def. $\underset{\sim}{\mu}(t) := A(t)\underset{\sim}{M}(t)$.

$$
\begin{aligned}
\dot{\underset{\sim}{\mu}} &= \dot{A}\underset{\sim}{M} + A\dot{\underset{\sim}{M}} \\
&= A\hat{\underset{\sim}{\Omega}}\underset{\sim}{M} + A(\underset{\sim}{M}\times\underset{\sim}{\Omega}) \\
&= A(\underset{\sim}{\Omega}\times\underset{\sim}{M} + \underset{\sim}{M}\times\underset{\sim}{\Omega}) = \underset{\sim}{0}.
\end{aligned}
$$

Red ODE: $\dot{A} = A\widehat{(\mathbb{I}^{-1}A^{-1}\underset{\sim}{\mu})}$,

$\underset{\sim}{\mu}$ const.

If we define $\mathbb{I}(A) := A\mathbb{I}A^{T}$, then,
$\mathbb{I}(A)^{-1} = (A^{T})^{-1}\mathbb{I}^{-1}A^{-1} = A\mathbb{I}^{-1}A^{T}$;

$$\mathbb{I}^{-1}A^{-1}\mu = A^{-1}\ell(A)^{-1}\mu = A^{T}\ell(A)^{-1}\mu$$

$$\widehat{A^{T}\ell(A)^{-1}\mu} = A^{T}\,\widehat{\ell(A)^{-1}\mu}\,A$$

$$\Rightarrow \dot{A} = A\left(\widehat{\mathbb{I}^{-1}A^{-1}\mu}\right)$$

$$= \widehat{\ell(A)^{-1}\mu}\,A \qquad (*)$$

1st order ODE on $SO(3)$, param by μ.

μ is spatial ang mom. (conserved); $\ell(A)^{-1}\mu$ is spatial ang vel.

~~Systems~~ Sols of one eq map onto sols of other. Neither is fund. superior.

Note: Conservation of μ is directly related to indep of Euler eq from A.

Analogs of this ~~pair~~ trio of systems exist for all Lie groups.
Analog of spheres is coadjoint orbits.

What do these eq have to do w/ rigid bodies?

Consider a 3-D ~~body~~ rigid body — rel. position of pts doesn't change.
Let B_{ref} denote ~~the region~~ a ref. pos. of the body & $\varphi_t : B_{ref} \to \mathbb{R}^3$ denote map to pos at time t.
Assume φ_0 is or. pres.; then φ_t is a Euc motion: $\exists$ curves $A(t)$ in $SO(3)$, $\underline{c}(t)$ in $\mathbb{R}^3$ s.t.

$$\varphi_t(\underline{X}) = A(t)\,\underline{X} + \underline{c}(t).$$

For convenience, assume ref coords $\underline{X}$ are ~~dist fr~~ rel to center of mass, & that $\underline{c} \equiv \underline{0}$. (This involves exp of transl symmetry.)
Let $\underline{\Omega}(t) \in \mathbb{R}^3$ be det by $\dot{A}(t) = A(t)\,\hat{\underline{\Omega}}(t)$,
so

$$\begin{aligned} \dot{\varphi}_t(\underline{X}) &= \dot{A}(t)\,\underline{X} \\ &= A(t)\,\hat{\underline{\Omega}}(t)\,\underline{X} \\ &= A(t)\,(\underline{\Omega}(t) \times \underline{X}). \end{aligned}$$

$$KE = \frac{1}{2}\int_{B_{ref}} \rho(\underline{X})\, \|\dot{\varphi}_t(\underline{X})\|^2 \, d\underline{X},$$

where ρ is density.

$\cancel{\frac{1}{2}\int_{B_{ref}}}$ Now,

$$
\begin{aligned}
\|\dot{\varphi}_t(\underset{\sim}{X})\|^2 &= \|A(t)(\underset{\sim}{\Omega}(t) \times \underset{\sim}{X})\|^2 \\
&= \|\underset{\sim}{\Omega}(t) \times \underset{\sim}{X}\|^2 \\
&= \|\underset{\sim}{\Omega}(t)\|^2 \|\underset{\sim}{X}\|^2 - \langle \underset{\sim}{\Omega}(t), \underset{\sim}{X} \rangle^2
\end{aligned}
$$

$$
\Rightarrow KE = \frac{1}{2}\int_{B_{ref}} \rho(\underset{\sim}{X}) \left(\|\underset{\sim}{\Omega}\|^2 \|\underset{\sim}{X}\|^2 - \langle \underset{\sim}{\Omega}, \underset{\sim}{X} \rangle^2 \right) d\underset{\sim}{X}
$$

$$
= \frac{1}{2} \left\langle \underset{\sim}{\Omega}, \left(\int_{B_{ref}} \rho(\underset{\sim}{X}) \left(\|\underset{\sim}{X}\|^2 \mathbb{1} - \underset{\sim}{X}\underset{\sim}{X}^T \right) d\underset{\sim}{X} \right) \underset{\sim}{\Omega} \right\rangle .
$$

Let $\mathbb{I} = \int_{B_{ref}} \rho(\underset{\sim}{X}) \left(\|\underset{\sim}{X}\|^2 \mathbb{1} - \underset{\sim}{X}\underset{\sim}{X}^T \right) d\underset{\sim}{X}$.

(This sets some restrictions on evalues of $\mathbb{I}$.)