Math. Z. 240, 37–51 (2002)
Digital Object Identifier (DOI) 10.1007/s002090100359

Mathematische Zeitschrift

Moment inequalities and central limit properties of isotropic convex bodies

Ulrich Brehm, Peter Hinow, Hendrik Vogt, Jürgen Voigt*

Fachrichtung Mathematik, Technische Universität Dresden, 01062 Dresden, Germany (e-mail: {brehm;vogt;voigt}@math.tu-dresden.de)

Received: 21 March 2001 / in final form: 17 October 2001 /
Published online: 4 April 2002 –

Abstract. The object of our investigations are isotropic convex bodies $K \subseteq \mathbb{R}^n$, centred at the origin and normed to volume one, in arbitrary dimensions. We show that a certain subset of these bodies – specified by bounds on the second and fourth moments – is invariant under forming 'expanded joins'. Considering a body K as above as a probability space and taking $u \in S^{n-1}$, we define random variables $X_{K,u} = x \cdot u$ on K. It is known that for subclasses of isotropic convex bodies satisfying a 'concentration of mass property', the distributions of these random variables are close to Gaussian distributions, for high dimensions n and 'most' directions $u \in S^{n-1}$. We show that this 'central limit property', which is known to hold with respect to convergence in law, is also true with respect to L_1-convergence and L_∞-convergence of the corresponding densities.

Mathematics Subject Classification (2000): 52A20, 60F25, 26B25

1 Introduction

Let $K \subseteq \mathbb{R}^n$ be a convex body, i.e. a convex compact set with nonempty interior. We say that K is *isotropic* if its ellipsoid of inertia is a Euclidean ball, i.e.,

$$L_K^2 := \int_K (x \cdot u)^2 \, dx$$

is independent of the unit vector $u \in S^{n-1}$. The number L_K is called the *radius of inertia* of the body K. We say that K is *normed* if its centroid is

* Partly supported by the DFG

at the origin and its n-dimensional volume $\lambda_n(K)$ equals 1. Note that every convex body with nonempty interior has an affine image that is normed and isotropic.

For more information on isotropic convex bodies, related notions, and further properties and applications we refer to [2], [7], [8].

The set of all normed isotropic convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}_n$, and we define

$$\mathcal{K} := \bigcup_{n=1}^{\infty} \mathcal{K}_n.$$

We regard a body $K \in \mathcal{K}_n$, with the measure λ_n restricted to K, as a probability space. For each unit vector $u \in S^{n-1}$ we define a random variable $X_{K,u}\colon K \to \mathbb{R}$ by

$$X_{K,u}(x) := x \cdot u.$$

The density of the distribution of this random variable is given by

$$\varphi_{K,u}(t) = \lambda_{n-1}\big(\{x \in K;\, x \cdot u = t\}\big) \quad (t \in \mathbb{R}).$$

In [4] it was shown that for Euclidean balls and for cubes these densities are close to Gaussian densities for large dimensions and most directions u, in the sense made precise below. Further, it was shown in [1] that an analogous property holds for the distribution functions of the random variables $X_{K,u}$ for classes of symmetric convex bodies satisfying the 'concentration of mass property' described in Theorem 4.1(i). Also, in [10] it was observed that this concentration of mass property is a consequence of inequality (1.3) below, and the application of [11] was shown to yield a 'central limit property'.

In order to make this more precise, we recall the following definition from [4]. Let $\mathcal{M} \subseteq \mathcal{K}$ be a set of normed isotropic convex bodies. We say that $\mathcal{M}$ has the *central limit property* if

$$\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\varphi_{K,u} - g_{L_K^2}\|_1 > \varepsilon\}\big) \to 0 \quad \text{as } n \to \infty, \text{ for all } \varepsilon > 0. \tag{1.1}$$

Here,

$$g_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

denotes the Gaussian density, and μ_{n-1} the surface measure on S^{n-1}, normed to a probability measure. (In fact, the central limit property was expressed in [4] slightly differently – but equivalently – by requiring

$$\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mathbf{E}\, \|\varphi_{K,u} - g_{L_K^2}\|_1 \to 0 \quad \text{as } n \to \infty,$$

where $\mathbf{E}$ denotes expectation with respect to the probability measure μ_{n-1}.)

On the one hand, for the examples in [4] the central limit property (1.1), which involves the L_1-distance of densities, was shown. On the other hand, in the more general results [11], [1] only weaker forms of closeness were obtained. This observation was one of the starting points of this paper. As our main result we show that the central limit property (1.1) can be derived from a seemingly weaker form of this property. This is mainly due to the logarithmic concavity of the densities $\varphi_{K,u}$.

We use the following notation. For $K \in \mathcal{K}_n$ let

$$m_2(K) := \int_K |x|^2\,dx \quad (= nL_K^2),$$
$$m_4(K) := \int_K |x|^4\,dx$$

denote the second and fourth moment of the body K, respectively. Let $\mathcal{T}_n \subseteq \mathcal{K}_n$ be the set of all normed isotropic bodies K that satisfy

$$m_2(K) \leqslant m_2(\Delta_n)\,, \tag{1.2}$$

$$\frac{m_4(K)}{m_2(K)^2} \leqslant \frac{m_4(\Delta_n)}{m_2(\Delta_n)^2}\,, \tag{1.3}$$

where Δ_n denotes the normed regular n-simplex in $\mathbb{R}^n$. Let

$$\mathcal{T} := \bigcup_{n=1}^{\infty} \mathcal{T}_n. \tag{1.4}$$

Inequalities (1.2), (1.3) appear in [10] and [3], with the right hand sides written as explicit expressions depending on n. The set $\mathcal{T}$ is invariant under certain operations such as forming cones, cartesian products, joins (cf. [10]), and p-products (cf. [3]). As a consequence, $\mathcal{T}$ contains all ℓ_p^n-balls (normed to volume 1). Particular p-products are the join ($p = 1$) and the cartesian product ($p = \infty$).

It has been shown in [9] that, for $n = 2$, inequality (1.2) holds for all $K \in \mathcal{K}_2$. It appears that one does not know a body in $\mathcal{K}$ not satisfying inequalities (1.2) and (1.3).

In Sect. 2 we introduce the expanded join of two bodies in $\mathcal{K}$ and show that $\mathcal{T}$ is invariant under forming normed isotropic expanded joins. Sections 3 and 4 are independent of Sect. 2.

Section 3 is devoted to showing properties of logarithmically concave functions. The main result, Theorem 3.3, states that closeness of logarithmically concave densities to Gaussian densities can be expressed in several equivalent ways: in L_∞-norm or L_1-norm of the difference of the densities, or in L_∞-norm of the difference of the corresponding distribution functions.

In Sect. 4 we show that a set $\mathcal{M} \subseteq \mathcal{K}$ satisfying a concentration of mass property (see property (i) of Theorem 4.1) will also satisfy the central limit property defined in (1.1). Our main contribution is showing that in the case treated here a result of von Weizsäcker [11] can be reinforced using the results of Sect. 3. The general result for $\mathcal{M}$ is then applied to the class $\mathcal{T}$ defined above.

2 The expanded join

Continuing the investigations in [10], [3], we introduce yet another operation on convex bodies, the expanded join.

Let $n_0, n_1 \in \mathbb{N}_0$, $K_j \subseteq \mathbb{R}^{n_j}$ $(j = 0, 1)$, and $n = n_0 + n_1 + 1$. We define the (unnormed) *expanded join* of K_0 and K_1 by

$$\begin{aligned} K_0 \hat{\bullet} K_1 &= \operatorname{conv}\big((K_0 \times \{0\} \times \{0\}) \cup (\{0\} \times K_1 \times \{1\})\big) \\ &= \{((1-\lambda)x', \lambda x'', \lambda);\, x' \in K_0,\, x'' \in K_1,\, \lambda \in [0,1]\} \\ &= \bigcup_{\lambda \in [0,1]} \tilde{K}_\lambda \subseteq \mathbb{R}^n, \end{aligned}$$

where $\tilde{K}_\lambda = (1-\lambda)K_0 \times (\lambda K_1) \times \{\lambda\}$.

The set of extreme points is

$$\operatorname{ext}(K_0 \hat{\bullet} K_1) = \big(\operatorname{ext}(K_0) \times \{0\} \times \{0\}\big) \cup \big(\{0\} \times \operatorname{ext}(K_1) \times \{1\}\big),$$

thus in particular the expanded join of two simplices is a simplex (merely count the vertices). In the special case $n_1 = 0$, $K_1 = \Delta_0 = \mathbb{R}^0$, the expanded join $K_0 \hat{\bullet} K_1$ is a cone with base $K_0 \times \{0\}$. Finally note that the orthogonal projection $P_n \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ (omit the last component) maps $K_0 \hat{\bullet} K_1$ onto the join of K_0 and K_1 as treated in [10]. This motivates the notion 'expanded join'.

In the following we only consider $n_0, n_1 \geqslant 1$. Our first aim is to find an affine image of $K_0 \hat{\bullet} K_1$ that is normed and isotropic. For the volume of $K_0 \hat{\bullet} K_1$ we have

$$\begin{aligned} \lambda_n(K_0 \hat{\bullet} K_1) &= \int_0^1 \lambda_{n-1}(\tilde{K}_\lambda)\, d\lambda = \int_0^1 (1-\lambda)^{n_0} \lambda^{n_1}\, d\lambda \\ &= \mathrm{B}(n_0+1, n_1+1) = \frac{\Gamma(n_0+1)\Gamma(n_1+1)}{\Gamma(n_0+n_1+2)} = \frac{n_0! n_1!}{n!}\,. \end{aligned}$$

The centroid of $K_0 \hat{\bullet} K_1$ is at $r := (0, \ldots, 0, \lambda_c)$ where

$$\begin{aligned} \lambda_c &= \frac{1}{\lambda_n(K_0 \hat{\bullet} K_1)} \int_0^1 \int_{\tilde{K}_\lambda} \lambda\, d(x', x'')\, d\lambda = \frac{n!}{n_0! n_1!} \int_0^1 (1-\lambda)^{n_0} \lambda^{n_1+1}\, d\lambda \\ &= \frac{n!}{n_0! n_1!} \frac{n_0! (n_1+1)!}{(n+1)!} = \frac{n_1+1}{n+1}\,. \end{aligned}$$

Since K_0 and K_1 are isotropic, the function

$$u \mapsto \int_{K_0 \hat{\bullet} K_1 - r} (x \cdot u)^2 \, dx$$

is constant on $S^{n_0-1} \times \{0\} \times \{0\}$ and $\{0\} \times S^{n_1-1} \times \{0\}$. Thus we obtain an isotropic image of $K_0 \hat{\bullet} K_1 - r$ by applying a linear mapping T of the form

$$T = \operatorname{diag}(\underbrace{\alpha_0, \ldots, \alpha_0}_{n_0-\text{times}}, \underbrace{\alpha_1, \ldots, \alpha_1}_{n_1-\text{times}}, \alpha_2).$$

This means that we have to determine $\alpha_0, \alpha_1, \alpha_2 > 0$ such that

$$\begin{aligned} \frac{1}{n_0} \int_{T(K_0 \hat{\bullet} K_1 - r)} |x'|^2 \, d(x', x'', \lambda) &= \frac{1}{n_1} \int_{T(K_0 \hat{\bullet} K_1 - r)} |x''|^2 \, d(x', x'', \lambda) \\ &= \int_{T(K_0 \hat{\bullet} K_1 - r)} \lambda^2 \, d(x', x'', \lambda). \end{aligned} \tag{2.1}$$

As the second moment in the x'-direction we obtain

$$\begin{aligned} \frac{1}{n_0} \int_{K_0 \hat{\bullet} K_1 - r} |x'|^2 \, d(x', x'', \lambda) &= \frac{1}{n_0} \int_0^1 \int_{\tilde{K}_\lambda} |x'|^2 \, d(x', x'') \, d\lambda \\ = \frac{1}{n_0} \int_0^1 \int_{K_0} |x'|^2 \, dx' (1-\lambda)^{n_0+2} \lambda^{n_1} \, d\lambda &= a_0 m_2(K_0), \end{aligned}$$

where $a_0 = \frac{(n_0+2)! n_1!}{n_0 (n+2)!}$. Analogously we have

$$\frac{1}{n_1} \int_{K_0 \hat{\bullet} K_1 - r} |x''|^2 \, d(x', x'', \lambda) = a_1 m_2(K_1)$$

and

$$\int_{K_0 \hat{\bullet} K_1 - r} \lambda^2 \, d(x', x'', \lambda) = \int_0^1 \int_{\tilde{K}_\lambda} d(x', x'') (\lambda - \lambda_c)^2 \, d\lambda = a_2,$$

where $a_1, a_2 > 0$ depend only on n_0, n_1. From (2.1) we now obtain the condition

$$\alpha_0^{n_0+2} \alpha_1^{n_1} \alpha_2 a_0 m_2(K_0) = \alpha_0^{n_0} \alpha_1^{n_1+2} \alpha_2 a_1 m_2(K_1) = \alpha_0^{n_0} \alpha_1^{n_1} \alpha_2^3 a_2$$

for α_0, α_1, and α_2, which simplifies to

$$\alpha_0^2 a_0 m_2(K_0) = \alpha_1^2 a_1 m_2(K_1) = \alpha_2^2 a_2. \tag{2.2}$$

From the requirement $\lambda_n(T(K_0\hat{\bullet}K_1 - r)) = 1$ we obtain the condition

$$\alpha_0^{n_0}\alpha_1^{n_1}\alpha_2 = \frac{n!}{n_0!n_1!}\,. \tag{2.3}$$

Solving (2.2) for α_0 and α_1 and inserting this into (2.3) yields

$$\alpha_2 = \left(\frac{n!}{n_0!n_1!}\left(\frac{a_0}{a_2}m_2(K_0)\right)^{\frac{n_0}{2}}\left(\frac{a_1}{a_2}m_2(K_1)\right)^{\frac{n_1}{2}}\right)^{\frac{1}{n}}. \tag{2.4}$$

For our further considerations we need not compute α_0, α_1 and α_2 more explicitly. We define the *normed isotropic expanded join* by

$$\begin{aligned} &K_0 \bullet K_1 \\ &= \operatorname{conv}\big((\alpha_0 K_0 \times \{0\} \times \{-\alpha_2\lambda_c\}) \cup (\{0\} \times \alpha_1 K_1 \times \{\alpha_2(1-\lambda_c)\})\big) \\ &= \{(\alpha_0(1-\lambda)x', \alpha_1\lambda x'', \alpha_2(\lambda-\lambda_c));\, x' \in K_0,\, x'' \in K_1,\, \lambda \in [0,1]\}. \end{aligned}$$

Theorem 2.1. *The set $\mathcal{T}$ defined in (1.4) is invariant under forming normed isotropic expanded joins.*

Proof. We have to show the persistence of inequalities (1.2) and (1.3). We may assume $n_0, n_1 \geqslant 1$ since the case of cones has been treated in [10, Sect. 2]. For the second moment of $K_0 \bullet K_1$ we obtain, using (2.1), (2.3), and (2.4),

$$\begin{aligned} m_2(K_0 \bullet K_1) &= n\int_{K_0\bullet K_1} \lambda^2\, d(x', x'', \lambda) = \alpha_0^{n_0}\alpha_1^{n_1}\alpha_2^3\, n a_2 \\ &= \frac{n!}{n_0!n_1!}\left(\frac{n!}{n_0!n_1!}\left(\frac{a_0}{a_2}m_2(K_0)\right)^{\frac{n_0}{2}}\left(\frac{a_1}{a_2}m_2(K_1)\right)^{\frac{n_1}{2}}\right)^{\frac{2}{n}} n a_2. \end{aligned}$$

Thus the validity of (1.2) for K_0 and K_1 implies

$$m_2(K_0 \bullet K_1) \leqslant m_2(\varDelta_{n_0} \bullet \varDelta_{n_1}) = m_2(\varDelta_n),$$

which is (1.2) for $K_0 \bullet K_1$.

The fourth moment is computed as

$$\begin{aligned} \int_{K_0\bullet K_1} |x|^4\, dx &= \int_{K_0\bullet K_1} (|x'|^2 + |x''|^2 + \lambda^2)^2\, dx \\ &= \alpha_0^{n_0+4}\alpha_1^{n_1}\alpha_2 b_0\, m_4(K_0) + \alpha_0^{n_0}\alpha_1^{n_1+4}\alpha_2 b_1\, m_4(K_1) \\ &\quad + \alpha_0^{n_0}\alpha_1^{n_1}\alpha_2^5 b_2 + \alpha_0^{n_0+2}\alpha_1^{n_1+2}\alpha_2 b_{01} m_2(K_0) m_2(K_1) \\ &\quad + \alpha_0^{n_0+2}\alpha_1^{n_1}\alpha_2^3 b_{02} m_2(K_0) + \alpha_0^{n_0}\alpha_1^{n_1+2}\alpha_2^3 b_{12} m_2(K_1), \end{aligned}$$

where again $b_0, b_1, b_2, b_{01}, b_{02}, b_{12} > 0$ depend only on n_0, n_1. Using (2.3) as well as

$$\begin{aligned}\frac{1}{n} m_2(K_0 \bullet K_1) &= \alpha_0^{n_0+2} \alpha_1^{n_1} \alpha_2 a_0\, m_2(K_0) \\ &= \alpha_0^{n_0} \alpha_1^{n_1+2} \alpha_2 a_1\, m_2(K_1) \\ &= \alpha_0^{n_0} \alpha_1^{n_1} \alpha_2^3 a_2\end{aligned}$$

appropriately, we obtain

$$\frac{m_4(K_0 \bullet K_1)}{m_2(K_0 \bullet K_1)^2} = c_0 \frac{m_4(K_0)}{m_2(K_0)^2} + c_1 \frac{m_4(K_1)}{m_2(K_1)^2} + c_2,$$

with suitable $c_0, c_1, c_2 > 0$ depending only on n_0, n_1. Similarly as above for the second moment, the validity of (1.3) for K_1, K_2 implies (1.3) for $K_0 \bullet K_1$. $\square$

3 Inequalities for logarithmically concave functions

The assertion of Lemma 3.2(b) below will be needed in Sect. 4. Here we prove rather general explicit inequalities for logarithmically concave functions, which should be of independent interest. In Sect. 4 we would only need a qualitative estimate for a specific case.

We recall that a function $f\colon I \to [0, \infty)$ is called *logarithmically concave* if $\ln f$ is concave on an interval $I \subseteq \mathbb{R}$ (with the usual convention $\ln 0 = -\infty$). For $a \in \mathbb{R}$ we denote $a^+ := \max\{a, 0\}$.

Lemma 3.1. *Let $0 \leqslant f \in L_1([0, \infty))$ be logarithmically concave, $f(0) > 0$, $\|f\|_1 > 0$, $a := \frac{\|f\|_1}{f(0)}$. Let $g\colon [0, \infty) \to \mathbb{R}$ be increasing. Assume that*

$$t_1 := \sup\{t \geqslant 0;\, f \geqslant g \text{ on } [0, t]\} < \infty.$$

Then
$$\int_0^{t_1} (f - g)(t)\, dt \geqslant \int_0^{\infty} \big(f(0) e^{-t/a} - g(t)\big)^+ dt.$$

Proof. Let $\tilde{f}(t) := f(0) e^{-t/a}$ ($t \geqslant 0$) and

$$t_2 := \sup\{t \geqslant 0;\, \tilde{f} \geqslant g \text{ on } [0, t]\} \ \big(\in [0, \infty]\big).$$

Then $\tilde{f} \geqslant g$ on $[0, t_2)$, and $\tilde{f} \leqslant g$ on (t_2, ∞) since $\tilde{f} - g$ is decreasing. We have to show $\int_0^{t_2} (\tilde{f} - g)(t)\, dt \leqslant \int_0^{t_1} (f - g)(t)\, dt$.

In case $f = \tilde{f}$ there is nothing to show, so we assume $f \neq \tilde{f}$. Note that $\int \tilde{f} = \int f$ by the definition of a. Thus, $f - \tilde{f}$ changes sign at least once. Since $\ln f - \ln \tilde{f}$ is concave and $f(0) = \tilde{f}(0)$, there exists $t_0 > 0$ such that

$f \geqslant \tilde{f}$ on $[0, t_0)$ and $f \leqslant \tilde{f}$ on (t_0, ∞). This implies $\int_0^s \tilde{f}(t)\,dt \leqslant \int_0^s f(t)\,dt$ for all $s \geqslant 0$ and hence

$$\int_0^{t_2} (\tilde{f} - g)(t)\,dt \leqslant \int_0^{t_2} (f - g)(t)\,dt.$$

Assume first that $t_2 \leqslant t_0$. Then $t_2 \leqslant t_1 \leqslant t_0$, so we obtain the assertion since $f - g \geqslant 0$ on (t_2, t_1). Now assume $t_2 > t_0$. Then $t_0 \leqslant t_1 \leqslant t_2$. We have $f(\leqslant \tilde{f}) \leqslant f(0)$ on (t_0, ∞), so f and thus $f - g$ is decreasing on (t_0, ∞). By the definition of t_1, this implies $f - g \leqslant 0$ on (t_1, t_2), and we obtain the assertion. □

Lemma 3.2. *Let* $0 \leqslant f \in L_1(\mathbb{R})$ *be logarithmically concave,* $\|f\|_1 \geqslant 1$, $g \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$*, and* F, G *antiderivatives of* f, g*, respectively.*

(a) Let $\alpha > 1$*. If* $\|F - G\|_\infty \leqslant \frac{1}{2} - \frac{1+\ln\alpha}{2\alpha}$ *then* $\|f\|_\infty \leqslant \alpha \operatorname{ess\,sup} g$*.*

(b) If g *is Lipschitz continuous with Lipschitz constant* L *then*

$$\|f - g\|_\infty \leqslant 2\sqrt{L + \|f\|_\infty^2}\,\|F - G\|_\infty^{\frac{1}{2}}.$$

(c) Let $\alpha > 1$*. If* $\|F - G\|_\infty \leqslant \frac{1}{2} - \frac{1+\ln\alpha}{2\alpha}$, $\|g\|_1 \leqslant 1$*, and* g *is Lipschitz continuous with Lipschitz constant* L *then*

$$\|f - g\|_\infty \leqslant 2\sqrt{L}\sqrt{1 + \alpha^2}\,\|F - G\|_\infty^{\frac{1}{2}}.$$

Proof. We will use the inequality $\left|\int_a^b (f(t) - g(t))\,dt\right| \leqslant 2\|F - G\|_\infty$, which is valid for all $a, b \in \mathbb{R}$.

(a) Observe that $M := \operatorname{ess\,sup} g \geqslant 0$. Without restriction assume that $f(0) = \|f\|_\infty > M$. Let I be the maximal open interval containing 0 with $f \geqslant g$ on I. Then

$$2\|F - G\|_\infty \geqslant \int_I (f - g)(t)\,dt \geqslant \int_{-\infty}^{\infty} (f(t) - M)^+ dt.$$

By Lemma 3.1, with $g = M$, we have

$$\int_0^{\infty} (f(t) - M)^+ dt \geqslant \int_0^{\infty} (f(0)e^{-t/a} - M)^+ dt,$$

where $a := \int_0^\infty f(t)\,dt / f(0)$. The above inequality makes sense in the case $a = 0$, too; then both sides are 0. In the same way we obtain

$$\int_{-\infty}^{0} (f(t) - M)^+ dt \geqslant \int_{-\infty}^{0} (f(0)e^{t/b} - M)^+ dt,$$

with $b := \int_{-\infty}^0 f(t)\,dt / f(0)$.

Let $t_0 := a \ln \frac{f(0)}{M}$ and $s_0 := -b \ln \frac{f(0)}{M}$. Then $f(0)e^{-t_0/a} = f(0)e^{s_0/b} = M$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} (f(t)-M)^+ dt &\geqslant \int_{s_0}^{0} (f(0)e^{t/b} - M)\, dt + \int_0^{t_0} (f(0)e^{-t/a} - M)\, dt \\ &= b(f(0)-M) - a(M - f(0)) - (t_0 - s_0)M \\ &= (a+b)\big(f(0) - M - M \ln \tfrac{f(0)}{M}\big).\end{aligned}$$

Since $a + b = \|f\|_1/f(0) \geqslant \frac{1}{f(0)}$ we arrive at

$$2\|F-G\|_\infty \geqslant \int_{-\infty}^{\infty} (f(t)-M)^+ dt \geqslant 1 - \frac{M}{f(0)}\left(1 + \ln \frac{f(0)}{M}\right).$$

Using $2\|F-G\|_\infty \leqslant 1 - \frac{1}{\alpha}(1+\ln\alpha)$, we deduce that $\|f\|_\infty = f(0) \leqslant \alpha M$.

(b) We have to estimate $|f(x) - g(x)|$ for all $x \in \mathbb{R}$. Since the assumptions are translation invariant it suffices to consider $x = 0$. First assume $f(0) \leqslant g(0)$. We suppose that f is decreasing on $[0,\infty)$. (If this is not the case then f is increasing on $(-\infty, 0]$; the latter case is treated analogously.) Let $\eta := \frac{g(0)-f(0)}{L}$. By the Lipschitz continuity of g we have

$$g(t) - f(t) \geqslant g(0) - Lt - f(0) = L(\eta - t) \quad (t \geqslant 0),$$

and thus

$$2\|F-G\|_\infty \geqslant \int_0^\eta (g(t) - f(t))\, dt \geqslant \int_0^\eta L(\eta - t)\, dt = \frac{(g(0)-f(0))^2}{2L},$$

or

$$g(0) - f(0) \leqslant 2\sqrt{L\|F-G\|_\infty}.$$

Now assume $h := f(0) - g(0) > 0$. We have to show $\|F-G\|_\infty \geqslant \frac{h^2}{4(L+\|f\|_\infty^2)}$.

Let $\tilde{g}(t) := g(0) + L|t|$ $(t \in \mathbb{R})$. Let I $(\tilde{I})$ be the maximal interval containing 0 with $f \geqslant g$ on I $(f \geqslant \tilde{g}$ on $\tilde{I})$. Then, by the Lipschitz continuity of g,

$$2\|F-G\|_\infty \geqslant \int_I (f-g)(t)\, dt \geqslant \int_{\tilde{I}} (f - \tilde{g})(t)\, dt.$$

Let a, b as in the proof of (a), and define $\tilde{f}$ by $\tilde{f}(t) := f(0)e^{-t/a}$ $(t \geqslant 0)$ and $\tilde{f}(t) := f(0)e^{t/b}$ $(t \leqslant 0)$. The first inequality in the following estimate follows from Lemma 3.1, whereas the second inequality is illustrated by Fig. 1 (the area between the graphs of $\tilde{g}$ and $\tilde{f}$ contains the shaded area):

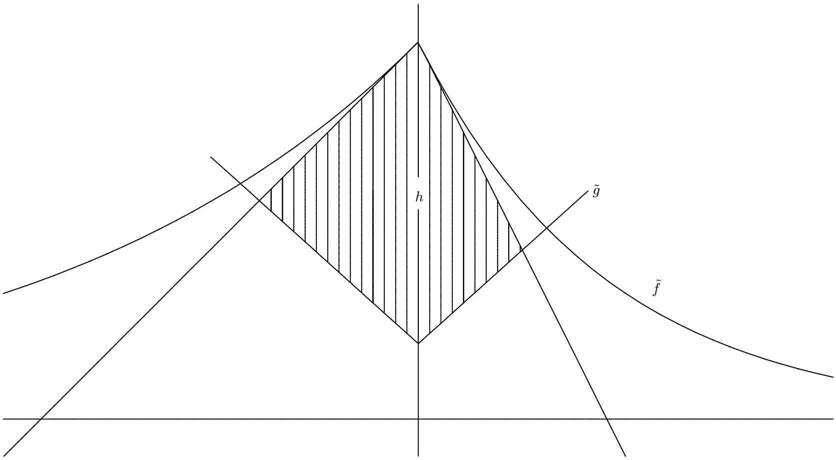

Fig. 1

$$\begin{aligned}\int_{\tilde{I}}(f-\tilde{g})(t)\,dt &\geqslant \int_{-\infty}^{\infty}\big(\tilde{f}(t)-\tilde{g}(t)\big)^{+}dt\\ &\geqslant \int_{0}^{\infty}\big(h-\tfrac{f(0)}{a}t-Lt\big)^{+}dt+\int_{-\infty}^{0}\big(h+\tfrac{f(0)}{b}t+Lt\big)^{+}dt\\ &=\frac{h^2}{2(L+\frac{f(0)}{a})}+\frac{h^2}{2(L+\frac{f(0)}{b})}.\end{aligned}$$

Recall that $c := a+b = \|f\|_1/f(0)$. Thus,

$$2\|F-G\|_\infty \geqslant \min_{0\leqslant a\leqslant c}\frac{h^2}{2}\left(\frac{1}{L+\frac{f(0)}{a}}+\frac{1}{L+\frac{f(0)}{c-a}}\right).$$

The mininum is attained at $a=0$ and at $a=c$, so

$$2\|F-G\|_\infty \geqslant \frac{h^2}{2(L+\frac{f(0)^2}{\|f\|_1})} \geqslant \frac{h^2}{2(L+\|f\|_\infty^2)},$$

which was to be shown.

(c) is a direct consequence of (a) and (b) since $\|g\|_\infty \leqslant \sqrt{L}$. □

We will apply Lemma 3.2 in form of the next theorem, where G_{σ^2} denotes the Gaussian distribution function with variance σ^2,

$$G_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{t}\exp\left(-\frac{s^2}{2\sigma^2}\right)\,ds.$$

Theorem 3.3. *There exists an increasing function* $\beta_1\colon (0,\infty) \to (0,2]$ *with*

$$\beta_1(t) = O\big(t(-\ln t)^{\frac{1}{2}}\big) \qquad \big(0 < t \leqslant \tfrac{1}{\sqrt{2\pi}}\big)$$

with the following property. Let $0 \leqslant f \in L_1(\mathbb{R})$ *be logarithmically concave,* $\|f\|_1 = 1$, *and* F *the distribution function of* f. *Let* $\sigma > 0$ *and assume that* $\|f\|_\infty \leqslant \frac{1}{\sigma}$. *Then*

$$\|F - G_{\sigma^2}\|_\infty \leqslant \|f - g_{\sigma^2}\|_1 \leqslant \beta_1(\sigma\|f - g_{\sigma^2}\|_\infty) \leqslant \beta_1(\sqrt{5}\|F - G_{\sigma^2}\|_\infty^{\frac{1}{2}}),$$

$$\|f - g_{\sigma^2}\|_\infty \leqslant \frac{\sqrt{5}}{\sigma}\|F - G_{\sigma^2}\|_\infty^{\frac{1}{2}}.$$

Proof. The first inequality is clear; the second one is proved in [4, Prop. 2.5]. The third inequality is a consequence of the last one. Note that g_{σ^2} is Lipschitz continuous with constant $\frac{1}{\sqrt{2\pi e \sigma^2}} < \frac{1}{4\sigma^2}$. Thus, the last inequality is a direct consequence of Lemma 3.2(b). □

Remark 3.4. If f is logarithmically concave then the assumption $\|f\|_\infty \leqslant \frac{1}{\sigma}$ in the above theorem is in particular satisfied if $\int_{-\infty}^{\infty} t f(t)\, dt = 0$ and σ^2 is the variance of f. This follows from [6, Thm. 8, eqn. (7)], applied with $\varphi(t) = t^2$.

4 A central limit theorem

The main result of this section (and the whole paper) is the following. As in Sect. 3, G_{σ^2} denotes the Gaussian distribution function with variance σ^2. Also,

$$\Phi_{K,u}(t) = \lambda_n(\{x \in K;\, x \cdot u \leqslant t\})$$

is the distribution function of the density $\varphi_{K,u}$, for $K \in \mathcal{K}_n$, $u \in S^{n-1}$.

Theorem 4.1. *Let* $\mathcal{M} \subseteq \mathcal{K}$ *be a set of normed isotropic convex bodies. Consider the following properties:*

(i) $\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \lambda_n\big(\{x \in K;\, \big||x|^2 - nL_K^2\big| > \varepsilon n L_K^2\}\big) \to 0$ *as* $n \to \infty$, *for all* $\varepsilon > 0$;

(ii) $\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\Phi_{K,u} - G_{L_K^2}\|_\infty > \varepsilon\}\big) \to 0$ *as* $n \to \infty$, *for all* $\varepsilon > 0$;

(iii) $\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\varphi_{K,u} - g_{L_K^2}\|_1 > \varepsilon\}\big) \to 0$ *as* $n \to \infty$, *for all* $\varepsilon > 0$, *i.e.,* $\mathcal{M}$ *has the central limit property (1.1);*

(iv) $\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\varphi_{K,u} - g_{L_K^2}\|_\infty > \varepsilon\}\big) \to 0$ *as* $n \to \infty$, *for all* $\varepsilon > 0$.

Then (i) ⇒ (ii) ⇔ (iii) ⇒ (iv). *If* $\sup_{K \in \mathcal{M}} L_K < \infty$ *then also* (iv) ⇒ (iii).

Theorem 4.2. *The set $\mathcal{T}$ defined in (1.4) satisfies properties* (i) – (iv) *of Theorem 4.1.*

Remark 4.3. In [1, Thm. 4] the implication (i) $\Rightarrow$ (ii) of Theorem 4.1 is shown for classes of centrally symmetric bodies, with quantitative estimates. By use of our Theorem 3.3, this result can be reinforced to a quantitative result concerning L_1-norms and L_∞-norms of the difference of the corresponding densities.

Before proceeding to the proof of Theorem 4.1 we want to recall a result of von Weizsäcker [11] which will be an essential ingredient of our proof.

For $K \in \mathcal{K}_n$ we define probability measures P_K and $\tilde{P}_K$ on the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel sets of $\mathbb{R}^n$ by

$$\begin{aligned} P_K(B) &:= \lambda_n(B \cap K), \\ \tilde{P}_K(B) &:= P_K(L_K B) \quad (B \in \mathcal{B}(\mathbb{R}^n)). \end{aligned}$$

For the marginal densities and the corresponding distribution functions of the measure $\tilde{P}_K$ we obtain

$$\begin{aligned} \tilde{\varphi}_{K,u}(t) &= L_K \varphi_{K,u}(L_K t), \\ \tilde{\Phi}_{K,u}(t) &= \Phi_{K,u}(L_K t) \quad (t \in \mathbb{R},\ u \in S^{n-1}), \end{aligned} \tag{4.1}$$

respectively.

Since K is isotropic the covariance matrix $\mathcal{C}$ of $\tilde{P}_K$ is computed as

$$\mathcal{C} = \left(\int_{\mathbb{R}^n} x_i x_j \, d\tilde{P}_K(x) \right)_{i,j=1,\ldots,n} = L_K^{-2} \left(\int_K x_i x_j \, dx \right)_{i,j=1,\ldots,n} = E_n.$$

Hence the conditions [11, (9) and (10)], i.e.

$$\begin{aligned} \operatorname{trace} \mathcal{C} &= O(n), \\ \sum_{i,j=1}^{n} \mathcal{C}_{ij}^2 &= o(n^2), \end{aligned}$$

are satisfied. Further, property (i) in Theorem 4.1 can be expressed as

$$\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \tilde{P}_K\left(\{x \in \mathbb{R}^N;\ \left|\tfrac{|x|}{\sqrt{n}} - 1\right| \geqslant \varepsilon\}\right) \to 0 \quad \text{as } n \to \infty, \text{ for all } \varepsilon > 0,$$

i.e., the condition [11, (11)] is satisfied.

In order to describe the topology of convergence in law on $\operatorname{Prob}(\mathbb{R})$, the probability measures on $\mathcal{B}(\mathbb{R})$, we use the Lévy metric; cf. [5, Sect. 8.1, Theorem 3]. For $P, Q \in \operatorname{Prob}(\mathbb{R})$, with distribution functions F, G, the Lévy distance $L(P,Q)$ is defined by

$$L(P,Q) := \inf\{\varepsilon > 0;\, F(t-\varepsilon) - \varepsilon \leqslant G(t) \leqslant F(t+\varepsilon) + \varepsilon \text{ for all } t \in \mathbb{R}\}.$$

Expressed with this metric, von Weizsäcker's result implies that, in case property (i) in Theorem 4.1 holds, we have

$$\sup_{K \in \mathcal{K}_n \cap \mathcal{T}} \mu_{n-1}\left(\{u \in S^{n-1};\, L(\tilde{\varphi}_{K,u}\lambda_1, g_1\lambda_1) > \varepsilon\}\right) \to 0 \tag{4.2}$$

as $n \to \infty$, for all $\varepsilon > 0$. (This follows from [11, Corollary 1]. Also, one has to use that, due to the concentration of mass, the mixed Gaussian can be replaced by the Gaussian with variance 1; cf. [11, p. 316, line 10].)

The above is the main part of the proof of the implication (i)⇒(ii) in Theorem 4.1. We further need the following preliminary facts.

Lemma 4.4. *Let $c > 0$, and let $P, Q \in \operatorname{Prob}(\mathbb{R})$, with associated distribution functions F, G, respectively. Assume that F, G are Lipschitz continuous with Lipschitz constant c. Then*

$$\|F - G\|_\infty \leqslant (1 + c)L(P,Q).$$

Proof. It suffices to show $G(t) - F(t) \leqslant (1+c)L(P,Q)$ for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$, $\varepsilon > L(P,Q)$. Then

$$G(t) \leqslant F(t+\varepsilon) + \varepsilon \leqslant F(t) + c\varepsilon + \varepsilon.$$

Letting $\varepsilon \to L(P,Q)$ we obtain $G(t) - F(t) \leqslant (1+c)L(P,Q)$. □

Remark 4.5. For all $K \in \mathcal{K}$ we have

$$L_K \geqslant \frac{1}{\sqrt{2\pi e}}\,.$$

Indeed, it is well-known that $L_K \geqslant L_{B_n}$ for all $K \in \mathcal{K}_n$, where B_n is the normed Euclidean ball (we have

$$nL_K^2 = \int_K |x|^2 dx \geqslant \int_{B_n} |x|^2 dx = nL_{B_n}^2,$$

since the norm of a vector in $K \setminus B_n$ is always greater than the norm of a vector in $B_n \setminus K$). The radius of inertia of the normalised ball is given by (see [4, Sect. 3], easy computation)

$$L_{B_n}^2 = \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}}{(n+2)\pi} \geqslant \frac{n\sqrt{2\pi\frac{n}{2}}^{\frac{2}{n}}}{2e(n+2)\pi} \geqslant \frac{1}{2\pi e}\,.$$

Proof of Theorem 4.1. We make use of the fact that the densities $\varphi_{K,u}$ are logarithmically concave. Indeed, by Brunn's theorem (cf. [2, Theorem 5.1]) the functions $\varphi_{K,u}(\cdot)^{\frac{1}{n-1}}$ are concave on their support. Since the logarithm is increasing and concave, $\ln \varphi_{K,u} = (n-1)\ln(\varphi_{K,u}{}^{\frac{1}{n-1}})$ is concave on $\mathbb{R}$.

(i) ⇒ (ii). From (4.1) and Remark 3.4 we know that $\|\tilde{\varphi}_{K,u}\|_\infty \leqslant 1$ for all $K \in \mathcal{K}$. By (4.1), Lemma 4.4, and (4.2) we obtain

$$\begin{aligned}&\sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\Phi_{K,u} - G_{L_K^2}\|_\infty > \varepsilon\}\big)\\ &= \sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\tilde{\Phi}_{K,u} - G_1\|_\infty > \varepsilon\}\big)\\ &\leqslant \sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, L(\varphi_{K,u}\lambda_1, g_{L_K^2}\lambda_1) > \frac{\varepsilon}{2}\}\big) \to 0\end{aligned}$$

as $n \to \infty$.

(ii) ⇔ (iii) is a direct consequence of the first chain of inequalities in Theorem 3.3, and Remark 3.4.

(ii) ⇒ (iv). Because of Remark 3.4, we can apply the last inequality of Theorem 3.3 to $f = \varphi_{K,u}$, with $\sigma = L_K \geqslant \frac{1}{\sqrt{2\pi e}}$ by Remark 4.5. Thus we have

$$\begin{aligned}&\sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\varphi_{K,u} - g_{L_K^2}\|_\infty > \varepsilon\}\big)\\ &\leqslant \sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \sqrt{5}\sqrt{2\pi e}\sqrt{\|\Phi_{K,u} - G_{L_K^2}\|_\infty} > \varepsilon\}\big)\\ &= \sup_{K\in\mathcal{K}_n\cap\mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\Phi_{K,u} - G_{L_K^2}\|_\infty > \frac{\varepsilon^2}{10\pi e}\}\big) \to 0\end{aligned}$$

as $n \to \infty$.

(iv) ⇒ (iii) under the additional assumption $\sup_{K\in\mathcal{M}} L_K < \infty$. This is similar to the proof of '(ii) ⇒ (iv)' above. (Instead of the last inequality of Theorem 3.3, use the second one: $\|\varphi_{K,u} - g_{L_K^2}\|_1 \leqslant \beta_1(L_K\|\varphi_{K,u} - g_{L_K^2}\|_\infty)$.) □

Proof of Theorem 4.2. As noted in [10, eqn. (1.1)], inequality (1.3) implies property (i) of Theorem 4.1 for $\mathcal{M} = \mathcal{T}$. More precisely, for all $\varepsilon > 0$ we have

$$\begin{aligned}&\sup_{K\in\mathcal{T}_n} \lambda_n\big(\{x \in K;\, \big||x|^2 - nL_K^2\big| \geqslant \varepsilon n L_K^2\}\big)\\ &\qquad \leqslant \frac{1}{\varepsilon^2}\left(\frac{(n+1)(n+2)}{(n+3)(n+4)}\left(1 + \frac{2}{n} + \frac{6}{n+1}\right) - 1\right) \to 0\end{aligned}$$

as $n \to \infty$. Thus the statements are a consequence of Theorem 4.1. □

References

1. M. Anttila, K. M. Ball, I. Perissinaki: The central limit problem for convex bodies. Preprint, 1998
2. K. M. Ball: A modern introduction to convex geometry. In: Silvio Levy (ed.) Flavors of Geometry, Cambridge University Press, Cambridge, 1997
3. U. Brehm, H. Vogt, J. Voigt: Permanence of moment estimates for p-products of convex bodies. Studia Math., to appear
4. U. Brehm, J. Voigt: Asymptotics of cross sections for convex bodies. Beitr. Algebra Geom./Contrib. Algebra Geom. **41**, 437–454 (2000)
5. Y. S. Chow, H. Teicher: Probability Theory (2nd ed.). Springer, Berlin, 1988
6. M. Fradelizi: Hyperplane sections of convex bodies in isotropic position. Beitr. Algebra Geom./Contrib. Algebra Geom. **40**, 163–183 (1999)
7. R. Gardner: Geometric Tomography. Cambridge University Press, Cambridge, 1995.
8. V. D. Milman, A. Pajor: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geometric Aspects of Functional Analysis. (J. Lindenstrauss, V. D. Milman, eds.) Lect. Notes Math., vol. 1376, Springer, Berlin, 1989, pp. 64–104
9. T. W. Ting: An isoperimetric inequality for moments of inertia of plane convex sets. Trans. Am. Math. Soc. **107**, 421–431 (1963)
10. J. Voigt: A concentration of mass property for isotropic convex bodies in high dimensions. Israel J. Math. **115**, 235–251 (2000)
11. H. von Weizsäcker: Sudakov's typical marginals, random linear functionals and a conditional central limit theorem. Probab. Theory Relat. Fields **107**, 313–324 (1997)