Solutions to First Midterm Exam

Advisory grades: at least 85 points: A, at least 70: B, at least 55: C, at least 40: D.

1) (10 points) The general solution of $y' + 2y = 1$ is

(a) $y(t) = 2e^{-ct} + \frac{1}{2}$
(b) $y(t) = \frac{1}{2}e^{-2t} + c$
(c) $y(t) = ce^{-2t} + \frac{1}{2}$
(d) $y(t) = ce^{2t} - \frac{1}{2}$
(e) None of the above.

Solution. The correct answer is (c). (Check by differentiation).

2) (10 points) Consider the following initial value problems and mark all those that have a unique solution near the initial data.

( ) $y' = \sqrt{y}$, $y(0) = 0$

(**X**) $y' = \sqrt{y}$, $y(1) = 1$

(**X**) $y' = \frac{yt}{y^2+t^2+1}$, $y(0) = 0$

(**X**) $y' = y^{5/3}$, $y(0) = 0$.

Solution. The Picard-Lindelöf theorem tells us that $y' = f(t, y)$, $y(t_0) = y_0$ has a unique solution if both f and $\frac{\partial}{\partial y} f$ are continuous at (t_0, y_0). That is true for all but the first example, where the theorem does not hold. As we learnt in the lecture, $y = 0$ and $y = \frac{1}{4}t^2$ are two distinct solutions.

3) (10 points). The radioactive half-life of the element Unobtainium is 1600 years, that of Phantasium 320 years. Suppose we are initially given 20g of Phantasium and 5g of Unobtainium. After how many years will there be equal amounts of both elements left?

(a) 100 years

(b) 500 years

(c) 800 years

(d) 5000 years

(e) Never

Solution. Amount of Uno: $5 \times 2^{-t/1600}$. Amount of Ph: $20 \times 2^{-t/320}$. Set them equal: $5 \times 2^{-t/1600} = 20 \times 2^{-t/320}$ so $2^{-t/1600} = 4 \times 2^{-t/320} = 2^{2-t/320}$. Taking base 2 logarithms on both sides, we have $-\frac{t}{1600} = 2 - \frac{t}{320}$. Multiplying with 1600, we see $-t = 3200 - 5t$ so $4t = 3200$, $t = 800$. Answer (c).

4) (20 points). Find the general solution of the differential equation

$$y' = 2ty + t.$$

Solution. Several different possibilities.

- See the equation as a separable equation: $y' = t(2y+1)$. Now for $y \neq -\frac{1}{2}$ we can divide by $2y+1$, so $\frac{dy}{2y+1} = tdt$, hence $\frac{1}{2}\ln|2y+1| = \frac{1}{2}t^2 + c$ so $|2y+1| = e^{2c}e^{t^2}$.
  It follows that $2y+1 = ke^{t^2}$, with $k = \pm e^{2c}$ or $k = 0$, hence $y = Ke^{t^2} - \frac{1}{2}$ (where $K = \frac{k}{2}$) is the general solution.
- See the equation as a linear equation: $y' + p(t)y = f(t)$, $p(t) = -2t$, $f(t) = t$.
  General solution of homogeneous equation: $Ce^{\int 2tdt} = Ce^{t^2}$.
  One particular solution: $y \equiv -\frac{1}{2}$ (see this by trying to find a solution with $y' \equiv 0$).

5) (25 points). Find the general solution of the differential equation

$$y' = \frac{y^4 + t^4}{ty^3}.$$

Hint: The right-hand side can be written as a function of $\frac{y}{t}$ ("Euler homogeneous differential equation"). Try the substitution $v = \frac{y}{t}$.

Solution. If $v = \frac{y}{t}$ then $y = vt$ so $y' = v't + v$. Here the ODE is $y' = \frac{y}{t} + \left(\frac{y}{t}\right)^{-3}$ so $v't + v = v + v^{-3}$. This implies (separation of variables) $v^3 dv = \frac{dt}{t}$ so $\frac{v^4}{4} = \ln|t| + C$ so $v = \pm\sqrt[4]{4\ln|t| + \tilde{C}}$. Finally, $y = tv = \pm t\sqrt[4]{4\ln|t| + \tilde{C}}$.

6) (25 points). Solve the initial value problem

$$y' + \frac{y}{t} = \frac{1}{t^5}, \quad y(1) = 1.$$

Solution. This is a linear ODE. The integrating factor is $e^{\int \frac{dt}{t}} = t$. Multiplying the equation by t, we have $(ty)' = ty' + y = t \cdot t^{-5}$ so $(ty)' = t^{-4}$. Integrating gives $ty = -\frac{1}{3}t^{-3} + C$ so $y = -\frac{1}{3}t^{-4} + Ct^{-1}$. The initial condition $y(1) = 1$ implies $1 = -\frac{1}{3} + C$ so $C = \frac{4}{3}$.

The solution is

$$y(t) = -\frac{1}{3t^4} + \frac{4}{3t}.$$