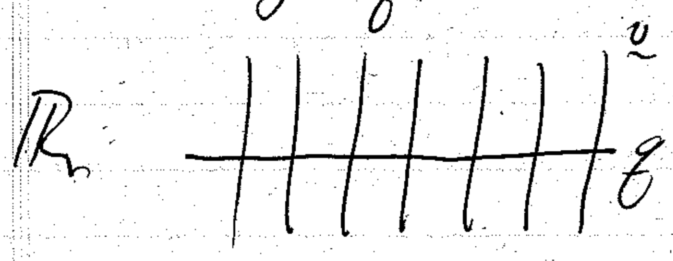
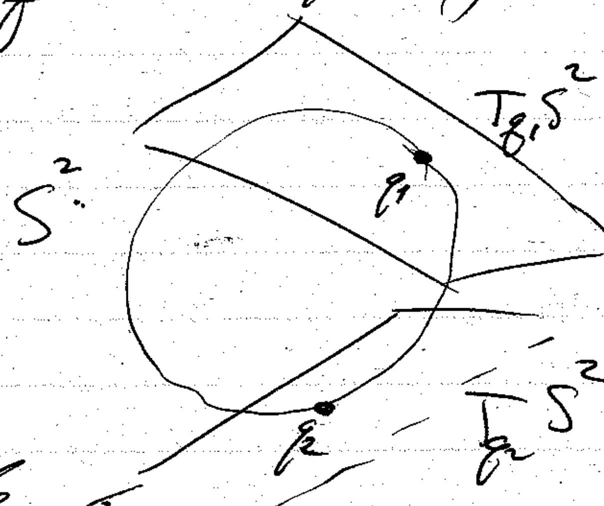
Variational problems on Lie groups

Key challenge in "naive" formulation of variational problems on nonlinear manifolds is the choice of a practical implementation of "$\frac{\partial L}{\partial q}$": how to vary the configuration q while "fixing" the velocity $\dot{q}$?





Want a 'natural' computationally tractable way of translating tangent vectors.

On Lie groups we can use the global left and right trivializations of the tangent bundle:

$$\tau_L, \tau_R : G \times \mathfrak{g} \to TG$$

$$\tau_L(g, \xi) := d_e L_g \cdot \xi \quad \in T_g G$$

$$\tau_R(g, \xi) := d_e R_g \cdot \xi, \ \circledast$$

Inverses are τ_L^{-1}, τ_R^{-1}.

$\circledast$ where $L_g(h) := gh$, $R_g(h) := hg$.

For matrix groups with mult being usual matrix mult, L_g, R_g are linear, so $\tau_L(g\xi) = g\xi$ etc.

More generally, since $\exp(\varepsilon\xi)$ is a curve through e in G, tangent to ξ at e,
$$\tau_L(g,\xi) = d_e L_g \cdot \xi = \frac{d}{d\varepsilon} L_g(\exp(\varepsilon\xi))\Big|_{\varepsilon=0}$$

Given a smooth function $f: G \to \mathbb{R}$, define $\frac{\delta f}{\delta g} : G \to \mathfrak{g}^*$ by
$$\frac{\delta f}{\delta g}(g) \cdot \xi := d_g f \cdot \tau_L(g,\xi) = \frac{d}{d\varepsilon} f(g\exp(\varepsilon\xi))\Big|_{\varepsilon=0}$$
analog., can use τ_R.

For a matrix group, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the inner product $\langle A, B\rangle = \text{trace}(A^T B)$ (componentwise Euclidean norm; if $G \leq GL(n)$, $A, B \in G$, then $\langle A, B \rangle = \sum_{i,j=1,\dots,n} a_{ij} b_{ij}$).

Given a curve of curves $\gamma_\varepsilon : [t_0, t_1] \to G$, there are unique maps $\eta : [t_0, t_1] \to \mathfrak{g}$ and $\omega : (-\varepsilon_0, \varepsilon_0) \times [t_0, t_1] \to \mathfrak{g}$ such that

$$\frac{d}{d\varepsilon}\gamma_\varepsilon(t)\Big|_{\varepsilon=0} = T_L(g(0,t),\eta(t))$$

and $\frac{d}{dt}\gamma_\varepsilon(t) = T_L(g(\varepsilon,t),\omega(\varepsilon,t))$

(Similar for T_R.)

On $\mathbb{R}^k$ mixed partials commute, but what is the relationship between $\dot\eta(t)$ and $\frac{d}{d\varepsilon}\omega(\varepsilon,t)|_{\varepsilon=0}$?

Work in matrix group case $G \subseteq gl(n)$, for notational and conceptual simplicity (informal optional exercise: do calculation in general case).

$$\frac{d}{dt}\frac{d}{d\varepsilon}\gamma_\varepsilon(t)\Big|_{\varepsilon=0} = \frac{d}{dt}\big(g(0,t)\eta(t)\big)$$

$$= \left(\frac{d}{dt}g(0,t)\right)\eta(t) + g(0,t)\dot\eta(t)$$

$$= g(0,t)\,\omega(0,t)\,\eta(t) + g(0,t)\,\dot\eta(t).$$

Letting $g(t) := g(0,t)$ ~~and~~, $\omega(t) := \omega(0,t)$ and $\omega'(t) := \frac{d}{d\varepsilon}\omega(\varepsilon,t)|_{\varepsilon=0}$

$$\frac{d}{dt}\frac{d}{d\varepsilon}\gamma_\varepsilon(t)\Big|_{\varepsilon=0} = g(t)\big(\omega(t)\eta(t) + \dot\eta(t)\big).$$

$$\frac{d}{d\varepsilon}\frac{d}{dt}\gamma_\varepsilon(t)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}g(\varepsilon,t)\,\omega(\varepsilon,t)\Big|_{\varepsilon=0}$$

$$= \left(\frac{d}{d\varepsilon}g(\varepsilon,t)\Big|_{\varepsilon=0}\right)\omega(0,t)$$

$$+ g(0,t)\left(\frac{d}{d\varepsilon}\omega(\varepsilon,t)\Big|_{\varepsilon=0}\right)$$

$$= g(t)\left(\eta(t)\omega(t) + \omega'(t)\right).$$

Since $G \subseteq GL(n)$, $g(t)$ is invertible, so

$$\begin{aligned}\omega(t)\eta(t) + \dot{\eta}(t) &= g(t)^{-1}\frac{d}{dt}\frac{d}{d\varepsilon}\gamma_\varepsilon(t)\Big|_{\varepsilon=0}\\ &= g(t)^{-1}\frac{d}{d\varepsilon}\frac{d}{dt}\gamma_\varepsilon(t)\Big|_{\varepsilon=0}\\ &= \eta(t)\omega(t) + \omega'(t),\end{aligned}$$

i.e.
$$\begin{aligned}\omega'(t) &= \dot{\eta}(t) + \omega(t)\eta(t) - \eta(t)\omega(t)\\ &= \dot{\eta}(t) + [\omega(t), \eta(t)].\end{aligned}$$

If R_g, T_{R_-} are used, commutator has opposite sign.

Given $\tilde{L} : (G \times \mathfrak{g}) \times \mathbb{R} \to \mathbb{R}$,
define $\mathcal{L}(\gamma) := \int_{t_0}^{t_1} \tilde{L}\left(T_L^{-1}(\dot{\gamma}(t)), t\right)dt.$

Given $\gamma_\varepsilon, \eta, \omega$ as above,

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{L}(\gamma_\varepsilon)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\tilde{L}\left((g(\varepsilon,t), \omega(\varepsilon,t)), t\right)dt\Big|_{\varepsilon=0}\\ &= \int_{t_0}^{t_1}\left(\frac{d}{d\varepsilon}\tilde{L}\left((g(\varepsilon,t), \omega(t)), t\right)\Big|_{\varepsilon=0} {}^{*} + \left\langle\frac{\partial\tilde{L}}{\partial\omega}(g(t), \omega(t)), \omega'(t)\right\rangle\right)dt\\ &= \int_{t_0}^{t_1}\left(\left\langle\frac{\delta\tilde{L}}{\delta g}(g(t), \omega(t), t), \eta(t)\right\rangle\right.\end{aligned}$$

$$+\left\langle \frac{\partial \tilde{L}}{\partial \omega}(g,\omega,t), \dot{\eta} + [\omega, \eta] \right\rangle \Big) dt$$

Integrating by parts and using the identity

$$\langle A, [B,C] \rangle = \operatorname{trace}(A^T(BC - CB))$$
$$= \operatorname{trace}((A^TB - BA^T)C)$$
$$= \langle B^TA - AB^T, C \rangle$$
$$= \langle [B^T, A], C \rangle,$$

we can rewrite the second term as

$$\int_{t_0}^{t_1} \left\langle -\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \omega}(g,\omega,t) + \left[\omega^T, \frac{\partial \tilde{L}}{\partial \omega}(g,\omega,t)\right], \eta \right\rangle dt$$
$$+ \left\langle \frac{\partial \tilde{L}}{\partial \omega}, \eta \right\rangle \Big]_{t_0}^{t_1}.$$

Thus the Euler-Lagrange equations for the left trivialization of a matrix group $G \leq GL(n)$ are

$$\dot{g} = g\omega$$
$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \omega}(g,\omega,t) = \left[\omega^T, \frac{\partial \tilde{L}}{\partial \omega}(g,\omega,t)\right] + \frac{\delta \tilde{L}}{\delta g}(g,\omega,t).$$

Example 2 $G = SO(3)$ Heavy top.

$$\tilde{L}(g,\omega) = \tfrac{1}{2}\langle \underline{\omega}, \mathbb{I}\underline{\omega}\rangle + \langle g\,\underline{m}, \underline{\Gamma}\rangle,$$

where $\underline{m} = \int_{B_{ref}} \rho_{ref}(\underline{X})\,\underline{X}\,d\underline{X}$,

$\underline{\Gamma}$ ~~gravity direction~~, mg gravity vector.

$$\left\langle \frac{\delta \tilde{L}}{\delta g}(g,\omega), \eta \right\rangle =$$

$$\frac{d}{d\varepsilon}\tilde{L}(g\exp(\varepsilon\hat{\eta}),\omega)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\langle g\exp(\varepsilon\hat{\eta})\,\underline{m}, \underline{\Gamma}\rangle\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\langle (\mathbb{1}+\varepsilon\hat{\eta}+O(\varepsilon^2))\,\underline{m}, g^T\underline{\Gamma}\rangle\Big|_{\varepsilon=0}$$

$$= \cancel{\tfrac{d}{d\varepsilon}\langle \underline{m}+\varepsilon\eta} \;\langle \hat{\eta}\,\underline{m}, g^T\underline{\Gamma}\rangle$$

$$= \langle \eta\times\underline{m}, g^T\underline{\Gamma}\rangle$$

$$= \langle \underline{m}\times g^T\underline{\Gamma}, \eta\rangle, \qquad \forall \eta\in\mathbb{R}^3$$

$$\Rightarrow \frac{\delta \tilde{L}}{\delta g}(g,\underline{\omega}) = \underline{m}\times g^T\underline{\Gamma}.$$

$$\frac{\partial \tilde{L}}{\partial \omega}(g,\omega) = \mathbb{I}\underline{\omega} \quad \text{as before.}$$

E-L eqn. $\quad \dot{g} = g\hat{\omega}$

$$\mathbb{I}\dot{\underline{\omega}} = \mathbb{I}\underline{\omega}\times\underline{\omega} + \underline{m}\times g^T\underline{\Gamma}$$