MATH 5652: Homework 1

(1.28, 2.18, 3.16, 3.28 in Ch probability review, and a., b. below, c. is a bonus problem)

The characteristic function of a random variable X is defined to be

$$\psi_X(t) := E(e^{itX}), \text{ for } t \in \mathbb{R}, \quad (\text{where } i = \sqrt{-1}).$$

This is a more general and useful notion then the moment-generating function $\phi_x(t) = E(e^{tX})$, and it has the same important property as (3.10) for ϕ:

$$\text{if } X \text{ and } Y \text{ are independent then } \psi_{X+Y}(t) = \psi_X(t)\psi_Y(t).$$

Another useful fact is that for any $a, b \in \mathbf{R}$

$$\psi_{aX+b}(t) = e^{itb}\psi_X(at).$$

If $\exists r > 0$ such that $|\phi_X(t)| < \infty$ for $t < r$ (or if all the moments $X^k, k = 1, 2, \ldots$ exist and satisfy a certain condition on how rapidly they grow), then

$$\psi_X(t) = \phi_X(it).$$

Most importantly, the characteristic function uniquely characterizes the distribution of a random variable, that is $\psi_X(t) = \psi_Y(t)$ iff X and Y have the same distribution.

a. Compute the characteristic function of all the following distributions: Poisson (rate λ), Geometric (parameter p), Bernoulli (parameter p), Bernoulli (number of trials n, parameter p), Uniform (on (a, b)), Exponential (mean λ), and Normal/Gaussian (mean μ, st.dev. σ).

b. Now either use the characteristic functions, or you can obtain directly from the distribution functions, the following approximation theorems for the discrete random variables above:

If the parameter p in the Binomial and Geometric distributions is replaced by p_n such that $p_n \to 0$ while $np_n \to \lambda$, then as $n \to \infty$

$$P[\text{Binomial}(n, p_n) = k] \to P[\text{ Poisson}(\lambda) = k]$$

$$nP[\text{Geometric}(p_n) = nk] \to P[\text{ Exponential}(\lambda) = k]$$

c.* Let (X_1, X_2) be a jointly Normal/Gaussian random vector with mean vector $\mu_X = (\mu_1, \mu_2)$ and variance covariance matrix Σ_X. Then the joint characteristic function of the random vector (X_1, X_2) is given by

$$\psi_{(X_1,X_2)}(t_1, t_2) := E(e^{i(t_1,t_2)\cdot(X_1,X_2)^T}) = e^{i(t_1,t_2)(\mu_1,\mu_2)^T - \frac{1}{2}(t_1,t_2)\cdot\Sigma\cdot(t_1,t_2)^T}.$$

Find the matrix $A \neq 0$ and vector b such that the components Y_1, Y_2 of the random vector $Y = AX + b$ are independent, that is the variance-covariance matrix Σ_Y of Y is diagonal. What is the distribution of such a random vector Y?

***** -this is a bonus problem