The canonical symplectic structure on a cotangent bundle

Define the map $\pi_Q : T^*Q \to Q$
by $\pi_Q(\mu_q) := q \quad \forall\, \mu_q \in T_q^*Q$,
and the canonical 1-form Θ_0 on T^*Q
by
$$\Theta_0(\mu_q)\cdot v_{\mu_q} := \underset{\in T_q^*Q}{\mu_q}\cdot \underbrace{(d_{\mu_q}\pi_Q \cdot v_{\mu_q})}_{\in T_qQ}.$$

Example 1 $\quad Q = \mathbb{R}^n$

$$\pi_{\mathbb{R}^n}(q,p) = q, \quad d_{(q,p)}\pi(q,p,v,w) = (q,v).$$
$$\Theta_0(q,p)\cdot(q,p,v,w) = (q,p)\cdot(q,v) = \langle p, v\rangle.$$

Example 2 $\quad Q = G \subseteq GL(n)$.

~~$\pi_G(\tilde{\tau}_L$~~ Define $\tilde{\tau}_{L,R} : G \times \mathfrak{g}^* \to T^*G$
by $\tilde{\tau}_L(g,\mu) := g^{-T}\mu$
and $\tilde{\tau}_R(g,\mu) := \mu g^{-T}$,
~~Then~~ If $\mathfrak{g}^*$ is identified with $\mathfrak{g}$
using the standard matrix inner product.

$$\tilde{\tau}_L(g,\mu)\cdot\tau_L(g,\xi) = \langle g^{-T}\mu, g\xi\rangle = \langle\mu,\xi\rangle;$$

similarly, $\tilde{\tau}_R(g,\mu)\cdot\tau_R(g,\xi\rangle = \langle\mu,\xi\rangle$.

$\pi_G(\tilde{\tau}_L(g,\mu)) = g$.

We'll compute $\tilde{\theta}_0 := \tilde{\tau}_L^*\theta_0$, a 1-form on $T(G\times\mathfrak{g}^*)\sim TG\times\mathfrak{g}^*\times\mathfrak{g}^*$.

$$d_{(g,\mu)}(\pi_G\circ\tilde{\tau}_L)\cdot(\tau_L(g,\xi),\mu,\nu)$$
$$= \tau_L(g,\xi).$$

Hence $\tilde{\theta}_0(g,\mu)\cdot(\tau_L(g,\xi),\mu,\nu)$

$$= \tilde{\tau}_L(g,\mu)\cdot d_{(g,\mu)}(\pi_G\circ\tilde{\tau}_L)\cdot(\tau_L(g,\xi),\mu,\nu))$$
$$= \tilde{\tau}_L(g,\mu)\cdot\tau_L(g,\xi)$$
$$= \langle\mu,\xi\rangle.$$

Given a 1-form β on Q, $\beta^*\theta_0 = \beta$.

Pf: $(\beta^*\theta_0)(q)\cdot v_q = \theta_0(\beta(q))\cdot d_q\beta\cdot v_q)$

$$= \beta(q)\cdot d_{\beta(q)}\pi_Q(d_q\beta\cdot v_q)$$
$$= \beta(q)\cdot(d_q(\pi_Q\circ\beta)\cdot v_q)$$
$$= \beta(q)\cdot v_q$$

since $\pi_Q \circ \beta = id$ $\quad$ $\begin{array}{c} T^*Q \\ \beta \uparrow \downarrow \pi_Q \\ Q \end{array}$.

Introduce construction $\mathrm{vert}_{\mu_q} : T_q^* Q \to T_{\mu_q} T^*Q$,

$\mathrm{vert}_{\mu_q}(\nu_q) := \frac{d}{d\varepsilon} \mu_q + \varepsilon \nu_q \big|_{\varepsilon=0}$,

<u>vertical lift</u> of ν_q.

$$d_{\mu_q}\pi_Q \cdot \mathrm{vert}_{\mu_q}(\nu_q) = \frac{d}{d\varepsilon} \pi_Q(\mu_q + \varepsilon \nu_q) \big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon} q \big|_{\varepsilon=0}$$
$$= 0.$$

Hence $\Theta_0(\mu_q) \cdot \mathrm{vert}_{\mu_q}(\nu_q) = 0$

for any $\mu_q, \nu_q \in T_q^* Q$, $q \in Q$.

vert_{μ_q} is 1-1, so $\mathrm{vert}_{\mu_q}(T_q^* Q)$ is isomorphic to $T_q^* Q$.

If $\mu_q = \beta(q)$ for some 1-form β on Q, then given $w_{\mu_q} \in T_{\mu_q}(T^*Q)$, let

$v_q := d_{\mu_q}\pi_Q \cdot w_{\mu_q} \in T_q Q$.

Claim: $\exists$ $\nu_q \in T_q^* Q$ such that

$w_{\mu_q} = d_q\beta \cdot v_q + \mathrm{vert}_{\mu_q}(\nu_q)$.

Sketch of proof:

$$d_{\mu_q}\Pi_Q\cdot(\omega_{\mu_q} - d_q\beta\cdot v_q) = v_q - d_q(\Pi_Q\circ\beta)\cdot v_q$$
$$= 0.$$

Hence it suffices to show that
$\mathrm{vert}_{\mu_q}(T_q^*Q) = \ker(d_{\mu_q}\Pi_Q)$.
We know $\mathrm{vert}_{\mu_q}(T_q^*Q) \subseteq \ker(d_{\mu_q}\Pi_Q)$,
and $\dim(\mathrm{vert}_{\mu_q}(T_q^*Q)) = \dim Q$,
since it's isomorphic to T_q^*Q.

Since $\mathrm{range}(d_{\mu_q}\Pi_Q) = T_qQ$,
$\mathrm{rank}(d_{\mu_q}\Pi_Q) = \dim Q$, so

$$\begin{aligned}\mathrm{nullity}(d_{\mu_q}\Pi_Q) &= \dim(T^*Q) - \mathrm{rank}(d_{\mu_q}\Pi_Q)\\ &= 2\dim Q - \dim Q\\ &= \dim(\mathrm{vert}_{\mu_q}(T_q^*Q)).\end{aligned}$$

Given a relevant / natural choice of β, we can regard $d_q\beta\cdot v_q$ as a plausible choice of a variation of configuration with "fixed" momentum — not intrinsic, though.

$\mathrm{vert}_{\mu_q}(\nu_q)$ is a entirely natural variation of momentum with fixed configuration.

Canonical symplectic structure on T^*Q

The canonical symplectic structure on T^*Q is $\omega := -d\theta_0$ i.e. minus the exterior derivative of the canonical 1-form. (Caution: This sign convention is not universal.)

Mini-review of exterior differentiation:
$\underset{\sim}{d}$ is the unique map taking k-forms to $(k+1)$-forms that satisfies

i) $\underset{\sim}{d} f = df$, the conventional differential, if f is a 0-form, i.e. scalar function.

ii) $\underset{\sim}{d}$ is an anti-derivation:

$$\underset{\sim}{d}(\lambda\beta) = \lambda \underset{\sim}{d}\beta \quad \forall \lambda \in \mathbb{R}, \text{ and } \forall k\text{-forms } \beta$$

$$\underset{\sim}{d}(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$$

$\forall$ k-forms α and l-forms β

iii) $\underset{\sim}{d}(\underset{\sim}{d}\beta) \equiv 0 \quad \forall$ forms β

iv) $\underset{\sim}{d}$ is natural with respect to restrictions: if β is defined on M and U is a subset of M, then

$$(\underset{\sim}{d}\beta)|_U = \underset{\sim}{d}(\beta|U).$$

Important property: $\underset{\sim}{d}$ commutes with pullback: $\varphi^*(\underset{\sim}{d}\beta) = \underset{\sim}{d}(\varphi^*\beta)$.

Very important property: "Cartan's magic formula": Given a vector field X and form β,

$$\mathcal{L}_X \beta = \iota_X \underset{\sim}{d}\beta + \underset{\sim}{d}\iota_X\beta.$$

Back to canonical symplectic structure:

Strategy for computing ω: Given $\mu_q \in T^*Q$ and $v_{\mu_q}, w_{\mu_q} \in T_{\mu_q}T^*Q$, choose 'convenient' vector fields X and Y such that $X(\mu_q) = v_{\mu_q}$, $Y(\mu_q) = w_{\mu_q}$. Then

$$\begin{aligned}
\omega(\mu_q)(v_{\mu_q}, w_{\mu_q}) &= \omega(\mu_q)(X(\mu_q), Y(\mu_q)) \\
&= (\iota_Y \iota_X \omega)(\mu_q) \\
&= -(\iota_Y \iota_X \underset{\sim}{d}\theta_0)(\mu_q) \\
&= -\iota_Y(\mathcal{L}_X\theta_0 - \underset{\sim}{d}(\iota_X\theta_0))(\mu_q).
\end{aligned}$$

Often convenient to further use identity

$$\mathcal{L}_X(\iota_Y\beta) = \iota_{\mathcal{L}_X Y}\beta + \iota_Y(\mathcal{L}_X\beta),$$

getting

$$\omega(\mu_q)(v_{\mu_q}, W_{\mu_q})$$
$$= \left(L_Y(i_X\theta_0) - L_X(i_Y\theta_0) + i_{L_X Y}\theta_0\right)(\mu_q).$$

For example, if $\mu_q = \beta(q)$ for some 1-form β on Q, then $\exists\, s_q, u_q \in T_qQ$ and $\nu_q, \tau_q \in T_q^*Q$ such that

$$v_{\mu_q} = d_q\beta\, s_q + \mathrm{vert}_{\mu_q}(\nu_q)$$
$$W_{\mu_q} = d_q\beta \cdot u_q + \mathrm{vert}_{\mu_q}(\tau_q).$$

$$\omega(\mu_q)(v_{\mu_q}, W_{\mu_q}) = \left(\beta^*(-\underset{\sim}{d}\theta_0)\right)(\cancel{\mu}_q)(s_q, u_q)$$
$$+\ \omega(\mu_q)(\mathrm{vert}_{\mu_q}(\nu_q),\ d_q\beta \cdot u_q)$$
etc.

First term satisfies

$$(\beta^*(-\underset{\sim}{d}\theta_0))(\cancel{\mu}_q)(s_q, u_q)$$
$$= -\underset{\sim}{d}(\beta^*\theta_0)(\cancel{\mu}_q)(s_q, u_q)$$
$$= -\underset{\sim}{d}\beta(q)(s_q, u_q).$$

Simplifications for terms of form
$\omega(\mu_q)(\mathrm{vert}_{\mu_q}(\nu_q), \cdot\,)$?

Choose 1-form γ such that $\nu_q = \gamma(q)$ and define the vector field X on T^*Q

by $\underline{X}(\mu_q) := \text{vert}_{\mu_q}(\gamma'(q))$.
Flow map $\hat{F}_\varepsilon$ of $\underline{X}$ is $\hat{F}_\varepsilon(\mu_q) = \mu_q + \varepsilon\gamma'(q)$;
note that flow never leaves the fiber —
hence $\pi_Q \circ \hat{F}_\varepsilon \equiv \pi_Q$.

$$(\mathcal{L}_{\underline{X}}\Theta_0)(\mu_q)(Z_{\mu_q}) = \frac{d}{d\varepsilon}(\hat{F}_\varepsilon^*\Theta_0)(\mu_q)(Z_{\mu_q})\Big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon}\hat{F}_\varepsilon(\mu_q)\cdot(d_{\mu_q}(\pi_Q\circ\hat{F}_\varepsilon)\cdot Z_{\mu_q})\Big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon}(\mu_q + \varepsilon\gamma'(q))\cdot(d_{\mu_q}\pi_Q\cdot Z_{\mu_q})\Big|_{\varepsilon=0}$$
$$= \gamma'(q)\cdot(d_{\mu_q}\pi_Q\cdot Z_{\mu_q}).$$

Since $i_{\underline{X}}\Theta_0 \equiv 0$, it follows that

$$\omega(\mu_q)(\text{vert}_{\mu_q}(v_q), Z_{\mu_q})$$
$$= -(\mathcal{L}_{\underline{X}}\Theta_0)(\mu_q)(Z_{\mu_q})$$
$$= -\gamma'(q)\cdot(d_{\mu_q}\pi_Q\cdot Z_{\mu_q}).$$
$$= -v_q\cdot(d_{\mu_q}\pi_Q\cdot Z_{\mu_q})$$

Substituting into the expansion of $\omega(\mu_q)(V_{\mu_q}, W_{\mu_q})$,
we find

$$\omega(\mu_q)(V_{\mu_q}, W_{\mu_q}) = -\underset{\wedge}{d}\beta(q)(s_q, u_q)$$
$$- v_q\cdot(d_{\mu_q}\pi_Q\cdot d_q\beta\cdot u_q)$$
$$+ t_q\cdot(d_{\mu_q}\pi_Q\cdot d_q\beta\cdot s_q) + 0$$

$$= \tau_q \cdot s_q - \nu_q \cdot u_q - \underset{\sim}{d}\beta(q)(s_q, u_q)$$

Exercise: $Q = \mathbb{R}^n$.
If $p_q = (q, p)$, set $\beta(q) = (q, p)$, so
$\beta(q) \cdot (q, \underset{\sim}{v}) = \langle p, \underset{\sim}{v} \rangle$.
Show the above formula reduces to

$$\omega(q,p)((q,p,\underset{\sim}{v}_1,\underset{\sim}{w}_1),(q,p,\underset{\sim}{v}_2,\underset{\sim}{w}_2)) = \langle \underset{\sim}{w}_2, \underset{\sim}{v}_1 \rangle - \langle \underset{\sim}{w}_1, \underset{\sim}{v}_2 \rangle,$$