EXERCISES

3.8A. (i) Give an example to show that J need not be minimized at a solution of Lagrange's equations. (*Hint.* Geodesics on the sphere.)

(ii) By considering second variations, show that, *locally*, J is minimized for geodesics of a Riemannian metric. (When minimization ceases one encounters conjugate points.)

3.8B. Suppose one begins with a variational formulation as in 3.8.3 as basic and then discovers the Lagrange equations. Show that by multiplying these equations by $\dot{q}^i$ and integrating you are led to the energy (Historically, this is a path to Hamilton's equations.)

3.8C. Let L be a Lagrangian depending on $q^i, \dot{q}^i, \ddot{q}^i, \ldots, \overset{(m)}{q}{}^i$.

(a) Derive the corresponding Euler–Lagrange equations by a variational argument.

(b) Show that these equations can be put into Hamiltonian form (see Whittaker [1959, pp. 265–267]).

(c) Formulate your results intrinsically on manifolds; you will need to find out about jet bundles. (See Sect. 5.5 and Rodrigues [1976].)

3.8D. Formulate and prove a principle of least action for (hyperregular) Hamiltonian systems on T^*Q.

3.8E. (A. Lichnerowicz, R. Jantzen and the authors). Let (P,ω) be a symplectic manifold and F_t the flow of a Hamiltonian vector field X_H. Let $G_t = TF_t$ be the tangent flow on TP and Y its generator. Show that Y is Hamiltonian with energy $\dot{H}(v) = -\omega(v, \tau_{TP} \cdot TX_H(v))$. (Use problems 1.6D and 3.3I). If (q^i, p_i) are canonical coordinates on P and (q^i, p_i, Q^i, P_i) are induced coordinates on TP show that

$$\dot{H}(q^i, p_i, Q^i, P_i) = Q^i \frac{\partial H}{\partial q^i} + P_j \frac{\partial H}{\partial p_j}.$$

(The system $X_{\dot{H}}$ on TP is called the *linearized Hamiltonian system of* X_H).

3.8F. (S. Shahshahani). Let $D \in \mathfrak{X}(TM)$ and define its *fiber differential* by $d_F D \in \mathfrak{X}^*(TM)$, $(d_F D) \cdot w_v = dD \cdot (T\tau_M \cdot w_v)^l_v$ (see 3.7.5). (a). In coordinates, show that $d_F D = \Sigma \frac{\partial D}{\partial \dot{q}^i} dq^i$. (b) Let ω be a symplectic form on TM which vanishes when pulled back to each fiber T_xM. Show that $\Delta = (d_F D)^\#$ is a vertical vector field. Let X_E be a second order Hamiltonian vector field on TM so that $Y = X_E + \Delta$ is also second order; Y is called a dissipative system with *Rayleigh dissipation function* D. (c) Show that this generalizes Exercise 3.7A(ii). (d) Let C denote the canonical vertical vector field on TM given by lifting vertically. Show that energy decreases along orbits of Y iff $C(D) > 0$. (e) Show that van der Pol's equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ is a dissipative system in this sense with E the harmonic oscillator energy and $D(x, \dot{x}) = \frac{1}{2}\mu\dot{x}^2(x^2 - 1)$. Use (d) to study when energy is decreasing and verify by a direct calculation.

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CHAPTER 4

Hamiltonian Systems with Symmetry

Associated with each one-parameter group of symmetries of a Hamiltonian system is a conserved quantity. For a group of symmetries we get thereby a vector-valued conserved quantity called the momentum. We shall discuss the properties of the momentum and how to construct it in Sect. 4.2, after summarizing the necessary topics from Lie group theory in Sect. 4.1. When symmetries are present the phase space can be reduced; that is, a number of variables eliminated. This topic is the subject of Sect. 4.3. Mechanical systems on Lie groups and the rigid body are discussed in Sect. 4.4. Smale's topological program for a mechanical system with symmetry is presented in Sect. 4.5 and this is applied to the rigid body problem in Sect. 4.6. A number of results presented in this chapter are new.

4.1. LIE GROUPS AND GROUP ACTIONS

In this section we develop the basic facts about Lie groups and actions of Lie groups on manifolds which we will need for applications to mechanics.

4.1.1 Definition. *A* ***Lie group*** *is a finite-dimensional smooth manifold G that is a group and for which the group operations of multiplication,* $\cdot : G \times G \to G : (g,h) \mapsto g \cdot h$, *and inversion,* $^{-1} : G \to G : g \mapsto g^{-1}$ *are smooth. Let e = identity.*

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4.1.2 Example. The group of linear isomorphisms of $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$, denoted $Gl(n,\boldsymbol{R})$, is a Lie group of dimension n^2. It is a smooth manifold, being an open subset of $\boldsymbol{R}^{n^2}$ and the group operations are smooth since the formulas for the product and inverse of matrices are smooth in the matrix components.

For every $g \in G$ the maps $L_g: G \mapsto G: h \mapsto gh$ and $R_g: G \mapsto G: h \mapsto hg$ are, respectively, *left* and *right translation* by g. Since $L_g \circ L_h = L_{gh}$ and $R_h \circ R_g = R_{gh}$, $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. Thus both L_g and R_g are diffeomorphisms. Moreover, $L_g \circ R_h = R_h \circ L_g$.

Actually, smoothness of inversion follows *automatically* from smoothness of multiplication. This is easily seen by applying the inverse function theorem to the map $(g,h) \mapsto (g,gh)$ of $G \times G$ to $G \times G$.

A vector field X on G is called *left invariant* if for every $g \in G$, $(L_g)_* X = X$, that is,

$$T_h L_g X(h) = X(gh) \qquad \text{for every } h \in G$$

Let $\mathfrak{X}_L(G)$ be the set of left-invariant vector fields on G; then the maps $\rho_1: \mathfrak{X}_L(G) \to T_eG: X \mapsto X(e)$ and $\rho_2: T_eG \to \mathfrak{X}_L(G): \xi \mapsto \{g \mapsto X_\xi(g) = T_e L_g \xi\}$ satisfy $\rho_1 \circ \rho_2 = id_{T_eG}$ and $\rho_2 \circ \rho_1 = id_{\mathfrak{X}_L(G)}$. Therefore, $\mathfrak{X}_L(G)$ and T_eG are isomorphic as vector spaces. Actually $\mathfrak{X}_L(G)$ is a Lie subalgebra of the set of all vector fields on G because if $X, Y \in \mathfrak{X}_L(G)$, then for every $g \in G$,

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y]$$

Defining a Lie bracket in T_eG by

$$[\xi,\eta] = [X_\xi, X_\eta](e) \qquad \text{for } \xi,\eta \in T_eG$$

makes T_eG into a Lie algebra (see 2.2.13). Note that $[X_\xi, X_\eta] = X_{[\xi,\eta]}$.

4.1.3 Definition. *The vector space T_eG with this Lie algebra structure is called the* ***Lie algebra*** *of G and is denoted by $\mathfrak{g}$ or if there is danger of confusion, by $\mathfrak{L}(G)$ or $\mathfrak{g}_G$.*

4.1.4 Example. For every $A \in L(\boldsymbol{R}^n, \boldsymbol{R}^n)$, $X_A: Gl(n,\boldsymbol{R}) \mapsto L(\boldsymbol{R}^n,\boldsymbol{R}^n): Y \mapsto YA$ is a left-invariant vector field on $Gl(n,\boldsymbol{R})$ because for every $Z \in Gl(n,\boldsymbol{R})$, $X_A(L_Z Y) = ZYA = T_Y L_Z X_A(Y)$ and $L_Z: Gl(n,\boldsymbol{R}) \to Gl(n,\boldsymbol{R}): Y \mapsto ZY$ is a linear mapping.

Therefore, by the local formula $[X,Y](x) = DY(x)\cdot X(x) - DX(x)\cdot Y(x)$,

$$[A,B] = [X_A, X_B](I) = DX_B(I)\cdot X_A(I) - DX_A(I)\cdot X_B(I)$$

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But $X_B(Z) = ZB$ is linear in Z, so $DX_B(I)\cdot Z = ZB$. Hence $DX_B(I)\cdot X_A(I$ $DX_B(I)\cdot A = AB$ and similarly $DX_A(I)\cdot X_B(I) = BA$. Thus, $L(\boldsymbol{R}^n,\boldsymbol{R}^n)$ *is Lie algebra of $Gl(n,\boldsymbol{R})$ with Lie bracket given by*

$$[A,B] = AB - BA$$

4.1.5 Proposition. *Let H and G be Lie groups and $f: H \to G$ a sm homomorphism. Then $T_ef: \mathfrak{L}(H) \to \mathfrak{L}(G)$ is a Lie algebra homomorphism.*

Proof. Since f is a homomorphism, $L_{f(h)} \circ f = f \circ L_h$ for every $h \in H$. ferentiation of this relation in h yields

$$X_{T_ef\cdot\xi} \circ f = Tf \circ X_\xi$$

that is,

$$f_* X_\xi = X_{T_ef\cdot\xi}$$

Therefore,

$$\begin{aligned} T_ef\cdot[\xi,\eta] &= T_ef\cdot[X_\xi, X_\eta](e) \qquad (e = e_H) \\ &= f_*[X_\xi, X_\eta](e) \qquad (e = e_G) \\ &= [f_*X_\xi, f_*X_\eta](e) \\ &= [X_{T_ef\cdot\xi}, X_{T_ef\cdot\eta}](e) \\ &= [T_ef\cdot\xi, T_ef\cdot\eta] \qquad \blacksquare \end{aligned}$$

For every $\xi \in T_eG$ let $\phi_\xi: \boldsymbol{R} \to G: t \mapsto exp\, t\xi$ denote the integral curve o passing through e at $t = 0$. Because X_ξ is left invariant, its flow is comp Indeed, the time of existence of the integral curve of X_ξ with initial condi g is the same as that with initial condition e since if $c(t)$ is an integral cur e, $g\cdot c(t)$ is an integral curve at g; see Exercise 2.1B. Therefore, ϕ_ξ is def for all $t \in \boldsymbol{R}$. The following argument shows that

$$\phi_\xi(s+t) = exp(s+t)\xi = exp\, s\xi\, exp\, t\xi = \phi_\xi(s)\phi_\xi(t)$$

for all $s, t \in \boldsymbol{R}$; that is, ϕ_ξ is a smooth homomorphism of the (additive) gr $\boldsymbol{R}$ into G and is therefore called a *one-parameter subgroup* of G. Fix $s \in \boldsymbol{R}$ define $\psi: \boldsymbol{R} \to G: t \mapsto \phi_\xi(s)\phi_\xi(t) = L_{\phi_\xi(s)}\phi_\xi(t)$; then ψ is an integral curve o passing through $\phi_\xi(s)$ at $t = 0$ by left invariance of X_ξ. Also $\theta: \boldsymbol{R}$ – $t \mapsto \phi_\xi(s+t)$ is an integral curve of X_ξ passing through $\phi_\xi(s)$ at $t = 0$ bec $\theta(0) = \phi_\xi(s)$ and

$$\frac{d\theta}{dt}(s+t) = \frac{d\theta}{d(s+t)}(s+t) = X_\xi(\theta(s+t))$$

Therefore, $\theta = \psi$ because the integral curve of X_ξ passing through $\psi_\xi(s$

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$t=0$ is unique. Consequently (2) holds. Notice that we have proven that the flow of X_ξ is $F_t^\xi(g)=g\,exp\,t\xi$.

If $\phi: \mathbf{R}\to G$ is a one-parameter subgroup of G, then $\phi=\phi_\xi$, where $\xi=d\phi/dt|_{t=0}$ because $\phi(t+s)=\phi(t)\phi(s)=L_{\phi(t)}\phi(s)$, which implies

$$\frac{d}{dt}\phi(t)=\frac{d}{ds}\phi(t+s)\Big|_{s=0}$$
$$=T_eL_{\phi(t)}\left(\frac{d\phi}{dt}\Big|_{t=0}\right)$$
$$=T_eL_{\phi(t)}\xi=X_\xi(\phi(t))$$

Thus ϕ is an integral curve of X_ξ passing through $e=\phi(0)$. But so is ϕ_ξ. Hence $\phi=\phi_\xi$ by uniqueness.

4.1.6 Definition. *The function $exp: T_eG\to G: \xi\mapsto\phi_\xi(1)$ is called the* ***exponential mapping*** *of the Lie algebra of G into G. If there is any danger of confusion we will write exp_G.*

The map exp is C^∞. Indeed, let Z be the vector field on $G\times\mathfrak{g}$ defined by $Z(g,\xi)=(X_\xi(g),0)$. Its flow is readily verified to be $F_t(g,\xi)=(g\,exp\,t\xi,\xi)$. From its definition, $X_\xi(g)=T_eL_g\xi$ is smooth in ξ and g and thus Z and hence F_1 are smooth maps. In particular $F_1(e,\xi)=(exp\,\xi,\xi)$ is smooth in ξ and so exp is smooth.

We have $T_0exp=id_{T_eG}$ because

$$T_0(exp)\xi=\frac{d}{dt}exp\,t\xi\Big|_{t=0}=\frac{d}{dt}\phi_\xi\Big|_{t=0}=X_\xi(\phi_\xi(0))=\xi$$

Therefore, by the inverse function theorem, exp is a local diffeomorphism. In general, exp is not a diffeomorphism onto G as is shown in Example 4.1.9(c) below. Before giving these examples we note a basic property of the exponential map.

4.1.7 Proposition. *If $f: H\to G$ is a smooth homomorphism of Lie groups, then for all $\eta\in\mathfrak{L}(H)$, $f(exp_H\eta)=exp_G(T_ef\cdot\eta)$.*

Proof. The mapping $\phi: \mathbf{R}\to G: t\mapsto f(exp_H\,t\eta)$ is a one-parameter subgroup of G. Therefore, $\phi(t)=exp_G t\xi$, where $\xi=(d/dt)\phi|_{t=0}=T_ef\cdot\eta$, which implies $f(exp_H\eta)=\phi(1)=\phi_\xi(1)=exp_G\xi=exp_G(T_ef\cdot\eta)$. ■

For every $g\in G$, let $I_g: G\to G: h\mapsto ghg^{-1}=R_{g^{-1}}L_gh$ be the inner automorphism associated with g. I_g is smooth and is a homomorphism because

$$I_g(hk)=ghkg^{-1}=ghg^{-1}gkg^{-1}=I_g(h)I_g(k)$$

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Let $Ad_g=T_eI_g=T_e(R_{g^{-1}}L_g): T_eG\to T_eG$, called the *adjoint mapping* associat with g. Using proposition 4.1.7 we have:

4.1.8 Corollary. *$exp(Ad_g\xi)=I_g\,exp\,\xi=g(exp\,\xi)g^{-1}$ for every $\xi\in T_eG$ a every $g\in G$.*

4.1.9 Examples. (a) Consider $\mathbf{R}^n$ with the additive structure as a] group. Then the Lie algebra of $\mathbf{R}^n$ is $\mathbf{R}^n$ and $exp: \mathbf{R}^n\to\mathbf{R}^n$ is the identity.

(b) For every $A\in L(\mathbf{R}^n,\mathbf{R}^n)$, $\phi_A: \mathbf{R}\to Gl(n,\mathbf{R})\subset\mathbf{R}^{n^2}: t\mapsto\Sigma_{n=0}^\infty t^nA^n/$ is a one-parameter subgroup because $\phi_A(0)=I$ and

$$\frac{d}{dt}\phi_A(t)=\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^n=\phi_A(t)A$$

which shows that ϕ_A is an integral curve of the left-invariant vector field . Therefore, the exponential mapping is given by $exp: L(\mathbf{R}^n,\mathbf{R}^n)\to Gl(n,\mathbf{R}$ $A\mapsto\phi_A(1)=\Sigma_{n=0}^\infty A^n/n!$. For $P\in Gl(n,\mathbf{R})$, $P\phi_AP^{-1}: \mathbf{R}\to Gl(n,\mathbf{R})$: t $P\phi_A(t)P^{-1}$ is an integral curve of $X_{PAP^{-1}}$ passing through I beca $P\phi_AP^{-1}(0)=I$ and

$$\frac{d}{dt}(P\phi_AP^{-1})(t)=P\frac{d\phi_A(t)}{dt}P^{-1}$$
$$=P\phi_A(t)AP=(P\phi_AP^{-1})(t)PAP^{-1}$$

But so is $\phi_{PAP^{-1}}$. Therefore, $\phi_{PAP^{-1}}=P\phi_AP^{-1}$, that is,

$$exp\,PAP^{-1}=P(exp\,A)P^{-1}$$

(c) Consider $exp: L(\mathbf{R}^2,\mathbf{R}^2)\to Gl(2,\mathbf{R})$. The following argument sh that

$$B=\begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}$$

is not in the image of exp. By the real canonical form from linear algebra (s e.g., Hirsch–Smale [1974], p. 129, Theorem 2) for every $A\in Gl(2,\mathbf{R})$, ther a $P\in Gl(2,\mathbf{R})$ such that PAP^{-1} is either

(i) $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ for some $\mu,\lambda\in\mathbf{R}$,

(ii) $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ for some $\alpha\in\mathbf{R}$, $\beta\in\mathbf{R}\backslash\{0\}$, or

(iii) $\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}$ for some $\lambda\in\mathbf{R}$.

Therefore, $exp\,PAP^{-1}$ is either

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(i) $\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$,

(ii) $e^{\alpha}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, or

(iii) $\begin{pmatrix} e^{\lambda} & 0 \\ 1 & e^{\lambda} \end{pmatrix}$.

Suppose that $B = \exp A$ for some $A \in L(\mathbf{R}^2, \mathbf{R}^2)$; then for any $P \in Gl(2,\mathbf{R})$, $PBP^{-1} = P(\exp A)P^{-1} = \exp(PAP^{-1})$. Now cases (i) and (iii) cannot occur since $\operatorname{trace} B = -3$ and

$$\operatorname{trace} B = \operatorname{trace}(PBP^{-1}) = \operatorname{trace}\exp(PAP^{-1})$$

which is >0 in these cases. Also, B cannot have the canonical form (ii) on p. 257 since that form has eigenvalues $\lambda = \alpha \pm i\beta$ which can never equal $-2, -1$.

Of course, whenever G is not connected, $\exp$ cannot be onto because $\exp(\mathfrak{g})$ is connected. Any matrix with negative determinant is not in the component of the identity of $Gl(n,\mathbf{R})$ (since $\det > 0$ on the component of the identity) and hence is not in the image of $\exp$. However, in the example just given, B *is* in the component of the identity; it is joined to I by the curve

$$\begin{pmatrix} (1+\theta/\pi)\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ for } 0 \leqslant \theta \leqslant \pi$$

Thus, we *cannot* conclude that $\exp$ is onto the component of the identity. However, if G admits a bi-invariant Riemannian metric (e.g., if G is compact) then this is true. See Exercise 4.4D. [It follows that $Gl(2,\mathbf{R})$ does not admit a bi-invariant metric.]

4.1.10 Definition. *A* ***Lie subgroup*** *H of a Lie group G is a subgroup of G for which the inclusion mapping $i: H \to G$ is an immersion, that is, $i(H)$ is an immersed submanifold of G.*

The next example shows that the manifold topology on $i(H)$ need not be the topology induced from G. In other words, i need not be an embedding.

4.1.11 Example. Let $\alpha \in [0,1) \setminus \mathbf{Q}$ and define $\phi: \mathbf{R} \to T^2 = S^1 \times S^1 \subset \mathbf{C}^2 : t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$. Then ϕ is a one-parameter subgroup of T^2. Moreover, ϕ is injective, for $(e^{2\pi i t}, e^{2\pi i \alpha t}) = (e^{2\pi i s}, e^{2\pi i \alpha s})$ if and only if for some $m, n \in \mathbf{Z}$, $t = s + n$ and $\alpha t = \alpha s + m$; if $m \neq 0$ and $n \neq 0$, then $m = \alpha(t-s) = \alpha n$, which contradicts $\alpha \not\in \mathbf{Q}$; hence either $m = 0$ or $n = 0$, which implies $t = s$. A similar argument shows that $T_t\phi(1) = d\phi/dt = 2\pi i(e^{2\pi i t}, \alpha e^{2\pi i \alpha t})$ is injective. Therefore,

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$\phi(\mathbf{R})$ is an injectively immersed submanifold of T^2. The following argume shows that $cl(\phi(\mathbf{R})) = T^2$, that is, $\phi(\mathbf{R})$ is *dense* in T^2. Let $p = (e^{2\pi i x}, e^{2\pi i y}) \in T$ then for all $m \in \mathbf{Z}$,

$$|\phi(x+m) - p| = |(0, e^{2\pi i \alpha x}(e^{2\pi i m\alpha} - e^{2\pi i z}))|$$

where $y = \alpha x + z$. It suffices to show that $C = \{e^{2\pi i m\alpha} \in S^1 | m \in \mathbf{Z}\}$ is dense S^1 because then there is a sequence $m_k \in \mathbf{Z}$ such that $e^{2\pi i m_k \alpha}$ converges $e^{2\pi i z}$. Hence, $\phi(x+m_k)$ converges to p. If for each $k \in \mathbf{Z}_+$ we divide S^1 intc arcs of length $2\pi/k$, then, because $\{e^{2\pi i m\alpha} \in S^1 | m = 1,2,\ldots,k+1\}$ are d tinct, for some $1 \leqslant n_k < m_k \leqslant k+1$, $e^{2\pi i m_k \alpha}$ and $e^{2\pi i n_k \alpha}$ belong to the same a Therefore, $|e^{2\pi i m_k\alpha} - e^{2\pi i n_k \alpha}| < 2\pi/k$, which implies $|e^{2\pi i p_k \alpha} - 1| < 2\pi/k$, wh $p_k = m_k - n_k$. Because

$$\bigcup_{j \in \mathbf{Z}_+} \{e^{2\pi i \alpha s} \in S^1 | s \in [jp_k, (j+1)p_k]\} = S^1$$

every arc of length less than $2\pi/k$ contains some $e^{2\pi i j p_k}$, which pro $cl(C) = S^1$. $\phi(\mathbf{R})$ is not an embedded submanifold of T^2 because it is locally closed in the topology of T^2.

The difficulty in the above example is the fact that the subgroup is closed.

4.1.12 Proposition. *If H is a closed subgroup of a Lie group G, then H i submanifold of G and in particular is a Lie subgroup.*

Proof. (Adams [1969]). Put a norm $\|\cdot\|$ on $\mathfrak{g}_G = T_eG$. Let $\xi_n \in \mathfrak{g}_G, \xi_n \neq 0$ such that $\exp \xi_n \in H$, $\xi_n \to 0$ and $\xi_n / \|\xi_n\| \to \xi \in \mathfrak{g}_G$. We will show that $\exp t\xi \in$ for every $t \in \mathbf{R}$. Since $\xi_n \to 0$, for any $t \in \mathbf{R}$ there is a sequence of integers such that $m_n\|\xi_n\| \to t$ as $n \to \infty$. Thus $\exp(m_n \xi_n) \to \exp t\xi$. But $\exp m_n \xi_n$ $(\exp \xi_n)^{m_n} \in H$ and H is closed, so $\exp t\xi \in H$.

Next, let $\mathfrak{g}_H = \{\xi \in \mathfrak{g}_G | \exp t\xi \in H \text{ for all } t \in \mathbf{R}\}$. We claim $\mathfrak{g}_H$ is a vec subspace of $\mathfrak{g}_G$. Clearly $\mathfrak{g}_H$ is closed under scalar multiplication. We need show it is closed under addition. Let $\xi_1, \xi_2 \in \mathfrak{g}_H$ and suppose $\xi_1 + \xi_2 \neq 0$. Fo sufficiently small, we can write $\exp t\xi_1 \cdot \exp t\xi_2 = \exp(f(t))$ since $\exp$ is a diff morphism of a neighborhood of e. Since

$$\xi_1 + \xi_2 = \frac{d}{dt} \exp t\xi_1 \exp t\xi_2 \Big|_{t=0}$$

$(1/t)f(t) \to \xi_1 + \xi_2$ as $t \to 0$. Therefore, letting $\xi_n = f(1/n)$ and $\xi = (\xi_1 + \xi$ $\|\xi_1 + \xi_2\|$, we see that $\exp t\xi \in H$, that is, $\xi_1 + \xi_2 \in \mathfrak{g}_H$.

Write $\mathfrak{g}_G = \mathfrak{g}_H \oplus \mathfrak{g}'$ and consider the diffeomorphism $\phi(\xi, \xi') = \exp \xi \cdot ex$ between a neighborhood of 0 in $\mathfrak{g}_G$ and a neighborhood of e in G. (It is a lc diffeomorphism by the implicit function theorem.) We use this map to sh that exp maps a neighborhood of 0 in $\mathfrak{g}_H$ to a neighborhood of e in H. If not, there would be a sequence $(\xi_n, \xi_n') \in \mathfrak{g}_H \oplus \mathfrak{g}'$ with $exp\, \xi_n \cdot exp\, \xi_n' \in H$, $exp\, \xi_n \cdot exp\, \xi_n' \to e$, and $\xi_n' \neq 0$. But $exp\, \xi_n \in H$, so $exp\, \xi_n' \in H$. There is a subsequence ξ_{n_k}' such that $\xi_{n_k}' / \|\xi_{n_k}'\| \to \xi \in \mathfrak{g}'$ (by compactness). But this would imply $\xi \in \mathfrak{g}_H$, which is impossible.

It follows from this that exp gives a submanifold chart for H near e modelled on $\mathfrak{g}_H$. By left translation, H is a submanifold around each of its points. ■

4.1.13 Proposition. *Let $i: H \to G$ be a Lie subgroup of G. Then for $\xi \in \mathcal{L}(G)$, $\xi \in T_e i(\mathcal{L}(H))$ if and only if $exp_G t\xi \in i(H)$ for all $t \in \boldsymbol{R}$.*

Proof. (Note that this was shown in the previous proof for closed subgroups.) If $\xi \in T_e i(\mathcal{L}(H))$, then 4.1.7 yields at once that $exp_G t\xi \in i(H)$. Conversely, if $exp_G t\xi \in i(H)$ for all t, write $exp_G t\xi = i\phi(t)$ for $\phi(t) \in H$. Since i is an injective immersion, ϕ is a smooth one-parameter subgroup of H and so $\phi(t) = exp_H t\eta$ for $\eta \in \mathcal{L}(H)$. Thus $\xi = T_e i \cdot \eta$. ■

Proposition 4.1.12 is just one of a number of "automatic smoothness" results for Lie groups. (See Varadarajan [1974] for further information.)

One can characterize Lebesgue measure up to a multiplicative constant on $\boldsymbol{R}^n$ by its invariance under translations. Similarly, on a locally compact group there is a unique left-invariant measure, called *Haar measure*. For Lie groups the existence of such measures is especially simple.

4.1.14 Proposition. *Let G be a Lie group. Then there is a volume form μ, unique up to nonzero multiplicative constants, which is left invariant. If G is compact, μ is right invariant as well.*

Proof. Pick any n-form μ_e on $T_e G$ that is nonzero and define an n-form on $T_g G$ by

$$\mu_g(v_1, \ldots, v_n) = \mu_e \cdot (TL_{g^{-1}} \cdot v_1, \ldots, TL_{g^{-1}} \cdot v_n)$$

Then μ_g is left invariant and smooth. For $n = \dim G$, μ_e is unique up to a scalar factor, so μ_g is as well.

Fix $g_0 \in G$ and consider $R_{g_0}^* \mu$. Since R_{g_0} and L_g commute, $R_{g_0}^* \mu$ is left invariant and hence $R_{g_0}^* \mu = c\mu$ for a constant c. Now if G is compact, this relationship may be integrated and by the change of variables formula, we deduce that $c = 1$. Hence μ is also right invariant. ■

This concludes our brief study of Lie groups as such, and we now turn to actions of groups on manifolds. Before proceeding the reader should be sure he understands some of the classical examples by consulting the exercises. For instance, later on, the fact that the Lie algebra of $SO(3)$ is $\boldsymbol{R}^3$ with the cross product as Lie bracket will be used without explicit mention.

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4.1.15 Definition. *Let M be a smooth manifold. An **action** of a Lie group on M is a smooth mapping $\Phi: G \times M \to M$ such that (i) for all $x \in M$ $\Phi(e, x) = x$ and (ii) for every $g, h \in G$, $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for all $x \in M$.**

4.1.16 Examples. (a) A complete flow F on M is an action of $\boldsymbol{R}$ on M.

(b) If H is a subgroup of a Lie group G, then $\Phi: H \times G \to G: (h, g) \mapsto hg$ an action of H on G.

For every $g \in G$ let $\Phi_g: M \to M: x \mapsto \Phi(g, x)$; then (i) becomes $\Phi_e = id$ while (ii) becomes $\Phi_{gh} = \Phi_g \circ \Phi_h$. Because $(\Phi_g)^{-1} = \Phi_{g^{-1}}$, Φ_g is a diffeomo phism. Definition 4.1.15 can be rephrased by saying that the mapping $g \mapsto \Phi$ is a homomorphism of G into the group of diffeomorphisms of M. If M is vector space and each Φ_g is a linear transformation, the action of G on M called a *representation* of G on M.

The following additional terminology regarding actions is useful.

4.1.17 Definitions. *Let Φ be an action of G on M. For $x \in M$, the **orbit** (Φ-orbit) of x is given by*

$$G \cdot x = \{\Phi_g(x) \mid g \in G\}$$

*An action is **transitive** if there is just one orbit. It is **effective** (or **faithful**) $\Phi_g = identity$ implies $g = e$; that is, $g \mapsto \Phi_g$ is one-to-one. An action is **free** if, f each $x \in M$, $g \mapsto \Phi_g(x)$ is one-to-one.*

The relation of belonging to the same Φ-orbit is an equivalence relatio on M. We let M/G be the set of equivalence classes, that is, M/G is the s of Φ-orbits. Let $\pi: M \to M/G: x \mapsto [x]$, where $[x]$ is the Φ-orbit containing Give M/G the quotient topology, that is, $U \subset M/G$ is open if and only $\pi^{-1}(U)$ is open in M.

4.1.18 Example. This example shows that the topology M/G need not b Hausdorff. Let $\Phi: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, $(t, x) \mapsto e^t x$, an action of the additive grou $G = \boldsymbol{R}$ on $M = \boldsymbol{R}$. There are three orbits, $[-1]$, $[0]$, and $[1]$. It is readil checked that the open sets in M/G are the empty set, the whole spac $\{[-1]\}$, $\{[1]\}$, and $\{[-1], [1]\}$. In particular $\{[0]\}$ is not open, so the topolog is not Hausdorff.

4.1.19 Proposition. *Let $\Phi: G \times M \to M$ be a smooth action and let $R =$ $\{(m, \Phi_g m) \in M \times M \mid (g, m) \in G \times M\}$. If R is a closed subset of $M \times M$, the the quotient topology on M/G is Hausdorff.*

*Strictly speaking, this is a *left action*. A *right action* is a map $\Phi: M \times G \to M$ such that $\Phi(x, e) =$ and $\Phi(\Phi(x, g), h) = \Phi(x, gh)$. For "automatic smoothness" results for group actions, see Bochn and Montgomery [1945] and Chernoff and Marsden [1970].

Proof. Suppose M/G is not Hausdorff. Then there are distinct $[x]$, $[y] \in M/G$ such that for any pair of neighborhoods U^x of $[x]$ and U^y of $[y]$, $U^x \cap U^y \neq \varnothing$.

Let V_i^x and V_i^y be nested bases of neighborhoods in M of x and y, $i = 1, 2, \ldots$. Let $W_i^x = \bigcup_{g \in G} \Phi_g V_i^x$ and $W_i^y = \bigcup_{g \in G} \Phi_g V_i^y$. Choosing $U_i^x = \pi(W_i^x)$ and $U_i^y = \pi(W_i^y)$, there must be $g_i, h_i \in G$ and $x_i \in V_i^x$, $y_i \in V_i^y$ such that

$$\Phi_{g_i} x_i = \Phi_{h_i} y_i, \quad \text{that is, } y_i = \Phi_{h_i^{-1} g_i} x_i$$

Now $y_i \to y$ and $x_i \to x$ as $i \to \infty$. Thus the points $(x_i, \Phi_{h_i^{-1} g_i} x_i) \in R$ converge, so the limit lies in R, as R is assumed closed. Thus $(x,y) \in R$; that is, $y = \Phi_g x$ for some $g \in G$ and so $[x] = [y]$. ■

The next theorem gives a necessary and sufficient condition for M/G to be a smooth manifold.

4.1.20 Theorem. *Let G act on M, and R be defined as in* 4.1.19. *Then R is a closed submanifold of $M \times M$ if and only if M/G has a smooth manifold structure such that $\pi: M \to M/G$ is a submersion.*

Proof. *Sufficiency.* Suppose that R is a closed submanifold of dimension r of $M \times M$ that has dimension $2n$. First we show that R is locally the graph of a smooth submersion of M into M, that is, for every $x \in M$ there is an open set $U \subset M$ with $x \in U$, a submanifold $N \subset M$ and a smooth submersion $\rho: U \subset M \to N \subset M$ such that for every $u \in U$, $\rho(u) \in N$ if and only if $(u, \rho(u)) \in R$. Since R is a submanifold of $M \times M$ and the map $R \to M: (x, \Phi_g(x)) \mapsto x$ is a submersion, by the local fibration theorem (Exercise 1.6G) find an open set $U_0 \subset M$ and a map $\eta: U_0 \times U_0 \subset M \times M \to \boldsymbol{R}^m$ such that $\eta^{-1}(\mathbf{0}) = (U_0 \times U_0) \cap R$ and $\eta_x = \eta_{1,x}: U_0 \subset M \to \boldsymbol{R}^m: y \mapsto \eta(x,y)$ is a submersion. This implies that $n \geqslant m$ and $\eta_x^{-1}(\mathbf{0})$ is a submanifold of U_0 with $T_x \eta_x^{-1}(\mathbf{0}) = E$, which has dimension $n - m$. Let F be a complement to E in $T_x M$. Shrinking U_0 if necessary, there is a submersion $\xi: U_0 \subset M \to \boldsymbol{R}^{n-m}$ such that $\xi^{-1}(\mathbf{0}) = N$ is a submanifold with $x \in N$ and $T_x \xi^{-1}(\mathbf{0}) = F$, which has dimension m. (See Exercise 1.6H.) Consider the mapping $\zeta: U_0 \times U_0 \subset M \times M \to \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$: $(y,z) \mapsto (\eta(y,z), \xi(z))$, then $\zeta(x,x) = (\mathbf{0}, \mathbf{0})$ and the partial mapping $\zeta_x: U_0 \subset M \to \boldsymbol{R}_i^n = \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}: z \mapsto (\eta_x(z), \xi(z))$; $T_x \zeta_x: T_x M \to T_{\zeta_x(x)} \boldsymbol{R}^n$ is bijective because $\ker T_x \zeta_x = \ker T_x \eta_x \cap \ker T_x \xi = E \cap F = \{\mathbf{0}\}$ and $\dim T_x M = n$. Therefore, the implicit function theorem applied to ζ gives an open set $U_1 \subset U_0$ with $x \in U_1$ and a smooth function $\rho: U_1 \subset M \to U_1 \subset M$ with $\rho(x) = x$ such that $\zeta^{-1}(\mathbf{0}, \mathbf{0}) = \{(u, \rho(u)) \in U_1 \times U_1 \subset U \times U \mid u \in U_1\}$. For all $u \in U_1$, $\eta(u, \rho(u)) = \mathbf{0}$, which implies $(u, \rho(u)) \in (U_1 \times U_1) \cap R$ and $\xi(\rho(u)) = \mathbf{0}$, that is, $\rho(u) \in N$. It remains to show that ρ is a submersion near x. Differentiating $\eta(u, \rho(u)) = \mathbf{0}$ at $(x,x) = (x, \rho(x))$ gives

$$\mathbf{0} = T\eta_{2,x} + T\eta_{1,x} \circ T_x \rho$$

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where $\eta_{2,x}: U_1 \subset M \to \boldsymbol{R}^m: z \mapsto \eta(z,x)$ and similarly $\eta_{1,x}(z) = \eta(x,z)$. Thus

$$\operatorname{rank} T\eta_{2,x} \leqslant \operatorname{rank} T_x \rho$$

But R is invariant under the diffeomorphism $j: M \times M \to M \times M: (y,z)$ (z,y) and $\eta_{2,x} = (\eta \circ j)_{1,x}$. Therefore,

$$\operatorname{rank} T\eta_{2,x} = \operatorname{rank} T\eta_{1,x} = m$$

But $\dim N = m$, so $\operatorname{rank} T_x \rho = m$, that is, ρ is a submersion near x.

Next we construct a chart at $[x]$ for M/G. Since R is closed, M/G i Hausdorff space in the quotient topology. Since $T_x \rho(T_x M) = T_x N$ and $\rho(x$ x, there is a chart (U, ϕ) at x of M such that $\phi \circ \rho \circ \phi^{-1}: \phi(U) = V \times W$ $\boldsymbol{R}^m \times \boldsymbol{R}^{n-m} \to V \times \{\mathbf{0}\} \subset \boldsymbol{R}^m \times \{\mathbf{0}\}: (v,w) \mapsto (v, \mathbf{0})$. Therefore, for all $u \in$ $u^1 = \rho(u)$ if and only if $\pi_1 \phi(u) = \pi_1 \phi(u^1)$, where $\pi_1: V \times W \to V: (v,w) \mapsto$ Define $\omega: V \subset \boldsymbol{R}^m \to \pi(U) \subset M/G: v \mapsto \pi \circ \phi^{-1}(v, \mathbf{0})$. Then ω is injective, if $\pi(\phi^{-1}(v, \mathbf{0})) = \pi(\phi^{-1}(v^1, \mathbf{0}))$, then $(\phi^{-1}(v, \mathbf{0}), \phi^{-1}(v^1, \mathbf{0})) \in (U \times U) \cap$ therefore, $\rho(\phi^{-1}(v, \mathbf{0})) = \phi^{-1}(v^1, \mathbf{0})$, which implies $v = \pi_1(\phi(\phi^{-1}(v, \mathbf{0})))$ $\pi_1(\phi(\phi^{-1}(v^1, \mathbf{0}))) = v^1$. Since π and ϕ are continuous and open mappings, ω a homeomorphism. Thus $(\pi(U), \omega^{-1})$ is a chart for M/G at $[x] = \pi(x)$.

Next we show that two charts $(\pi(U), \omega^{-1})$ and $(\pi(\tilde{U}), \tilde{\omega}^{-1})$ at $[x]$ compatible. Let $Y = \pi^{-1}(\pi(U) \cap \pi(\tilde{U})) \cap U \subset U \cap \tilde{U}$ and $\tilde{Y} = \pi^{-1}(\pi(U)$ $\pi(\tilde{U})) \cap \tilde{U} \subset U \cap \tilde{U}$. The following argument shows that for every $v \in \pi_1 \phi($ $= V$ there is a unique $\tilde{v} \in \tilde{\pi}_1 \tilde{\phi}(\tilde{Y}) = \tilde{V}$ such that

$$\pi(\phi^{-1}(v, W)) = \pi(\tilde{\phi}^{-1}(\tilde{v}, \tilde{W}))$$

where $W = \pi_2(\phi(Y))$, $\tilde{W} = \tilde{\pi}_2(\tilde{\phi}(\tilde{Y}))$ and $\pi_2: V \times W \to W: (v,w) \mapsto w$. For a $w, w' \in W$, $v = \pi_1(\phi\phi^{-1}(v,w)) = \pi_1(\phi\phi^{-1}(v, w^1))$, which implies $\pi(\phi^{-1}(v,w)$ $\pi(\phi^{-1}(v, w^1))$. Therefore, for any $w \in W$, $\pi(\phi^{-1}(v,w)) = \pi\phi^{-1}(v, W)$. Si larly, for any $\tilde{w} \in \tilde{W}$, $\pi(\tilde{\phi}^{-1}(\tilde{v}^1, \tilde{w})) = \pi(\tilde{\phi}^{-1}(\tilde{v}^1, \tilde{W}))$. Since $Y = \phi^{-1}(V \times$ $\tilde{Y} = \tilde{\phi}^{-1}(V \times W)$ and $\pi Y = \pi \tilde{Y}$, for every $(v,w) \in V \times W$ there is $(\tilde{v}, \tilde{w}$ $\tilde{V} \times \tilde{W}$ such that

$$\pi(\phi^{-1}(v, W)) = \pi(\phi^{-1}(v, w)) = \pi(\tilde{\phi}^{-1}(\tilde{v}, \tilde{w})) = \pi(\tilde{\phi}^{-1}(\tilde{v}, \tilde{W}))$$

Suppose there is $(\tilde{v}^1, \tilde{w}^1) \in \tilde{V} \times \tilde{W}$ such that $\pi(\tilde{\phi}^{-1}(\tilde{v}^1, \tilde{w}^1)) = \pi(\tilde{\phi}^{-1}(\tilde{v},$ which implies $\tilde{v}^1 = \pi_1(\tilde{\phi}(\tilde{\phi}^{-1}(\tilde{v}^1, \tilde{w}^1)) = \pi_1(\tilde{\phi}(\tilde{\phi}^{-1}(\tilde{v}, \tilde{w})) = \tilde{v}$. Thus $\tilde{v}$ is uniqu determined and so we can define a function $\psi: V \subset \boldsymbol{R}^m \to \tilde{V} \subset \boldsymbol{R}^m: v \mapsto \tilde{v}$. now show that ψ is smooth. Since $\pi(\tilde{\phi}^{-1}(\tilde{v}, \tilde{w})) = \pi(\phi^{-1}(v,$ $(\phi^{-1}(v,w), \tilde{\phi}^{-1}(\tilde{v}, \tilde{w})) \in R$. Therefore, for some $g \in G$, $\Phi_g(\phi^{-1}(v,w))$ $\tilde{\phi}^{-1}(\tilde{v}, \tilde{w})$. Since Φ_g is a diffeomorphism, there is an open set $U \subset M$ w $x \in U \subset Y$ and $\Phi_g U \subset \tilde{Y}$. Therefore, the map

$$\tilde{\phi} \circ \Phi_g \circ \phi^{-1}: \phi(Y) \subset V \times W \subset \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}$$

$$\to \tilde{\phi}(\tilde{Y}) \subset \tilde{V} \times \tilde{W}: (v,w) \mapsto (\psi(v), \theta(v,w))$$

is smooth, which implies that ψ is smooth. Therefore, the charts $(\pi(U), \omega^{-1})$ and $(\pi(\tilde{U}), \tilde{\omega}^{-1})$ at $[x]$ are compatible, that is, M/G is a smooth manifold.

Let (U, ϕ) and $(\pi(U), \omega^{-1})$ be charts at $x \in M$ and $[x] \in M/G$, respectively, then $\omega^{-1} \circ \pi \circ \phi^{-1}: \phi(U) \subset V \times W \subset \boldsymbol{R}^m \times \boldsymbol{R}^n \to V \subset \boldsymbol{R}^m: (v, w) \mapsto v$ is a smooth submersion, which implies that $\pi: M \to M/G$ is a smooth submersion. This proves sufficiency.

Necessity. Since $\Delta_{M/G} = \{([x],[x]) \in M/G \times M/G,\ [x] \in M/G\}$ is a closed submanifold of $M/G \times M/G$ and $\pi \times \pi: M \to M \to M/G \times M/G: (x,y) \mapsto (\pi(x), \pi(y))$ is a submersion, $(\pi \times \pi)^{-1} \Delta_{M/G} = R$ is a closed submanifold of $M \times M$. ■

We note the resemblance between the construction of charts on M/G and the proof of Exercise 1.6F.

A corollary of this argument, whose proof we leave to the reader, is the following useful technical remark: A map $\phi: M/G \to N$ is smooth if and only if $\phi \circ \pi: M \to N$ is smooth.

This yields the following criterion of smoothness on quotient manifolds. Assume $\Phi: G \times M \to M$, $\Psi: G \times N \to N$ are two smooth actions such that M/G, N/G are manifolds and $\pi_M: M \to M/G$, $\pi_N: N \to N/G$ are submersions. Let $f: M \to N$ be equivariant, that is, $f \circ \Phi_g = \Psi_g \circ f$ for all $g \in G$. This induces naturally a map $\hat{f}: M/G \to N/G$. Then smoothness of f implies smoothness of $\hat{f}$. This criterion is often called the *passage to quotients.*

The next result is a corollary of 4.1.20 concerning Lie groups themselves.

4.1.21 Corollary. *Let H be a closed subgroup of the Lie group G. If Φ: $H \times G \to G: (h, g) \mapsto hg$, then G/H is a smooth manifold and $\pi: G \to G/H$ is a submersion.*

Proof. Consider the mapping $\xi: G \times G \to G: (g,k) \mapsto kg^{-1} = m(i(g), k)$, where $m: G \times G \to G: (g,k) \mapsto gk$ and $i: G \to G: g \mapsto g^{-1}$. Since $T_{(g,k)}\xi(r,s) = T_k m_{i(g)} T_g i(r) + T_g m_k(s)$, where $m_g: G \to G: k \mapsto gk = L_g k$ and $m_k: G \to G: g \mapsto gk = R_k g$, $T_{(g,k)}\xi(r,s) = T_k L_{i(g)} \circ T_g i(r) + T_g R_k(s)$. Therefore, $T_{(g,k)}\xi(0,s) = T_g R_k(s)$. Thus $T_{(g,k)}\xi$ is surjective, because $T_g R_k$ is an isomorphism. Hence ξ is a submersion. Since H is a closed subgroup of G, H is a closed submanifold of G by 4.1.12. Hence $\xi^{-1}(H)$ is a closed submanifold of $G \times G$. But $(g,k) \in \xi^{-1}(H)$ if and only if $kg^{-1} \in H$, that is, if and only if $(g,k) \in R = \{(g, hg) \in G \times G \mid g \in G,\ h \in H\}$. Thus $\xi^{-1}(H) = R$ is closed submanifold of $G \times G$, which implies by Theorem 4.1.20 that G/H is a manifold and π: $G \to G/H$ is a submersion. ■

An action $\Phi: G \times M \to M$ is called *proper* if and only if $\tilde{\Phi}: G \times M \to M \times M$ defined by $\tilde{\Phi}(g,x) = (x, \Phi(g,x))$ is a proper mapping, that is, if $K \subset M \times M$ is compact, then $\tilde{\Phi}^{-1}(K)$ is compact. Equivalently, if x_n converges in M and $\Phi_{g_n} x_n$ converges in M, then g_n has a convergent subsequence in G. For instance, if G is compact this condition is automatically satisfied. However, the $\boldsymbol{R}$ action in Example 4.1.18 is free but not proper.

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If $\Phi: G \times M \to M$ is a smooth action and $x \in M$, $G_x = \{g \in G \mid \Phi_g x = x\}$ called the *isotropy group* of Φ at x. Since $G_x = \Phi_x^{-1}(x)$ and $\Phi_x: G \to M$ $g \mapsto \Phi(g,x)$ is continuous, G_x is a closed subgroup of G and hence by 4.1.1[illegible] smooth submanifold. If the action is proper, G_x is compact. Because $\Phi_x(g$ $= \Phi_g \circ \Phi_h x = \Phi_g x$ for every $h \in G_x$, Φ_x induces a mapping $\tilde{\Phi}_x: G/G_x \to Gx$ $M: gG_x \mapsto \Phi_g x$. This map is injective because if $\Phi_g x = \Phi_h x$, then $g^{-1}h \in$ that is, $gG_x = hG_x$.

4.1.22 Corollary. *If $\Phi: G \times M \to M$ is an action and $x \in M$, then $\tilde{\Phi}_x: G/$ $\to G{\cdot}x \subset M$ is an injective immersion. If Φ is proper, the orbit $G{\cdot}x$ is a clos submanifold of M and $\tilde{\Phi}_x$ is a diffeomorphism.*

Proof. First of all, $\tilde{\Phi}_x: G/G_x \to G{\cdot}x$ is smooth because $\tilde{\Phi}_x \circ \pi = \Phi_x$ is (see t Remark following 4.1.20). As we have already noted, $\tilde{\Phi}_x$ is one-to-one. show it is an immersion, we show that $T\tilde{\Phi}_x([g]) \cdot [\xi]$ is one-to-one. B $T\tilde{\Phi}_x([g]) \cdot [\xi] = T\Phi_x(g) \cdot \xi$. Thus $T\tilde{\Phi}_x([g])$ will be one-to-one if we can sh that $T_g G_x = \{\xi \in T_g G \mid T\Phi_x(g) \cdot \xi = 0\}$. The inclusion $\subset$ is obvious. For t opposite inclusion, we first suppose $g = e$. Thus, let $\xi \in \mathfrak{g}$ satisfy $T\Phi_x(e) \cdot \xi =$ Then

$$\frac{d}{dt} \Phi_x(\exp t\xi) = T\Phi_x(\exp t\xi) \cdot T(L_{\exp t\xi}) \cdot \xi$$

The defining property of an action may be written as

$$\Phi_x \circ L_g = \Phi_g \circ \Phi_x \quad \text{for all } g \in G,\ x \in M$$

Differentiating at e,

$$T\Phi_x(g) \cdot TL_g \cdot \xi = T\Phi_g(x) \cdot T\Phi_x(e) \cdot \xi$$

Taking $g = \exp t\xi$,

$$\frac{d}{dt} \Phi_x(\exp t\xi) = T\Phi_{\exp t\xi}(x) \cdot T\Phi_x(e) \cdot \xi = 0$$

Thus $\Phi_x(\exp t\xi) = \Phi_x(e) = x$, so $\exp t\xi \in G_x$, and thus $\xi \in T_e G_x$. This shows t inclusion $\supset$ for $g = e$. For the general case, note that the isomorphism $T_e L$ $T_e G \to T_g G$ satisfies $T_e L_g(T_e G_x) = T_g G_x$, $T_e L_g(\{\xi \in T_e G \mid T\Phi_x(e) \cdot \xi = 0\})$ $\{\eta \in T_g G \mid T\Phi_x(g) \cdot \eta = 0\}$ and the inclusion $\supset$ is proved. This completes t proof that $\tilde{\Phi}_x$ is an immersion.

If the action is proper, then $\tilde{\Phi}_x$ is a closed mapping and hence is homeomorphism onto its image. ■

One can alternatively prove that the orbits $G{\cdot}x$ are immersed subma folds by writing $G{\cdot}x = \Phi_x(G)$ and appealing directly to Exercise 1.6F((Here, $\ker \Phi_x$ is a subbundle of TG since it is $\ker T\Phi_x(e)$ made left invarian

Notice that if Φ is a transitive action of G on M, then for any $x \in M$, $G\cdot x = M$, so $M \approx G/G_x$. In this case M is called a *homogeneous space*. Conversely, if H is a subgroup of G and we set $M = G/H$ with G acting by left translation, M is a homogeneous space with $G_x = H$ for any $x \in M$. (See Exercises 4.1L, M.)

4.1.23 Proposition. *If $\Phi: G \times M \to M$ is a proper free smooth action, then M/G is a smooth manifold and $\pi: M \to M/G$ is a submersion.*

Proof. Let $\tilde{\Phi}: G \times M \to M \times M: (g,x) \mapsto (x, \Phi_g x)$, then the following argument shows that $\tilde{\Phi}$ has constant rank. Define the actions $\Lambda: G \times (M \times M) \to M \times M: (g,(x,y)) \mapsto (x, \Phi_g y)$ and $\Xi: G \times (G \times M) \to G \times M: (g,(h,x)) \mapsto (gh,x)$. Then $\Lambda_g: M \times M \to M \times M: (x,y) \mapsto (x, \Phi_g y)$ and $\Xi_g: G \times M \to G \times M: (h,x) \mapsto (gh,x)$ are diffeomorphisms for every $g \in G$. Also

$$\Lambda_g \circ \tilde{\Phi} = \tilde{\Phi} \circ \Xi_g \qquad \text{for every } g \in G \tag{1}$$

because

$$\Lambda_g \circ \tilde{\Phi}(h,x) = \Lambda_g(x, \Phi_h x) = (x, \Phi_g(\Phi_h x)) = (x, \Phi_{gh} x) = \tilde{\Phi}(gh,x) = \tilde{\Phi} \circ \Xi_g(h,x)$$

Taking the tangent of (1) at (e,x) gives

$$T_{\tilde{\Phi}(e,x)}\Lambda_g \circ T_{(e,x)}\tilde{\Phi} = T_{\Xi_g(e,x)}\tilde{\Phi} \circ T_{(e,x)}\Xi_g$$

which is equivalent to

$$T_{(x,x)}\Lambda_g \circ T_{(e,x)}\tilde{\Phi} = T_{(g,x)}\tilde{\Phi} \circ T_{(e,x)}\Xi_g$$

Therefore, the rank of $T_{(g,x)}\tilde{\Phi}$ equals the rank of $T_{(e,x)}\tilde{\Phi}$ since $T_{(x,x)}\Lambda_g$ and $T_{(e,x)}\Xi_g$ are isomorphisms. Thus the rank of $T_{(g,x)}\tilde{\Phi}$ is independent of $g \in G$. It remains hence to show that *rank* $T_{(e,x)}\tilde{\Phi}$ is independent of $x \in M$. We have

$$T_{(e,x)}\tilde{\Phi}: T_e G \times T_x M \to T_x M \times T_x M$$

$$T_{(e,x)}\tilde{\Phi}(\xi, v) = (v, T_e\Phi_x(\xi) + v)$$

and hence *rank* $T_{(e,x)}\tilde{\Phi} = n + \textit{rank}\, T_e\Phi_x$, where $n = \dim M$. But by 4.1.22, the map $\Phi_x: G \mapsto G\cdot x$ is a diffeomorphism ($G\cdot x$ being an immersed submanifold in M) and hence

$$T_e\Phi_x(T_e G) = T_x(G\cdot x)$$

so that *rank* $T_e\Phi_x = \dim G$, which is independent of x. Since Φ is a free action, $\tilde{\Phi}$ is injective, for if $(x, \Phi_g x) = (y, \Phi_h y)$, then $x = y$ and $\Phi_g x = \Phi_h y = \Phi_h x$, which implies $x = \Phi_{h^{-1}g} x$; hence $h^{-1}g = e$. By the local fibration theorem

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(Exercise 1.6G) image $\tilde{\Phi} = R$ is an injectively immersed submanifold of $M \times M$. Because $\tilde{\Phi}$ is proper, it is a closed mapping. Thus R is closed and $\tilde{\Phi}^{-1}$ is continuous. Therefore, $\tilde{\Phi}$ is a homeomorphism, which implies that R a submanifold of $M \times M$. By Theorem 4.1.20, M/G is a smooth manifol and $\pi: M \to M/G$ is a submersion. ■

Next we turn to the infinitesimal description of an action, which will b crucial for mechanics.

4.1.24 Definition. *Suppose $\Phi: G \times M \to M$ is a smooth action. If $\xi \in T_e G$ then $\Phi^\xi: \mathbf{R} \times M \to M: (t,x) \mapsto \Phi(\exp t\xi, x)$ is an $\mathbf{R}$-action on M, that is, Φ^ξ is flow on M. The corresponding vector field on M given by*

$$\xi_M(x) = \frac{d}{dt}\Phi(\exp t\xi, x)\bigg|_{t=0}$$

*is called the **infinitesimal generator** of the action corresponding to ξ.*

Remark. From the proof of 4.1.22, we find that in the language of infinitesimal generators

$$T_x(G\cdot x) = \{\xi_M(x) \mid \xi \in \mathfrak{g}\}$$

4.1.25 Examples (a) Let $\Phi: G \times G \to G: (g,h) \mapsto gh = L_g h$, then Φ is smooth action. If $\xi \in T_e G$, then $\Phi^\xi: \mathbf{R} \times G \to G: (t,h) \mapsto (\exp t\xi)h = R_h \exp t\xi$. Therefore, $\xi_G(g) = T_e R_g \xi$. Because $\Phi^\xi(t, R_g h) = (\exp t\xi)hg = R_g \Phi^\xi(t,h)$, ξ_G is *right* invariant and is therefore not equal to $X_\xi(g) = T_e L_g \xi$ which is *left* invariant, unless G is abelian.

(b) Let $\Phi: G \times T_e G \to T_e G: (g,\eta) \mapsto Ad_g \eta = T_e(R_{g^{-1}} L_g)\eta$, then Φ is smooth action called the *adjoint action* of G on $T_e G$. If $\xi \in T_e G$, we claim th $\xi_{T_e G} = ad_\xi$, where $ad: T_e G \times T_e G \to T_e G: (\xi,\eta) \mapsto [\xi,\eta]$.

Indeed, let $\phi_t(g) = g \exp t\xi = R_{\exp t\xi} g$, the flow of X_ξ. Then

$$\begin{aligned}
[\xi,\eta] &= [X_\xi, X_\eta](e) \\
&= \frac{d}{dt} T_{\phi_t(e)}\phi_{-t} \cdot X_\eta(\phi_t(e))\Big|_{t=0} \\
&= \frac{d}{dt} T_{\exp t\xi} R_{\exp(-t\xi)} X_\eta(\exp t\xi)\Big|_{t=0} \\
&= \frac{d}{dt} T_{\exp t\xi} R_{\exp(-t\xi)} T_e L_{\exp t\xi}\eta\Big|_{t=0} \\
&= \frac{d}{dt} T_e(L_{\exp t\xi} R_{\exp(-t\xi)})\eta\Big|_{t=0}
\end{aligned}$$

Therefore

$$\xi_{T_eG}(\eta)=\frac{d}{dt}Ad_{\exp t\xi}\eta\Big|_{t=0}=[\xi,\eta]=ad_\xi\eta$$

(c) Rephrasing, the result (b) says that

$$\frac{d}{dt}Ad_{\exp t\xi}\eta\Big|_{t=0}=[\xi,\eta]$$

that is, holding η fixed and differentiating,

$$T_e(Ad.\eta)\cdot\xi=[\xi,\eta]$$

where $Ad.\eta: G\to T_eG: g\mapsto Ad_g\eta$. More generally, from

$$Ad_{hg}\eta=Ad_h(Ad_g\eta)$$

we get, by differentiating in h,

$$T_g(Ad_g\eta)\cdot(T_eR_g\cdot\xi)=[\xi,Ad_g\eta],\quad \xi,\eta\in T_eG$$

that is,

$$T_g(Ad_g\eta)\cdot\xi_g=[TR_{g^{-1}}\cdot\xi_g,Ad_g\eta],\quad \xi_g\in T_gG,\quad \eta\in T_eG$$

Therefore, if $x(t)$ is a smooth curve in G and $\xi\in T_eG$ and we let $\xi(t)=Ad_{x(t)}\xi$, the chain rule gives

$$\frac{d\xi(t)}{dt}=\left[T_eR_{x(t)^{-1}}\frac{dx}{dt},\xi(t)\right]$$

a formula that we will need in Sect. 4.3. See (d) and Exercise 4.1H for the version on $\mathfrak{g}^*$.

(d) Let $\Phi: G\times(T_eG)^*\to(T_eG)^*: (g,\alpha)\mapsto Ad^*_{g^{-1}}\alpha$, where $Ad^*_g: (T_eG)^*\to(T_eG)^*: \beta\mapsto\{\eta\mapsto\beta(Ad_g\eta)\}$, then Φ is a smooth action called the *co-adjoint action* of G on $(T_eG)^*$. If $\xi\in T_eG$, then the following calculation shows that $\xi_{(T_eG)^*}=-ad^*_\xi$. Indeed, for $\eta\in T_eG$,

$$\begin{aligned}(\xi_{(T_eG)^*}(\alpha))\eta&=\frac{d}{dt}(Ad^*_{\exp(-t\xi)}\alpha)(\eta)\Big|_{t=0}\\&=\frac{d}{dt}\alpha(Ad_{\exp(-t\xi)}\eta)\Big|_{t=0}\\&=\alpha\left(\frac{d}{dt}Ad_{\exp(-t\xi)}\eta\Big|_{t=0}\right)\\&=\alpha([-\xi,\eta])=-ad^*_\xi\alpha(\eta)\end{aligned}$$

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The next proposition gives the basic properties of infinitesimal generato

4.1.26 Proposition. *Let $\Phi: G\times M\to M$ be a smooth action. For every $g\in$ and $\xi,\eta\in T_eG$ we have*

(i) $(Ad_g\xi)_M=\Phi^*_{g^{-1}}\xi_M$ *and*
(ii) $[\xi_M,\eta_M]=-[\xi,\eta]_M$.

Proof. (i) For $x\in M$

$$\begin{aligned}(Ad_g\xi)_M(x)&=\frac{d}{dt}\Phi(\exp tAd_g\xi,x)\Big|_{t=0}\quad\text{(by Definition 4.1.24)}\\&=\frac{d}{dt}\Phi(g(\exp t\xi)g^{-1},x)\Big|_{t=0}\quad\text{(by 4.1.8)}\\&=\frac{d}{dt}\Phi_g\circ\Phi(\exp t\xi,\Phi_{g^{-1}}(x))\Big|_{t=0}\quad\text{(because }\Phi\text{ is an action)}\\&=T_{\Phi_{g^{-1}}(x)}\Phi_g\frac{d}{dt}\Phi(\exp t\xi,\Phi_{g^{-1}}(x))\Big|_{t=0}\quad\text{(chain rule)}\\&=T_{\Phi_{g^{-1}}(x)}\Phi_g\xi_M(\Phi_{g^{-1}}(x))=(\Phi^*_{g^{-1}}\xi_M)(x)\end{aligned}$$

(ii) Let $g=\exp t\eta$ in (i), so that

$$(Ad_{\exp t\eta}\xi)_M=\Phi^*_{\exp(-t\eta)}\xi_M$$

Now $\Phi_{\exp(-t\eta)}$ is the flow of $-\eta_M$, so differentiating in t at $t=0$, right-hand side gives $[\xi_M,\eta_M]$. The derivative of the left-hand side at $t=0$ by 4.1.25(b), $([\eta,\xi])_M$. Thus (ii) follows. ■

Let $\tilde{X}_\xi(g)=T_eR_g\xi$. By 4.1.25(a), $\tilde{X}_\xi=\xi_G$ and so $[\tilde{X}_\xi,\tilde{X}_\eta]=-\tilde{X}_{[\xi,\eta]}$. The following ideas will play an important role in subsequent sections.

4.1.27 Definition. *Let M and N be manifolds and G a Lie group. Let Φ and Ψ be actions of G on M and N, respectively, and $f: M\to N$ a smooth map. say f is **equivariant** with respect to these actions if for all $g\in G$,*

$$f\circ\Phi_g=\Psi_g\circ f$$

that is, the following diagram commutes.

$$\begin{array}{ccc}M&\xrightarrow{f}&N\\ {\scriptstyle\Phi_g}\downarrow&&\downarrow{\scriptstyle\Psi_g}\\ M&\xrightarrow{f}&N\end{array}$$

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4.1.28 Proposition. *Let $f: M \to N$ be equivariant with respect to actions Φ and Ψ of G on M and N, respectively. Then for any $\xi \in \mathfrak{g}$,*

$$Tf \circ \xi_M = \xi_N \circ f$$

where ξ_M and ξ_N denote the infinitesimal generators on M and N, respectively, associated with ξ: in other words, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \xi_M \downarrow & & \downarrow \xi_N \\ TM & \xrightarrow[\quad Tf \quad]{} & TN \end{array}$$

Proof. By equivariance,

$$f \circ \Phi_{\exp t\xi} = \Psi_{\exp t\xi} \circ f$$

Differentiating with respect to t at $t=0$ and using the chain rule gives

$$Tf \circ \left(\frac{d}{dt} \Phi_{\exp \xi} \Big|_{t=0} \right) = \left(\frac{d}{dt} \Psi_{\exp t\xi} \Big|_{t=0} \right) \circ f$$

that is, $Tf \circ \xi_M = \xi_N \circ f$. ■

We conclude this section with a few supplementary remarks on actions that help to unify ideas.

If we think of an action Φ of G on M as a homeomorphism of G to $\mathfrak{D}(M)$, the diffeomorphism group of M, by $g \mapsto \Phi_g$, we can ask what the induced homomorphism of Lie algebras is [see 4.1.5 and Exercise 4.1G for a discussion of why $\mathfrak{X}$ is the Lie algebra of $\mathfrak{D}(M)$]. It is exactly the map

$$\Phi' = T_e\Phi: \mathfrak{g} \to \mathfrak{X}; \qquad \xi \mapsto \xi_M$$

[In 4.1.26(ii) we saw that Φ' is an *anti-homomorphism*; this is because the Lie algebra of $\mathfrak{D}(M)$ is $\mathfrak{X}(M)$ with bracket $-[X, Y]$; see Exercise 4.1G.] The action is called *essential* if this homomorphism $G \to \mathfrak{D}(M)$: $g \mapsto \Phi_g$ is injective.

It is not difficult to see that a homomorphism of connected Lie groups is determined by its tangent at the identity (Chevalley [1946, p. 113]). The analog of this for the maps $g \mapsto \Phi_g$ and Φ' is the following result of Palais [1957, Chapters II and III].

4.1.29 Theorem (Palais). *Let G be a simply connected Lie group, M a compact manifold, and φ: $\mathfrak{g} \to \mathfrak{X}(M)$ a Lie algebra homomorphism. Then there exists a unique action Φ: $G \to \mathfrak{D}(M)$ such that $\Phi' = \varphi$.*

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It follows that in the context of this theorem, the actions of G on M are i bijection with the potential infinitesimal generators, or homomorphisms of into $\mathfrak{X}(M)$. In our analogy $\Phi' = T_e\Phi$ above, it is clear that Φ should b essential iff Φ' is a monomorphism, and this is in fact made rigorous by th proof of 4.1.29 (omitted). Thus the essential actions of G on M are parame trized by isomorphisms of $\mathfrak{g}$ onto subalgebras of $\mathfrak{X}(M)$. These in turn may b parametrized as follows. Choose an ordered basis $(x_1, \ldots, x_k)$ for the re vector space $\mathfrak{g}$. The **constants of structure** $\{c^\gamma_{\alpha\beta}\}$ are defined by the **commut tion relations**

$$[x_\alpha, x_\beta] = c^\gamma_{\alpha\beta} x_\gamma$$

(summed on $\gamma = 1, \ldots, k$). Then a monomorphism φ: $\mathfrak{g} \to \mathfrak{X}(M)$ is uniquel determined for any ordered linearly independent set $(Y_1, \ldots, Y_k) \subset \mathfrak{X}(M$ satisfying the same commutation relations

$$[Y_\alpha, Y_\beta] = c^\gamma_{\alpha\beta} Y_\gamma$$

by the condition $\varphi(x_\alpha) = Y_\alpha$, and linear extension over $\mathfrak{g}$.

In this way we obtain a bijection between essential actions of G on M an k-tuples of vector fields on M that are linearly independent and satisfy fixed system of commutation relations. In case G is not simply connected o M is not compact, the parametrization of essential actions is considerabl more complicated. This aspect of the theory is not explicitly needed in th usual examples of actions in mechanics, but is useful for the intuition i provides. For more details, see Palais [1957] and Hermann [1966].

Some other investigations of Palais are also worth noting. Namely, he ha a general result which asserts that the diffeomorphisms of a manifold whic preserve a "geometric structure" form a Lie group. This generalizes the classi result of Myers and Steenrod [1939], which states that the isometries of Riemannian manifold form a finite-dimensional Lie group. Of course th Euclidean group, $O(n, \mathbf{R}) \times \mathbf{R}^n$, the isometry group of $\mathbf{R}^n$, is a special case See Kobayashi [1973] for the proof and discussion.

EXERCISES

4.1A. (i) Identify the Lie algebra of $SO(3, \mathbf{R}) = SO(3)$ with $\mathbf{R}^3$ as follows; defin the map:

$$\mathbf{R}^3 \to T_e SO(3): x = (x^1, x^2, x^3) \mapsto \hat{x} = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}$$

Show that $(x \times y)\hat{} = [\hat{x}, \hat{y}]$, where $\times$ is the usual vector (cross) product o $\mathbf{R}^3$. Thus the Lie algebra of $SO(3)$ may be viewed as $\mathbf{R}^3$ with vect product as Lie bracket. Note that $x \cdot y = -\frac{1}{2}$ *trace* $(\hat{x}\hat{y})$.