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ON NON-EXISTENCE OF VORTEX SOLUTIONS TO THE LANDAU-LIFSHITZ MAGNETIZATION EQUATION

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Abstract. In many cases the nonlinear Schrödinger equation supports existence of vortex or vortex-like solutions. The Landau-Lifshitz equation written using the stereographic projection is a nonlinear Schrödinger equation, but does not admit any such non-stationary solutions with finite energy. The only dynamically evolving vortex-like solutions have infinite energy.

1. Introduction

A typical phenomena in nonlinear partial differential equations is the presence of solitary wave solutions. Particularly, the structure of the nonlinear Schrödinger equation supports existence of so-called "vortex solutions" [1].

Vortices are localized topological defects. They are characterized by a nontrivial winding number of a flow around some point (or points) in a domain. Their name originates from the analogy to fluid flow singularities. Even as local structures, they can dominate the geometry and topology of the flow and in many cases they are the key to understanding the underlying physics.

Although the Landau-Lifshitz equation (LLE)

$$m_t = m \times \triangle m - \lambda m \times m \times \triangle m, \tag{1}$$

($\lambda \in \mathbb{R}$, $m = (m_1, m_2, m_3)$, $m : \mathbb{R}^2 \to \mathbb{S}^2$) does not have a priori the structure of the nonlinear Schrödinger equation, written using the stereographic projection, it transforms to such a form [2].

On the other hand, in the present paper we will show that the Landau-Lifshitz equation does not support existence of any vortices or localized vortex-like solutions (e.g. encapsulated vortices) with finite energy except for stationary solutions. While this is clear if dissipation is present ($\lambda > 0$), surprisingly it remains true in the absence of dissipation (when $\lambda = 0$). In fact, all the vortex-like non-stationary solutions have radially oscillating profile of infinite energy.

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The Landau-Lifshitz equation. The Landau-Lifshitz equation, often called also the magnetization equation, describes the time evolution of the density of the magnetic moment (magnetization) m in a ferromagnetic medium. The magnetic moment is primarily created by electron spins. Its magnitude is approximately constant for temperatures below the Curie point. The dynamics of the Landau-Lifshitz equation is derived from the general torque equation [3]

$$m_t = \gamma_0 L = \gamma_0 m \times h \,,$$

where L is the magnetic torque, h is the effective intensity of the magnetic field and $\gamma_0 = \frac{ge}{2m_e c}$ is a constant (e is the electron charge, m_e is the unit electron mass, c is the speed of light and g is the gyromagnetic ratio, $g \approx 2$). To preserve the magnitude $|m|$ of the magnetic moment it is traditional to include the Ginzburg-Landau phenomenological dissipative term [4]

$$m_t = \gamma_0 m \times h - \lambda m \times (m \times h) \,, \tag{2}$$

where λ is a dissipation constant — typically $\gamma_0 > \lambda$ (often $\gamma_0 >> \lambda$). The equation (2) is referred to as the Landau-Lifshitz equation [5] and is formally equivalent to the Gilbert equation:

$$m_t = \gamma(L - \eta m \times m_t) \,.$$

The effective intensity h of the magnetic field is the negative variational derivative of the free energy $E(m)$

$$h = -\frac{\delta E}{\delta m} \,.$$

The free energy $E(m)$ in two-dimensional models consists in general of four different terms [6]:

$$E(m) = \int_\Omega \left(\frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \frac{\partial m}{\partial x_i} \cdot \frac{\partial m}{\partial x_j} + \Phi(m) - H_{app} \cdot m + \frac{1}{8\pi} |H_{dem}(m)|^2 \right) dx \tag{3}$$

where the individual contributions in (3) represent the exchange, the anisotropy, the applied magnetic field and the demagnetizing field energies. The symmetric tensor (a_{ij}) is assumed to be positive definite, the function $\Phi : \mathbb{S}^2 \to \mathbb{R}$ depends on the internal structure of the ferromagnet and is assumed to be smooth and convex (as defined on $\mathbb{R}^3$), H_{app} is a prescribed divergence-free applied magnetic field and H_{dem} is the demagnetizing magnetic field given by the magnetostatic equations (a simpler form of the Maxwell equations)

$$\nabla \cdot (H_{dem} + 4\pi m) = 0 \qquad \text{and} \qquad \nabla \times H_{dem} = 0 \,.$$

In what follows we will consider only the two-dimensional model (i.e. $m : \mathbb{R}^2 \to \mathbb{S}^2$) in the whole plane $\Omega = \mathbb{R}^2$. We will also assume that the leading term of the free energy — the exchange energy — is uniform and diagonal and dominates all the other sources of the free energy, so

$$E(m) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla m|^2 dx \,. \tag{4}$$

Then (2) yields (1). An easy calculation shows that the constraint $|m| = 1$ is preserved in time by (1). A numerical scheme for (1) was recently proposed by W. E and X.-P. Wang [7].

Let us note that that if $\lambda = 0$ the equation (1) reduces to

$$m_t = m \times \triangle m \tag{5}$$

and describes the Hamiltonian (symplectic) flow of harmonic mappings into $\mathbb{S}^2$. If $\lambda = \infty$ the equation (1) reduces to $m_t = \triangle m + |\nabla m|^2 m$ (since $-m \times (m \times \triangle m) = \triangle m + |\nabla m|^2 m$ for $|m| = 1$) and describes heat flow of harmonic maps into $\mathbb{S}^2$.

A detailed discussion of the Landau-Lifshitz equation can be found in Komineas and Papanicolaou [4], Visintin [6] and Kosevich, Ivanov and Kovalev [8]. Another common form of (1) is the so-called easy-axis magnetization, for which the one-direction anisotropy energy term dominates the total free energy (3),

$$m_t = m \times \triangle m + \lambda m \times (m_3 \hat{z})\,, \tag{6}$$

where $\hat{z}$ is the z-direction unit vector [9].

Let us now discuss the dynamics of (1). First, since $-m \times (m \times h) = h - (m \cdot h)m$ for $|m| = 1$, the second term on the right-hand side of (1) is a projection and forces m to tend to the direction of $h = \triangle m$. It dissipates the energy because

$$\frac{d}{dt} E(m) = -\int m_t \cdot h dx = -\lambda \int |m \times h|^2 dx\,.$$

On the other hand, the term $m \times h$ is not dissipative and is perpendicular to both m and h and forces m to rotate around h. The total effect of terms on the right hand side of (1) results in a non-planar spiral revolution of a unit vector m asymptotically tending to $h/|h|$.

Landau-Lifshitz equation in spherical coordinates. For our further study it is convenient to transform the Landau-Lifshitz equation using spherical coordinates:

$$m_1 = \cos\psi \sin\phi, \qquad m_2 = \sin\psi \sin\phi, \qquad m_3 = \cos\phi\,.$$

The functions ϕ and ψ naturally map the plane $\mathbb{R}^2$ into $[0, \pi)$ and $[0, 2\pi)$ respectively, but we will allow their values to be any real numbers and always consider the value modulo the appropriate constant (π and 2π respectively). In this setting (1) turns into

$$\phi_t = -F + \lambda G\,, \qquad \psi_t \sin\phi = G + \lambda F\,, \tag{7}$$

where

$$F = \sin\phi \triangle\psi + 2\cos\phi \nabla\phi \cdot \nabla\psi \qquad \text{and} \qquad G = \triangle\phi - \sin\phi\cos\phi(\nabla\psi)^2\,.$$

Our goal is to search for standing wave solutions represented in polar coordinates (r, θ) as

$$\phi = \phi(r), \qquad \psi = n\theta + \omega t + \psi_0\,, \tag{8}$$

where n is a vortex degree, ω is the angular velocity (frequency) and ψ_0 is the initial phase. Then $F = 0$ and $\phi_t = 0$, so (7) becomes $\lambda G = 0$ and $\psi_t \sin\phi = G$. Thus non-trivial solutions may exist only for $\lambda = 0$ and (1) reduces to (5). Then (7) yields

$$\phi_{rr} + \frac{1}{r}\phi_r - \sin\phi\cos\phi\frac{n^2}{r^2} = \omega\sin\phi\,. \tag{9}$$

Note the similarity with the structure of the cubic nonlinear Schrödinger equation [10].

To avoid the singularity at $r = 0$ we can exploit with a minor modifications the technique of Iaia and Warchal [10]. The proper initial conditions are

$$\lim_{r\to 0^+} \frac{1}{r^n}\phi(r) = d \quad \text{and} \lim_{r\to 0^+} \frac{1}{r^{n-1}}\phi'(r) = nd\,. \tag{10}$$

In this paper we will always consider $d > 0$. From the form of these conditions we can infer that to specify a solution one only needs to prescribe one of them. The energy E of a solution of the form (8) is given by

$$E = \pi \int_0^\infty \left(\phi_r^2 + \frac{n^2}{r^2}\sin^2\phi\right) r dr\,. \tag{11}$$

Similarly, following [10] it is possible to justify the global existence of the solutions and uniqueness for the initial value problem (9)–(10) such that $\lim_{r\to\infty}\phi(r) = 0, \pi$ or 2π. All such solutions have locally finite energy over any finite interval $(0, R)$. Any other initial condition not in agreement with $\phi(0) = 0$ (e.g. $\phi(0) = \frac{\pi}{2}$) clearly leads to a locally infinite energy in some neighborhood of 0.

Let us state our main result here.

Theorem 1. *For any $\omega \neq 0$, any $\lambda \in \mathbb{R}$ and any $d > 0$ the only solution $\phi(r)$ to (9)–(10) oscillates infinitely many times around π as $r \to \infty$ and has an infinite energy (11).*

The statement of Theorem 1 is particularly interesting if $\lambda = 0$ since in the case of $\lambda \neq 0$ one can use the Derrick-Pohozaev scaling arguments to rule out any stationary solutions. Unfortunately for $\lambda = 0$ the energy associated with (1) is invariant in such a scaling and the argument cannot be used here.

Let us note that one can use similar arguments for the Bessel functions to show that the natural energy associated with the Bessel functions

$$E_B = \int_0^\infty r u_r^2(r) + \frac{1}{r}u^2(r) dr$$

is infinite for every Bessel function, a solution to the Bessel equation

$$r^2 u'' + ru' + (r^2 - n^2)u = 0\,.$$

Vortices for nonlinear Schrödinger equation. After introduction of vortices by Neu [1], A. de Bouard [11] studied the existence and stability of the vortex solutions to nonlinear Schrödinger equation:

$$iw_t = -\Delta w - F(|w|^2)w \tag{12}$$

with the boundary condition $w(x) \to 1$ as $|x| \to \infty$. Except for stationary solutions she also discussed so-called stationary-state solutions of the form

$$w(x,t) = e^{i\omega t}u_\omega(x) \tag{13}$$

in one dimensional system (12). In more dimensions de Bouard proved a sufficient condition for the existence of stationary bubble solutions of the form

$$w(x,t) = e^{i\omega t}e^{in\theta}U(r) \tag{14}$$

for a certain class of nonlinearities.

Let us also point out that the solution (13) of (12) gives a rise to a solution of the same equation *traveling* with an arbitrary velocity v by the means of the Galilean boost transformation

$$w(x,t) = e^{(i/2)(v\cdot v)-(i/4)|v|^2 t + it\omega + i\theta_0} u_\omega(x - vt - x_0)\,.$$

For cubic and cubic-quintic nonlinear Schrödinger equations the existence of vortex and encapsulated vortex solutions was proved by Iaia and Warchall [10], [12]. Their stability was extensively studied in [13]. One of the currently most active areas of investigation in the field is another nonlinear Schrödinger equation, the Gross-Pitaevsky equation, which models the mean-field approximation of Bose-Einstein condensates. This equation supports the existence of vortices in agreement with a recent experiments. Stability of these vortices will be a subject of our further study [14].

The connection between solutions to (1) of the structure (8) and works of Neu and de Bouard is straightforward. One needs to transform the solution (8) written in the spherical coordinates by the stereographic projection which transforms (1) into nonlinear Schrödinger equation

$$iw_t = -\triangle w - 2\,\frac{(\nabla w)^2}{1+|w|^2}\bar{w}\,. \tag{15}$$

The solution w to (15) is given by $w = u + iv$, where u and v are defined by

$$m_1 = \frac{2u}{1+u^2+v^2}\,, \quad m_2 = \frac{2v}{1+u^2+v^2}\,, \quad m_3 = \frac{u^2+v^2-1}{1+u^2+v^2}\,.$$

Here m is a solution to (1), $|m| = 1$. Then

$$u = \frac{\cos\psi \pm |\cos\psi||\cos\phi|}{\sin\phi}, \qquad v = u\tan\psi\,. \tag{16}$$

For simplicity let us ignore the absolute values in (16). The two choices of the sign ± in (16) yield two different formulas for a solution w to (15)

$$w = e^{i\psi}\cot\frac{\phi}{2}, \qquad \text{and} \qquad w = e^{i\psi}\tan\frac{\phi}{2}\,.$$

We see that it is the exactly same setting as in (14),

$$w(r,\theta,t) = e^{i(n\theta+\omega t+\theta_0)}U(r)\,. \tag{17}$$

The appropriate choice of the form of the function $U(r)$ in (17) is given by the sign of ω as we will see in Lemma 1.

The outline of the paper. The organization of this paper is following. First, in Section 2 we will discuss the basic properties of solutions to (9)–(10). Then in Section 3 we determine the shape of the profile of a solution to that system by ruling out all the other possibilities (for $\omega \neq 0$). Finally, in Section 4, we will prove the Theorem 1 by energy estimates using the polar coordinates in the phase space.

2. Basic Properties of the Vortex Solutions

Let us now state without proof a few simple facts about the symmetries and scaling properties of the solutions to (9). Note that in the following proposition we do not take into account the initial condition (10).

Proposition 1.

- *If $\phi(r)$ is a solution of (9) then $\phi^*(r) = -\phi(r)$ is a solution too.*
- *If $\phi(r)$ is a solution of (9) then $\phi^*(r) = 2\pi + \phi(r)$ is a solution too.*
- *If $\phi(r)$ is a solution of (9) for a parameter ω then $\phi^*(r) = \pi \pm \phi(r)$ is a solution of the same equation for a parameter $-\omega$.*

- *If $\phi(r)$ is a solution of (9) for a parameter ω then $\phi^*(r) = \phi(\lambda r)$ is a solution of the same equation for a parameter $\lambda^2\omega$. (Thus one can always assume that $\omega = 1$, $\omega = -1$ or $\omega = 0$.)*

First, we will consider the case $\omega = 0$. In two dimensions there exist stationary solutions to (1), the Belavin-Polyakov instantons. These solutions are harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^2$. The equation (9) becomes integrable and all the solutions satisfying (10) are

$$\phi(r) = 2\tan^{-1}\left(\frac{d}{2}r^n\right) . \tag{18}$$

Note that such a solution has a finite energy (11).

The next lemma shows that a solution of the problem (9)–(10) is bounded.

Lemma 1. *Let $\phi(r)$ be a solution to (9)–(10) for $\omega = 1$ or $\omega = -1$ respectively. Then $\phi(r)$ is bounded, $0 < \phi(r) < 2\pi$ for all $r > 0$ or $-\pi < \phi(r) < \pi$ respectively.*

Proof. First, let $\omega > 0$. We multiply (9) by the term $r^2\phi_r(r)$ to obtain

$$\frac{\partial}{\partial r}\left(\frac{r^2\phi_r^2}{2}\right) + \frac{\partial}{\partial r}\left(\frac{n^2}{4}\cos 2\phi\right) = -\omega r^2\frac{\partial}{\partial r}\left(\cos\phi\right) .$$

By integrating the last identity on interval (R, r) and by using integration by parts we derive "the Pohozaev identity"

$$\begin{aligned} &\frac{1}{2}\left(r^2\phi_r^2(r) - R^2\phi_r^2(R)\right) + \frac{n^2}{4}(\cos 2\phi(r) - \cos 2\phi(R)) \\ &= 2\omega\int_R^r s(\cos\phi(s) - \cos\phi(r))ds + \omega R^2(\cos\phi(R) - \cos\phi(r)) . \end{aligned} \tag{19}$$

Let us set $R = 0$ in (19) and use $\lim_{r\to 0}\phi(r) = 0$. The initial condition (10) implies $\phi(s) > 0$ on $(0, r)$ for some $r > 0$. Then

$$\frac{1}{2}r^2\phi_r^2(r) - \frac{n^2}{2}\sin^2\phi(r) = 2\omega\int_0^r s(\cos\phi(s) - \cos\phi(r))ds . \tag{20}$$

If there exists $r > 0$ such that $\phi(r) = 0$ (we may assume that $\phi(s) > 0$ on $(0, r)$) then by (20)

$$0 \le r^2\phi^2(r) = 4\omega\int_0^r s(\cos\phi(s) - 1)ds < 0 ,$$

which yields a contradiction.

Hence $\phi(r) > 0$ for all $r > 0$. The very same argument can be used to prove that $\phi(r) < 2\pi$ for all $r > 0$. Similarly in the case $\omega < 0$ one can prove $-\pi < \phi(r) < \pi$ for all $r > 0$. □

Lemma 2. *Let $\phi(r)$ be a solution to (9)–(10) for $\omega > 0$ and let r be a point of local maximum (a local minimum) of $\phi(r)$, $r > \frac{n}{\sqrt{\omega}}$. Then $\phi(r) > \pi$ (or $\phi(r) < \pi$ respectively). If $\omega < 0$ then $\phi(r) > 0$ (or $\phi(r) < 0$ respectively).*

Proof. Since $r^2\omega > n^2$, it follows $\omega + \cos\phi(r)\frac{n^2}{r^2} > 0$. Then by (9)

$$\phi_{rr} = -\frac{1}{r}\phi_r + \sin\phi\left(\omega + \cos\phi\frac{n^2}{r^2}\right) = \sin\phi\left(\omega + \cos\phi\frac{n^2}{r^2}\right) .$$

Thus the sign of ϕ_{rr} is the same as the sign of $\sin\phi$. The statement immediately follows by Lemma 1. Similar arguments prove the statement for $\omega < 0$. □

Let us list all the possible behaviors of radial profiles of vortex-like solutions to (9)–(10) for $\omega > 0$ using Lemma 1–2 (we omit the trivial solutions $\phi(r) = k\pi$):

- "pure vortex solution" – growing for all $r > \frac{n}{\sqrt{\omega}}$, and satisfying
$$\lim_{r\to\infty} \phi(r) = \pi \quad \text{or} \quad 2\pi\,,$$
- "encapsulated vortex solution" – any solution satisfying
$$\lim_{r\to\infty} \phi(r) = 0\,,$$
- "finitely oscillating solution" – any solution satisfying
$$\lim_{r\to\infty} \phi(r) = \pi\,,$$
which is monotone on some interval (R, ∞);
- "infinitely oscillating solution" (also called "oscillating solution") – any solution satisfying
$$\lim_{r\to\infty} \phi(r) = \pi$$
which is not monotone on any interval (R, ∞).

The similar classifications exists also for $\omega < 0$. All the possible behaviors will be analyzed in the next section.

3. Non-existence of the Vortex Solutions

In this section we demonstrate that the only possible type of a solution to (9)–(10) is an "infinitely oscillating solution" by eliminating all the other types.

Theorem 2. *There are no solutions to (9)–(10) for $\omega > 0$ with the property $\lim_{r\to\infty} \phi(r) = 0$, i.e. there are no "encapsulated vortex solutions".*

Proof. Let us assume a contrary, let $\phi(r)$ be a solution to (9)–(10) such that $\lim_{r\to\infty} \phi(r) = 0$. Since $\phi(r) > 0$ for some small positive r, it has at least one local maximum. Fix R to be any of them. Then (19) yields

$$\begin{aligned}&\frac{1}{2} r^2 \phi_r^2(r) + \frac{n^2}{4}(\cos 2\phi(r) - \cos 2\phi(R))\\ &= 2\omega \int_R^r s(\cos\phi(s) - \cos\phi(r))ds + \omega R^2(\cos\phi(R) - \cos\phi(r))\,.\end{aligned} \tag{21}$$

Let us moreover assume that $\phi(R) \neq \pi$ ($\phi(R) \neq 0$ by Lemma 1). So $\cos 2\phi(R) < 1$. By Lemma 2 such a solution must be decreasing for r large enough so we can choose r to satisfy both $\cos\phi(s) < \cos\phi(r)$ for all $s \in (R, r)$ and $\cos 2\phi(R) < \cos 2\phi(r)$. Then the left hand side of (21) is positive while the right hand side is negative yielding a contradiction.

In the case of $\phi(R) = \pi$ the equation (21) becomes

$$\begin{aligned}\frac{r^2\phi_r^2(r)}{2} &= 2\omega \int_R^r s(\cos\phi(s) - \cos\phi(r))ds\\ &\quad + \omega R^2(-1 - \cos\phi(r)) - \frac{n^2}{4}(\cos 2\phi(r) - 1)\,.\end{aligned} \tag{22}$$

Again we can choose r large enough to satisfy $\cos\phi(s) < \cos\phi(r)$ for all $s \in (R, r)$ so the integral in (22) becomes negative. Then

$$\begin{aligned} \frac{1}{2}r^2\phi_r^2(r) &\leq -\omega R^2(1+\cos\phi(r)) + \frac{n^2}{4}(1-\cos 2\phi(r)) \\ &= 2\cos^2\frac{\phi(r)}{2}\left(n^2\sin^2\frac{\phi(r)}{2} - \omega R^2\right). \end{aligned} \tag{23}$$

On the other hand, for r large enough $\phi(r)$ is positive and close to zero. Hence $\cos^2\frac{\phi(r)}{2} > 0$ and $n^2\sin^2\frac{\phi(r)}{2} < 2\omega R^2$. This contradicts (23). ☐

Similar statement is true for $\omega < 0$ but first we need to prove the following lemma which will estimates the possible rate of decay of such solutions.

Lemma 3. *If there exists a positive solution* $\phi(r)$ *to (9)–(10) for* $\omega < 0$, *such that* $\lim_{r\to\infty}\phi(r) = 0$ *then* $n\phi(r) + r\phi_r(r) \geq 0$, *i.e.* $\phi(r) \geq \frac{1}{r^n}\phi(r_0)r_0^n$ *for any* $r > r_0 > 0$.

Proof. For the sake of brevity we set $\omega = -1$. The equation (9) can be rewritten as

$$\phi_{rr} + \frac{1}{r}\phi_r - \frac{n^2}{r^2}\phi = -\sin\phi + \frac{n^2}{r^2}\left(\frac{\sin 2\phi}{2} - \phi\right). \tag{24}$$

Since $\sin 2\phi < 2\phi$ for $\phi > 0$ and $\phi(r) < \pi$ by Lemma 1, the right hand side of (24) is always negative. Hence

$$\phi_{rr} + \frac{1}{r}\phi_r - \frac{n^2}{r^2}\phi < 0.$$

Following the same calculation as in [10] we set $\phi(r) = r^n u(r)$. Then for any $r > r_0 > 0$

$$u(r) < u(r_0) + u_r(r_0)\frac{r_0}{2n}\left[1 - \left(\frac{r_0}{r}\right)^{2n}\right].$$

After substitution $\phi(r) = r^n u(r)$ we finally obtain

$$\phi(r) < \left(\frac{r}{r_0}\right)^n\left[\phi(r_0) + \left(\frac{r_0\phi_r(r_0)}{2n} - \frac{\phi(r_0)}{2}\right)\left(1 - \left(\frac{r_0}{r}\right)^{2n}\right)\right]. \tag{25}$$

The inequality (25) holds for any $r > r_0 > 0$. If there exists a positive solution to (9) then the term in the bracket must be positive for every such a pair (r, r_0). Therefore

$$\phi(r_0) + \left(\frac{r_0\phi_r(r_0)}{2n} - \frac{\phi(r_0)}{2}\right)\left[1 - \left(\frac{r_0}{r}\right)^{2n}\right] > 0.$$

Because r can be arbitrarily large, the difference $1 - \left(\frac{r_0}{r}\right)^{2n}$ can be arbitrarily close to one. Thus it is necessary that

$$\phi(r_0) + \left(\frac{r_0\phi_r(r_0)}{2n} - \frac{\phi(r_0)}{2}\right) \geq 0.$$

Integration over (r_0, r) then implies

$$\phi(r) \geq \frac{1}{r^n}\phi(r_0)r_0^n.$$

☐

Theorem 3. *There is no solution to the system (9)–(10) for* $\omega < 0$ *with the property* $\lim_{r\to\infty} \phi(r) = 0$ *which is monotone on some interval* (R, ∞)*, i.e. there are no "encapsulated vortex solutions".*

Proof. First, we will prove that there are no positive solutions to (9)–(10) for $\omega = -1$ such that $\lim_{r\to\infty} \phi(r) = 0$.

We use the Pohozaev identity (19) on interval $(0, r)$:

$$r^2 \phi_r^2(r) - n^2 \sin^2 \phi(r) = -4 \int_0^r s\,(\cos\phi(s) - \cos\phi(r))\, ds\,. \tag{26}$$

By Lemma 2 a solution $\phi(r)$ may not attain a local minimum for r large enough. Hence, if there exists a positive solution to (9)–(10) such that $\lim_{r\to\infty} \phi(r) = 0$ it must decrease monotonically to zero on (r, ∞) for some r large enough. Then Lemma 3 implies $0 > r\phi_r \geq -n\phi$ and $r^2\phi_r^2 \leq n^2\phi^2$. Combining it with (26) yields

$$-4 \int_0^r s\,(\cos\phi(s) - \cos\phi(r))\, ds \leq n^2(\phi^2 - \sin^2\phi)\,. \tag{27}$$

We pass to the limit $r \to \infty$ on both sides of (27). Then

$$\lim_{r\to\infty} n^2(\phi^2 - \sin^2\phi) = 0$$

on the right hand side. On the other side

$$\lim_{r\to\infty} 4 \int_0^r s\,(\cos\phi(r) - \cos\phi(s))\, ds = 4 \int_0^\infty s\,(1 - \cos\phi(s))\, ds > 0$$

(the integrand is a positive quantity which justifies the limit) yields a contradiction for any nontrivial solution.

If $\phi(r)$ is not positive, it is by the assumption monotone on some interval (R, ∞). Let us assume $\phi(r)$ decreases to zero at infinity (if it increases, one needs to perform the transformation $\tilde{\phi}(r) = -\phi(r)$ first). If $\phi(r)$ is not positive for all $r \in (0, \infty)$, then there exist $R > 0$, such that $\phi(R) = 0$ and $\phi(r) > 0$ for $r > R$. Hence we can exploit Lemma 3 on the interval (R, ∞) and repeat the above arguments for positive solutions to prove the statement of the theorem just by replacing zero by R in the proof. ☐

Theorem 4. *There is no solution to the system (9)–(10) for* $\omega > 0$ *with the property* $\lim_{r\to\infty} \phi(r) = \pi$ *or* $\lim_{r\to\infty} \phi(r) = 2\pi$ *growing for all* $r > \frac{n}{\sqrt{\omega}}$*, i.e. there are no "pure vortex solutions".*

Proof. First, let us assume that $\lim_{r\to\infty} \phi(r) = \pi$. Then $\tilde{\phi} = \pi - \phi$ solves (9) for $\omega < 0$ with the initial condition $\tilde{\phi}(0) = \pi$. Clearly $\lim_{r\to\infty} \tilde{\phi}(r) = 0$. The monotonicity ensures that the solution is decreasing on the interval $\left(\frac{n}{\sqrt{\omega}}, \infty\right)$. The non-existence of such a solution was already proved in Theorem 3.

Therefore let us assume that $\lim_{r\to\infty} \phi(r) = 2\pi$. We will slightly modify the argument used in the proof of Theorem 3.

We apply the Pohozaev identity (19) on a interval $(0, r)$ to obtain

$$r^2 \phi_r^2(r) = n^2 \sin^2 \phi(r) + 4\omega \int_0^r s\,(\cos\phi(s) - \cos\phi(r))\, ds\,. \tag{28}$$

The left hand side of (28) is a positive quantity $r^2\phi_r^2(r) \geq 0$. The integral can be rewritten as a sum of two integrals

$$4\omega\int_0^R s\,(\cos\phi(s) - \cos\phi(r))\,ds + 4\omega\int_R^r s\,(\cos\phi(s) - \cos\phi(r))\,ds\,.$$

Let R be chosen such that $\phi(r) > \frac{3}{2}\pi$ for all $r > R$. Then for $r > R$ the argument of the second integral is a negative quantity and we can pass to the limit $r \to \infty$:

$$4\omega\int_0^R s\,(\cos\phi(s) - \cos\phi(r))\,ds \to 4\omega\int_0^R s\,(\cos\phi(s) - 1)\,ds < 0$$

and $n^2\sin^2\phi(r) \to 0$, $4\omega\int_R^r s\,(\cos\phi(s) - \cos\phi(r)) \to L < 0$. Hence the limit of the right hand side of (28) is negative – a contradiction. $\square$

Theorem 5. *There is no solution to the system (9)–(10) for* $\omega < 0$ *with the property* $\lim_{r\to\infty}\phi(r) = \pi$ *or* $\lim_{r\to\infty}\phi(r) = -\pi$ *monotone for all* $r > \frac{n}{\sqrt{\omega}}$, *i.e. there are no "pure vortex solutions".*

Proof. First, assume that $\lim_{r\to\infty}\phi(r) = \pi$ (for $\lim_{r\to\infty}\phi(r) = -\pi$ the proof is analogical using the transformation $\tilde{\phi} = -\phi$).

We will transform the solution to $\tilde{\phi} = \pi - \phi$ to obtain the solution $\tilde{\phi}$ to (9) for $\omega > 0$ satisfying the initial condition $\tilde{\phi}(0) = \pi$. For simplicity we will drop tildes in the rest of this proof. One can use the same arguments as in the proof of Theorem 4 to show the non-existence of such solutions: apply the Pohozaev identity on $(0, r)$ to get

$$r^2\phi_r^2 = n^2\sin^2\phi + 4\omega\int_0^r s\,(\cos\phi(s) - \cos\phi(r))\,ds$$

and then send $r \to \infty$. The left-hand side has a positive limit while the right-hand side has a negative limit (justification of the existence of the limit can be done exactly as in the proof of Theorem 4). $\square$

Theorem 6. *There is no solution to the system (9)–(10) for* $\omega > 0$ *with the property* $\lim_{r\to\infty}\phi(r) = \pi$ *monotone on some interval* (R, ∞), $R > 0$, *i.e. there are no "finitely oscillating solutions".*

Proof. One can again combine the same arguments as in the proofs of previous theorems. First, by the transformation $\tilde{\phi} = \pi + \phi$ we obtain a solution $\tilde{\phi}$ to (9) for $\omega < 0$ satisfying the initial condition $\tilde{\phi}(0) = -\pi$ with the property

$$\lim_{r\to\infty}\tilde{\phi}(r) = 0\,.$$

The non-existence of such a solution is then guaranteed by Theorem 3. The only difference is in the initial condition but the second part of the proof of Theorem 3 is independent on initial data. $\square$

Let us note that by Lemma 2 there are no "finitely oscillating solutions" for $\omega < 0$ which were not covered by Theorem 3 or 5.

4. Energy of the Oscillating Solutions

The aim of this final section is to prove Theorem 1. The first statement of Theorem 1, the oscillating character of the solution, follows from Theorem 2-6 of the previous section where we ruled out all the other possible profile behaviors of $\phi(r)$ for both $\omega > 0$ and $\omega < 0$. Thus we only need to show that such a solution has an infinite energy (11). This fact is not a priori clear (the stationary solutions (18) for $\omega = 0$ all have finite energy) because both terms in the energy formula (11) have the same scaling properties. This suggests a strong energetic instability of any vortex structures for $\omega \neq 0$.

The outline of the proof is following: We will prove the statement for $\omega > 0$ (the proof for $\omega < 0$ is analogical). By the scaling properties of the energy and the scaling invariance of the equation (9) (see Proposition 1) we may assume without loss of generality that $\omega = 1$. We will assume that the contrary is true – the solution has finite energy. Using that fact we show that the solution must "monotonically" decay to π (Lemma 4). Then we shift the solution to oscillate around 0 instead of π by the transformation $\phi \to \phi - \pi$. The transformed solution solves (9) with $\omega = -1$. We also show that for such a solution the energy (11) has the same character as the energy $E^* = \int r\phi^2 + r\phi_r^2 dr$. Finally, using the polar coordinates in the phase plane, we will show that the energy E^* (and so E) is infinite.

Let $\phi(r)$ be a solution to (9)–(10) for fixed parameter d in (10). It is convenient to introduce $(a_i)_1^\infty$, $(b_i)_1^\infty$ and $(c_i)_1^\infty$, the increasing infinite sequences of the intercepts of $\phi(r)$ with $y = \pi$ and the sequences of local maxima and local minima of $\phi(r)$ respectively. By neglecting first few terms and by Lemma 2 we may assume $a_1 < b_1 < a_2 < c_1 < a_3 < b_2 < \dots$, i.e. $a_{2i-1} < b_i < a_{2i} < c_i < a_{2i+1}$ for $i \geq 1$. For simplification we also define $(r_i)_1^\infty$, $r_1 = a_1$, $r_2 = b_1$, $r_3 = a_2$, $r_4 = c_1$, etc., i.e. the increasing sequence (r_i) is the "ordered" union of the sequences (a_i), (b_i) and (c_i).

Lemma 4. *Assume that the solution $\phi(r)$ to (9)–(10) has finite energy. Then*

$$\lim_{r\to\infty} \phi(r) = \pi \tag{29}$$

and the convergence is "monotone"

$$|\phi(b_i) - \pi| > |\pi - \phi(c_i)| > |\phi(b_{i+1}) - \pi| \qquad \textit{for } i \geq i_0,$$

for some $i_0 \geq 1$, where (b_i), (c_i) are defined above.

Written in terms of the sequence (r_i):

$$|\phi(r_{2i}) - \pi| > |\pi - \phi(r_{2i+2})| \qquad \textit{for } i \geq i_0.$$

Proof. The inequality $|ab| \leq \frac{a^2+b^2}{2}$ implies

$$\left| \int_a^b \phi_r(r)\sin\phi(r) dr \right| \leq \int_a^b |\phi_r(r)\sin\phi(r)|\, dr \leq \frac{1}{2}\int_a^b r\phi_r^2(r) + \frac{\sin^2\phi(r)}{r} dr\,.$$

Thus

$$|\cos\phi(b) - \cos\phi(a)| \leq \frac{1}{2}\int_a^b r\phi_r^2(r) + \frac{\sin^2\phi(r)}{r} dr\,. \tag{30}$$

Since the problem (9)–(10) is well-posed the integrals on the right-hand side of (30) are finite for $0 \leq a < b < \infty$. Then by (30)

$$|\cos\phi(r_{i+1}) - \cos\phi(r_i)| \leq \frac{1}{2}\int_{r_i}^{r_{i+1}} r\phi_r^2(r) + \frac{\sin^2\phi(r)}{r} dr$$

and

$$\sum_{i=1}^{\infty} |\cos(r_i) - \cos(r_{i+1})| \leq \frac{1}{2}\int_{r_1}^{\infty} r\phi_r^2(r) + \frac{\sin^2\phi(r)}{r} dr < \infty\,,$$

where the integral on the right-hand side converges by the assumption of the lemma

$$\int_0^{\infty} r\phi_r^2(r)dr < \infty \qquad \text{and} \qquad \int_0^{\infty} \frac{\sin^2\phi(r)}{r} dr < \infty\,.$$

Since $\cos\phi(r_{2i-1}) = -1$, for all $i \geq 1$, we have

$$|\cos\phi(r_{2i-1}) - \cos\phi(r_{2i})| + |\cos\phi(r_{2i}) - \cos\phi(r_{2i+1})| = 2|1 + \cos\phi(r_{2i})|\,.$$

Thus

$$\sum_{i=1}^{\infty} |1 + \cos(r_{2i})| < \infty\,, \qquad \lim_{i\to\infty} \cos\phi(r_{2i}) = -1$$

and (29) follows. The oscillating solution $\phi(r)$ converges to π and we may assume that $\pi/2 < \phi(r) < 3/2\pi$ for $r > R_0$ for some $R_0 > 0$.

Next we consider two consecutive local extremes at R and r (i.e. $(R, r) = (b_i, c_i)$ or $(R, r) = (c_i, b_{i+1})$), $r > R > R_0$, of $\phi(r)$. First, assume that

$$|\phi(R) - \pi| < |\phi(r) - \pi|\,. \tag{31}$$

The requirement $r > R > R_0$ assures that

$$0 > \cos\phi(r) > \cos\phi(R)\,. \tag{32}$$

By Pohozaev identity (19) applied on interval (R, r)

$$\begin{aligned} (\cos\phi(r) - \cos\phi(R))\left[\frac{n^2}{2}(\cos\phi(r) + \cos\phi(R)) + R^2\right] = \\ 2\int_R^r s\,(\cos\phi(s) - \cos\phi(r))\,ds\,. \end{aligned} \tag{33}$$

For R large enough ($R > |n|$) it holds

$$R^2 + \frac{n^2}{2}(\cos\phi(r) + \cos\phi(R)) \geq R^2 - n^2 > 0\,.$$

The inequality (32) implies $\cos\phi(r) - \cos\phi(R) > 0$, so the left-hand side of the equation (33) is positive. On the other hand, $\cos\phi(r)$ is the maximum value of $\cos\phi(s)$ on the interval (R, r) so the integral on the right-hand side of (33) is negative. This yields a contradiction with the assumption (31). One can easily derive by the same argument that $|\pi - \phi(r)| = |\pi - \phi(R)|$ is not possible as well.

Simple consideratons prove that for two consecutive local extremes located at R and r, $r > R > R_1$, $R_1 = \max\{R_0, |n|\}$, of $\phi(r)$ the following inequality holds.

$$|\pi - \phi(r)| < |\pi - \phi(R)|\,.$$

Let us mention two other consequences of a "monotone" convergence

$$0 > \cos\phi(R) > \cos\phi(r) \qquad \text{and} \qquad \cos\phi(r_{2i}) \searrow -1\,.$$

□

It is very convenient to consider a solution $\phi(r)$ oscillating around $y = 0$ instead of around $y = \pi$. Therefore let us introduce

$$\tilde{\phi} = \phi - \pi\,.$$

Then $\tilde{\phi}$ solves (see also Proposition 1)

$$\tilde{\phi}_{rr} + \frac{1}{r}\tilde{\phi}_r - \sin\tilde{\phi}\cos\tilde{\phi}\frac{n^2}{r^2} = -\sin\tilde{\phi}\,,$$

i.e. $\tilde{\phi}$ solves (9) for $\omega = -1$ with the initial condition $\tilde{\phi}(0) = -\pi$. Since the initial condition does not enter the arguments in this section (it was only used to prove the oscillating character of $\phi(r)$), we drop tildes and from now on we will assume only

$$\phi_{rr} + \frac{1}{r}\phi_r - \sin\phi\cos\phi\frac{n^2}{r^2} = -\sin\phi \tag{34}$$

and that $\phi(r)$ oscillates around 0. Clearly, the location of the points (r_i), (a_i), (b_i) and (c_i) does not change in this transformation and Lemma 4 implies

$$\phi(r) \to 0 \qquad \text{and} \qquad |\phi(r_{2i})| > |\phi(r_{2i+2})| \quad \text{for } i \geq i_0.$$

Lemma 5. *Let $\phi(r)$ be a non-trivial solution to (34) and (a_i) and i_0 be defined as above. Then for every $\varepsilon > 0$ there exists $i_1 \geq i_0$ such that for every $i \geq i_1$,*

$$\frac{5}{6}(1-\varepsilon)\int_{a_i}^{a_{i+1}} r\phi^2(r)dr \leq \int_{a_i}^{a_{i+1}} r\phi_r^2(r)dr \leq (1+\varepsilon)\int_{a_i}^{a_{i+1}} r\phi^2(r)dr\,. \tag{35}$$

Proof. First, by the integration by parts

$$\int_{a_i}^{a_{i+1}} r\phi_r^2 dr = \phi r\phi_r|_{a_i}^{a_{i+1}} - \int_{a_i}^{a_{i+1}} \phi(\phi_r + r\phi_{rr})dr\,.$$

Since $\phi(a_i) = 0$, (34) gives

$$\phi_r + r\phi_{rr} = r\left(\frac{n^2}{r^2}\sin\phi\cos\phi - \sin\phi\right)\,.$$

Hence

$$\int_{a_i}^{a_{i+1}} r\phi_r^2(r)dr = \int_{a_i}^{a_{i+1}} r\phi\sin\phi\left(1 - \frac{n^2}{r^2}\cos\phi\right)dr\,. \tag{36}$$

For any fixed n there exists $R_1 > 0$ such that for every r, $r > R_1$ the inequality $\left|\frac{n^2}{r^2}\cos\phi\right| < \varepsilon$ is true independently of a behavior of $\phi(r)$. Thus by (36)

$$(1-\varepsilon)\int_{a_i}^{a_{i+1}} r\phi\sin\phi dr \leq \int_{a_i}^{a_{i+1}} r\phi_r^2(r)dr \leq (1+\varepsilon)\int_{a_i}^{a_{i+1}} r\phi\sin\phi dr\,. \tag{37}$$

The Taylor expansion of $\sin x$ implies the following simple calculus inequality

$$\frac{5}{6}x^2 < x\sin x < x^2 \quad \text{for every } |x| \leq 1,\ x \neq 0\,.$$

Setting $x = \phi(r)$ and shifting R_1 if necessary (we need $|\phi(r)| \leq 1$) we get by (37)

$$\frac{5}{6}(1-\varepsilon)\int_{a_i}^{a_{i+1}} r\phi^2 dr \leq \int_{a_i}^{a_{i+1}} r\phi_r^2(r)dr \leq (1+\varepsilon)\int_{a_i}^{a_{i+1}} r\phi^2 dr\,.$$

$\square$

Note that the fraction $\frac{5}{6}$ can be removed from (35) by a small modification of the proof.

Corollary 1. *Let $\phi(r)$ be a solution to (34) with finite energy. Then the integral $\int_0^\infty r\phi^2(r)dr$ must be finite too.*

Proof. Using Lemma 5 we only need to show that $\int_0^\varepsilon r\phi^2 dr$ and $\int_0^\varepsilon r\phi_r^2(r)dr$ are finite for some $\varepsilon > 0$. That fact follows immediately from well-posedness of the problem. $\square$

Based on Corollary 1 we may introduce a new "energy" quantity

$$E^* = \int_0^\infty r\phi^2(r) + r\phi_r^2(r)dr\,. \tag{38}$$

For $\psi(r)$ a solution to (9)–(10) the energy E^* has then the same character as the energy E, i.e. both quantities are both finite or both infinite (the term $\int_1^\infty r\phi^2(r)dr$ clearly dominates the term $\int_1^\infty \frac{\phi^2(r)}{r}dr$). Therefore it is enough to show that E^* is infinite for an (oscillating) solution $\phi(r)$ and in the rest of the work when referring to energy we will always refer to E^*.

Let us introduce new "polar" coordinates in the phase plane of $\phi(r)$:

$$\rho^2(r) = \phi^2(r) + \phi_r^2(r) \qquad \text{and} \qquad \theta(r) = \tan^{-1}\frac{\phi_r(r)}{\phi(r)} \quad \text{for } \phi(r) \neq 0. \tag{39}$$

For $\phi(r) = 0$ we define

$$\theta(r) = \frac{\pi}{2}\operatorname{sgn}|\phi_r(r)|\,.$$

Furthermore, let us denote $e(r) = r\rho^2$ the energy density

$$E^* = \int_0^\infty r\rho^2 dr = \int_0^\infty e(r)dr\,.$$

Lemma 6. *Let $\phi(r)$ be a non-trivial solution to (34) and let (a_i), (b_i) and (c_i) and i_0 be defined as above. Let (a,b) be either an interval (a_{2i-1}, b_i) or (a_{2i}, c_i) for $i \geq i_2$ for some $i_2 \geq i_0$. Then*

$$\int_a^b e(r)dr \geq a^2\phi_r^2(a)\ln\left(\frac{b}{a}\right)\,. \tag{40}$$

Proof. First note that by the choice of the interval (a,b) we have $\phi\phi_r > 0$ on (a,b). Then by (39)

$$\frac{\partial}{\partial r}e(r) = \frac{\partial}{\partial r}r\rho^2 = \phi^2 + \phi_r^2 + 2r\phi_r(\phi + \phi_{rr})\,.$$

Using the equation (34) we obtain

$$\frac{\partial}{\partial r}e(r) \;=\; \phi^2 - \phi_r^2 + 2r\phi_r\left(\phi - \sin\phi + \frac{n^2}{r^2}\sin\phi\cos\phi\right)\,. \tag{41}$$

Since $\phi\phi_r > 0$ on (a,b), we also have

$$\phi_r(\phi - \sin\phi) > 0\,.$$

The choice of i_0 in Lemma 4 ensures that $|\phi(r)| < \pi/2$ for all $r > a_i$. Then on (a,b)

$$\phi_r\sin\phi\cos\phi > 0\,.$$

The last inequality combined with (41) yields

$$\frac{\partial}{\partial r}e(r) > -\phi^2 - \phi_r^2 = -\rho^2 = -\frac{e(r)}{r}\,.$$

Integrating the last inequality on (a, R), $R \le b$ we get

$$e(R) \ge e(a)\frac{a}{R}\,.$$

Finally, an integration over $R \in (a, b)$ proves the statement of the lemma:

$$\int_a^b e(r)dr \ge ae(a)\ln\left(\frac{b}{a}\right)\,.$$

□

In Lemma 6 we estimated the energy E^* on subintervals (a_{2i-1}, b_i) and (c_i, a_{2i+1}) where the product $\phi\phi_r$ is positive. To get an estimate in terms of a_i only one needs to prove a uniformity of the "'angular velocity" θ in the phase plane (Lemma 7).

Lemma 7. *Let $\phi(r)$ be a non-trivial solution to (34) and let θ and ρ be defined by (39). Then for every $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that for all $r > R(\varepsilon)$, ($\phi(r) \neq 0$) it holds*

$$-1-\varepsilon < \theta_r(r) < -1+\varepsilon\,.$$

Proof. Since a short calculation gives

$$\theta_r = \frac{1}{1+\frac{\phi_r^2}{\phi^2}}\cdot\frac{\phi_{rr}\phi - \phi_r^2}{\phi^2} = \frac{\phi_{rr}\phi - \phi_r^2}{\rho^2} = -1 + \frac{\phi_{rr}\phi + \phi^2}{\rho^2}\,,$$

we will prove

$$|S(\phi)| = \left|\frac{\phi(\phi+\phi_{rr})}{\rho^2}\right| < \varepsilon\,, \tag{42}$$

for $r > R$ for some $R > 0$ which will immediately prove the lemma. By (34) the quantity $S(\phi)$ introduced in (42) is equivalent to

$$\begin{aligned} S(\phi) &= \frac{\phi\left(-\frac{1}{r}\phi_r + \phi - \sin\phi + \frac{n^2}{r^2}\sin\phi\cos\phi\right)}{\rho^2} \\ &= -\frac{1}{r}\frac{\phi\phi_r}{\phi^2+\phi_r^2} + \frac{n^2}{r^2}\frac{\phi\sin\phi}{\phi^2+\phi_r^2} + \frac{\phi^2 - \phi\sin\phi}{\phi^2+\phi_r^2}\,, \end{aligned}$$

which may be estimated by

$$\begin{aligned} |S(\phi)| &\le \frac{1}{2r} + \frac{n^2}{r^2}\frac{|\phi||\sin\phi|}{\phi^2+\phi_r^2} + \frac{|\phi||\phi-\sin\phi|}{\phi^2+\phi_r^2} \\ &\le \frac{1}{2r} + \frac{n^2}{r^2}\frac{\phi^2}{\phi^2+\phi_r^2} + \frac{\phi^2}{\phi^2+\phi_r^2}\frac{\phi^2}{6} \\ &\le \frac{1}{2r} + \frac{n^2}{r^2} + \frac{\phi^2}{6}\,. \end{aligned}$$

By Lemma 4 there exists $R = R(\varepsilon) > 0$ such that

$$\frac{1}{2r} + \frac{n^2}{r^2} + \frac{\phi^2(r)}{6} < \varepsilon\,,$$

for all $r > R$. This proves (42) and hence the whole statement of Lemma 7. □

Corollary 2. *Let $\phi(r)$ be a non-trivial solution to (34) and let (a_i), (b_i) and (c_i) be defined as above. Then each b_i is approximately in the middle of the interval (a_{2i-1}, a_{2i}), i.e. there exists $1 > K > \frac{1}{2} > k > 0$ such that for every i large enough*

$$K > \frac{b_i - a_{2i-1}}{a_{2i} - a_{2i-1}} > k > 0\,.$$

(Similar statement holds for c_i inside the interval (a_{2i}, a_{2i+1})).

Proof. By the definition of $\theta(r)$, (a_i) and (b_i)

$$\int_{a_{2i-1}}^{b_i} \theta_r dr = \theta(b_i) - \theta(a_{2i-1}) = -\frac{\pi}{2}\,.$$

Then by Lemma 7 for any $\varepsilon > 0$ for all i large enough

$$(1-\varepsilon)(b_i - a_{2i-1}) < \frac{\pi}{2} < (1+\varepsilon)(b_i - a_{2i-1})\,. \tag{43}$$

Similarly

$$(1-\varepsilon)(a_{2i} - b_i) < \frac{\pi}{2} < (1+\varepsilon)(a_{2i} - b_i)\,. \tag{44}$$

Combining (43) with (44) we obtain

$$\frac{1}{2}\frac{1-\varepsilon}{1+\varepsilon} \le \frac{b_i - a_{2i-1}}{a_{2i} - a_{2i-1}} \le \frac{1}{2}\frac{1+\varepsilon}{1-\varepsilon}\,.$$

The statement of the corollary immediately follows. □

Now we can combine Corollary 2 with the estimate (40) of Lemma 6 to get the following corollary.

Corollary 3. *Let $\phi(r)$ be a non-trivial solution to (34), let (a_i), (b_i), (c_i) be defined as above, let k be defined as in Corollary 2 and let i_0 be defined as in Lemma 4. Then*

$$\int_{a_i}^{a_{i+1}} e(r)dr \ge k \ln\left(\frac{a_{i+1}}{a_i}\right) a_i^2 \phi_r^2(a_i)\,,$$

for all $i \ge i_2$ for some $i_2 \ge i_0$.

Proof. For simplicity, let the interval (a_i, a_{i+1}) contains b_j, i.e. $a_i < b_j < a_{i+1}$ (clearly $i = 2j-1$), in the other case ($a_i < c_j < a_{i+1}$) the proof is analogical. Then by (40)

$$\int_{a_i}^{b_j} e(r)dr \ge a_i^2 \phi_r^2(a_i) \ln\left(\frac{b_j}{a_i}\right)\,.$$

Since $\frac{b_j - a_i}{a_{i+1} - a_i} \ge k$ by Corollary 2 for i large enough ($i \ge i_2$) we have

$$\ln\frac{b_j}{a_i} \ge k \ln\frac{a_{i+1}}{a_i}$$

by a simple calculus inequality. Thus

$$\begin{aligned}\int_{a_i}^{a_{i+1}} e(r)dr &\ge \int_{a_i}^{b_j} e(r)dr \ge a_i^2 \phi_r^2(a_i) \ln\left(\frac{b_j}{a_i}\right) \\ &\ge k a_i^2 \phi_r^2(a_i) \ln\left(\frac{a_{i+1}}{a_i}\right)\,.\end{aligned}$$

□

Next we prove that the quantity $a_i^2\phi_r^2(a_i)$ is increasing with an increasing index i.

Proposition 2. *Let $\phi(r)$ be a non-trivial solution to (34) and let (a_i) be defined as above. Then*

$$a_i^2\phi_r^2(a_i) < a_{i+1}^2\phi_r^2(a_{i+1})\,.$$

Proof. By Pohozaev identity applied to (34) on the interval (a_i, a_{i+1}) we obtain

$$\frac{1}{2}\left(a_{i+1}^2\phi_r^2(a_{i+1}) - a_i^2\phi_r^2(a_i)\right) = -2\int_{a_i}^{a_{i+1}} s(\cos\phi(s)-1)ds > 0\,,$$

which proves the statement of Proposition 2. □

Finally we can prove Theorem 1.

Proof. We prove the theorem by a contradiction. Let us assume that the solution $\phi(r)$ has finite energy. Then also the modified solution $\tilde{\phi}(r) = \phi(r) - \pi$ has finite energy and solves (34). Let us drop the tildes and consider $\phi(r)$ to be a solution of (34). By Lemma 1 it has also finite modified energy E^* defined by (38). Let (a_i) be defined as above – the infinite sequence of the zero points of $\phi(r)$. Then by Corollary 3

$$\int_{a_i}^{a_{i+1}} e(r)dr \geq k\ln\left(\frac{a_{i+1}}{a_i}\right) a_i^2\phi_r^2(a_i)\,,$$

for $i \geq i_2$. Thus

$$E^* \geq \sum_{j=i_2}^{\infty}\int_{a_j}^{a_{j+1}} e(r)dr \geq \sum_{j=i_2}^{\infty} k\ln\left(\frac{a_{j+1}}{a_j}\right) a_j^2\phi_r^2(a_j)\,.$$

Also by Proposition 2

$$a_j^2\phi_r^2(a_j) \geq a_1^2\phi_r^2(a_1) = C > 0\,,$$

so

$$E^* \geq Ck\sum_{j=i_2}^{\infty}\ln\left(\frac{a_{j+1}}{a_j}\right) = Ck\lim_{j\to\infty}\left(\ln(a_j) - \ln(a_{i_2})\right) = \infty\,,$$

which yields the contradiction. □

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