Homework #1

Conservation laws for the Euler-Lagrange equations.

1) Given a time-independent Lagrangian $L:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, define the energy $E:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$E(q,v) := \left\langle \frac{\partial L}{\partial v}(q,v), v\right\rangle - L(q,v)$$

Show that E is constant along solution curves of the Euler-Lagrange equations determined by L.

2) (Noether's Theorem - special case)
Let $G\subseteq GL(n)$ be a matrix group with algebra $\mathfrak{g}=T_I G$ (infinitesimal group elements).
Given $\xi\in\mathfrak{g}$, define the infinitesimal generator $\xi_{\mathbb{R}^n}$ by

$$\xi_{\mathbb{R}^n}(q) := \frac{d}{d\varepsilon}\exp(\varepsilon\xi)\,q\Big|_{\varepsilon=0} \quad \forall q\in\mathbb{R}^n.$$

(Here $\exp:\mathfrak{g}\to G$ is the usual matrix exponential.)

Assume that a time-independent Lagrangian $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is invariant under the action of G on $\mathbb{R}^n$, i.e. $L(Aq, A\underline{v}) = L(q, \underline{v})$ $\forall q, v \in \mathbb{R}^n$, $A \in G$.

Show that for any $\xi \in \mathfrak{g}$, the scalar function $J_\xi(q, \underline{v}) := \langle \frac{\partial L}{\partial \underline{v}}(q, \underline{v}), \xi_{\mathbb{R}^n}(q) \rangle$ is constant along solution curves of the Euler-Lagrange equation det. by L.

Angular momentum: Show that if $G = SO(3)$, then — using the identification of $so(3)$ with $\mathbb{R}^3$ — $\xi_{\mathbb{R}^3}(q) = \underset{\sim}{\xi} \times q$ and conservation of $J_{\underset{\sim}{\xi}}$ for all $\xi \in \mathbb{R}^3$ is equivalent to conservation of angular momentum.

3) Particle in a central force field:

Given a scalar function $\varphi: \mathbb{R} \to \mathbb{R}$, ~~define~~ and positive constant m, define the Lagrangian $L: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $L(q, \underline{v}) := \frac{m}{2}\|\underline{v}\|^2 - \varphi(\frac{1}{2}\|q\|^2)$.

i) Show that angular momentum is

conserved along trajectories of the Euler-Lagrange equations determined by L.

ii) Show that if the angular momentum is 0, the E-L equations are equivalent to a 2^{nd} order ODE on $\mathbb{R}$. Use conservation of energy to obtain qualitative information about the trajectories with 0 ang. mom.

iii) Show that if the angular momentum is nonzero, the E-L equations are equivalent to a 2^{nd} order ODE on $\mathbb{R}^2$. Express these equations in polar coordinates; find expressions for the energy and angular momentum in polar coords. Use the conservation laws to solve the system as fully as possible, obtaining a reduced/simplified system of ODEs.

4) Consider the Lagrangian $L: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$
$L(q,v) = \frac{v^2}{2} - \frac{q^2}{2}$.
Determine the Euler-Lagrange equations, the sols, and the energy. (easy!)
Determine the approximate solutions

using the time-discretizations / numerical methods described below, and describe the behavior of the energy along the discrete trajectories.

Given a vector field $\underline{X}$ on $\mathbb{R}^n$ and time step Δt, ~~define~~ and an initial condition x_0, define discrete trajectories $\{x_0, x_1, \ldots\}$ using the following rules:

i) $\dfrac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \underline{X}(\underline{x}_n)$

ii) $\dfrac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \underline{X}(\underline{x}_{n+1})$

iii) $\dfrac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \underline{X}(\underline{x}_{n+1/2})$,

where $\dfrac{\underline{x}_{n+1/2} - \underline{x}_n}{\Delta t/2} = \underline{X}(\underline{x}_n)$

iv) $\dfrac{\underline{x}_{n+1} - \underline{x}_n}{\Delta t} = \dfrac{1}{2}\left(\underline{X}(\underline{x}_n) + \underline{X}(\underline{x}_{n+1})\right)$.