Flux-gradient and source term balancing for certain high resolution shock-capturing schemes

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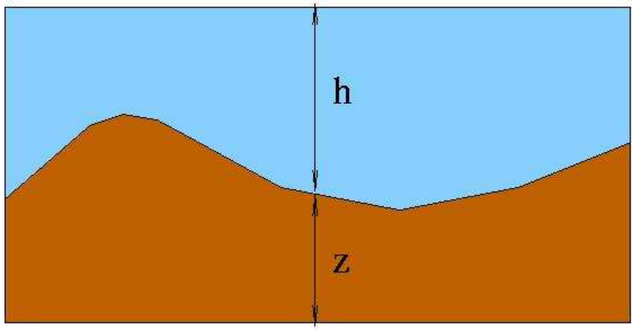
Shallow water equations

Derived from the Navier-Stokes model after:



- depth averaging
- hydrostatic hypothesis
- neglecting viscosity and turbulence
- not considering wind effects nor Coriolis force.

$\longrightarrow$ System of conservation laws plus a source term:

$$U_t + F(U)_x + E(U)_y = S$$

$$\begin{pmatrix} h \\ q_1 \\ q_2 \end{pmatrix}_t + \begin{pmatrix} q_1 \\ \frac{q_1^2}{h} + \frac{1}{2}gh^2 \\ \frac{q_1 q_2}{h} \end{pmatrix}_x + \begin{pmatrix} q_2 \\ \frac{q_1 q_2}{h} \\ \frac{q_2^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}_y = \begin{pmatrix} 0 \\ -ghz_x \\ -ghz_y \end{pmatrix}$$

Depth positivity condition: $u_R - u_L \le 2\sqrt{gh_L} + 2\sqrt{gh_R}$

(For flat topographies)

Numerical implementation - Homogeneous system

$$\mathbf{U_t} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = \mathbf{0}$$

[Fedkiw et al.] Method of high accuracy without spurious oscillations and shock capturing, based on

Shu-Osher finite difference Essentially Non-Oscillatory (ENO)

+

Marquina's flux splitting technique

$\mathbf{ENO-RF}$ construction:

$$\mathsf{F}_{i+1/2} = \begin{cases} \hat{f}^{Roe}_{i+1/2} & \text{if } f' \text{ does not change sign in } [u_i, u_{i+1}] \\ \hat{f}^{LLF}_{i+1/2} & \text{else} \end{cases}$$

Numerical implementation - Homogeneous system

Extension to nonlinear systems

Given $U^L = U^L_{i+1/2}$ and $U^R = U^R_{i+1/2}$,
we compute the **'sided' local characteristic** variables and fluxes

$$\begin{aligned} u^p_{L,j} &= L^p(U^L)\cdot U_j & f^p_{L,j} &= L^p(U^L)\cdot F_j \\ u^p_{R,j} &= L^p(U^R)\cdot U_j & f^p_{R,j} &= L^p(U^R)\cdot F_j \end{aligned} \qquad \text{for } p = 1,2,3$$

$L^p(U^{L,R})$, $R^p(U^{L,R})$ left and right eigenvectors
$\lambda^p(U^{L,R})$ eigenvalues
of the Jacobian Matrices $J(U^{L,R}) = F'(U^{L,R})$.

Then, the **numerical flux** is

$$\mathsf{F}_{i+1/2} = \sum_p (\mathcal{F}^p_{i+1/2})^L R^p(U^L_{i+1/2}) + (\mathcal{F}^p_{i+1/2})^R R^p(U^R_{i+1/2})$$

Numerical implementation - Source term

Following ideas in [Gascón and Corberán] we propose an extension of Marquina's flux splitting scheme [Fedkiw et al.] for the numerical approximation of the nonhomogneous system.

Split source term, $S = S_1 + S_2 = \begin{pmatrix} 0 \\ -ghz_x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ghz_y \end{pmatrix}$

Define functions:

$$B(x,y,t) = -\int_0^x S_1(s,y,t)ds = \int_0^x gh(s,y,t)z_x(s,y)ds$$

$$C(x,y,t) = -\int_0^y S_2(x,s,t)ds = \int_0^y gh(x,s,t)z_y(x,s)ds$$

The original system can then be written in 'conservation form' as follows,

$$\mathbf{U_t} + (\mathbf{F}+\mathbf{B})_\mathbf{x} + (\mathbf{E}+\mathbf{C})_\mathbf{y} = \mathbf{0}$$

→ **new flux: physical flux + primitive of source term**

Numerical implementation - Source term

Consider the scalar, 1D version of system $\mathbf{w_t + g(w, x)_x = 0}$ where,

$$g(w, x) = f(w) + b(x, t), \quad b(x, t) = -\int_0^x s(y, w(y, t))dy$$

We seek a semi-discrete formulation of the type

$$\boxed{w_t + \frac{G_{i+1/2} - G_{i-1/2}}{\Delta x} = 0}$$

$\longrightarrow$ $\mathbf{G_{i+1/2}}$: **numerical 'modified flux' function** (interpolated via ENO)

Numerical implementation - Source term

Construction of $G_{i+1/2}$: upwind direction is determined by $f(w)$.

$\longrightarrow$ We use $ENO-RF$ construction directly on the modified flux data g_i

$$G_{i+1/2} = \mathcal{G}_{i+1/2} + HOT_{i+1/2}$$

- $\mathcal{G}_{i+1/2}$ is the first order contribution,
- $HOT_{i+1/2}$ are the high order terms obtained from the ENO reconstruction.

$\mathcal{G}_{i+1/2}$ involves point-values of the convective flux (f_i or f_{i+1}) and integral expressions of the form $b_l = \int_{x_{-1/2}}^{x_l} s(w(x),x)dx$.

$\longrightarrow$ b_l **hard to handle computationally**

HOT involve only divided differences of g:

$$g[l,l+1] = \frac{g_{l+1}-g_l}{\Delta x} = f[l,l+1] + \frac{1}{\Delta x}\underbrace{\int_{x_l}^{x_{l+1}} s(w(x),x)dx}_{b_{l,l+1}}$$

$\longrightarrow$ $b_{l,l+1}$ **much more convenient computationally**

Numerical implementation - Source term

Carrying out the algebra, we easily arrive at

$$\mathcal{G}^{+}_{i+1/2} = \begin{cases} f_i & \text{if } f' > 0 \text{ in } [w_i, w_{i+1}] \\ f_{i+1} + b_{i,i+1} & \text{if } f' < 0 \text{ in } [w_i, w_{i+1}] \\ \frac{1}{2}(f_i + \alpha_{i+1/2} w_i) + \frac{1}{2}(f_{i+1} - \alpha_{i+1/2} w_{i+1}) + \frac{1}{2} b_{i,i+1} & \text{else} \end{cases}$$

$$\mathcal{G}^{-}_{i+1/2} = \begin{cases} f_i - b_{i,i+1} & \text{if } f' > 0 \text{ in } [w_i, w_{i+1}] \\ f_{i+1} & \text{if } f' < 0 \text{ in } [w_l, w_r] \\ \frac{1}{2}(f_i + \alpha_{i+1/2} w_i) + \frac{1}{2}(f_{i+1} - \alpha_{i+1/2} w_{i+1}) - \frac{1}{2} b_{i,i+1} & \text{else} \end{cases}$$

In such a way that $\boxed{\mathcal{G}_{i+1/2} - \mathcal{G}_{i-1/2} = \mathcal{G}^{+}_{i+1/2} - \mathcal{G}^{-}_{i-1/2}}$

'Modified Flux' computation at p-th charac. field

- If $\lambda^p(U^L_{i+1/2}) > 0$ and $\lambda^p(U^R_{i+1/2}) > 0$: **upwind from the left**,

  $(\tilde{G}^{p,\pm})^L = \mathcal{G}^{\pm}_{i+1/2} + HOT^L_{i+1/2}$ using ENO-Roe with projection $L^p(U^L)$

  $(\tilde{G}^{p,\pm})^R = 0.$

- If $\lambda^p(U^L_{i+1/2}) < 0$ and $\lambda^p(U^R_{i+1/2}) < 0$: **upwind from the right**,

  $(\tilde{G}^{p,\pm})^L = 0.$

  $(\tilde{G}^{p,\pm})^R = \mathcal{G}^{\pm}_{i+1/2} + HOT^R_{i+1/2}$ using ENO-Roe with projection $L^p(U^R)$

- If $\lambda^p(U^L_{i+1/2})\lambda^p(U^R_{i+1/2}) \leq 0$: **sonic point nearby**

  Define $\alpha_{i+1/2} = \max(|\lambda^p(U^L_{i+1/2})|, |\lambda^p(U^R_{i+1/2})|)$

  $(\tilde{G}^{p,\pm})^L$ using ENO-LLF with projection $L^p(U^L)$

  $(\tilde{G}^{p,\pm})^R$ using ENO-LLF with projection $L^p(U^R)$

Numerical implementation - Study of the steady state

The scheme is said to satisfy the:

- **Exact C-property** if it is exact when applied to the stationary case $q \equiv 0$ and $h + z \equiv constant$
- **Approximate C-property** if it is not exact but accurate to order $O(\Delta x^2)$

Using two 'sided' Jacobians and ENO of order $n \geq 2 \longrightarrow$ Approx. C-property.
Using a single Jacobian and any interpolation order $\longrightarrow$ Exact C-property.

Numerical implementation - Combined 1J2J scheme

1. Compute $U^L = U^L_{i+1/2}$ and $U^R = U^R_{i+1/2}$ with ENO-biased.
2. Compute $\lambda^p(U^L)$ and $\lambda^p(U^R)$.
3. If $||U^L - U^R|| < \Delta x^{s\,\diamond}$ and $\lambda^p(U^L)\lambda^p(U^R) > 0$ (for $p = 1, 2$) then (contiguous states are very close and no sonic point nearby: Use 1J)

   Define average state $U^A_{i+1/2} = \frac{1}{2}(U^L + U^R)$, compute: $L(U^A)$, $R(U^A)$.

   Then $G^{\pm}_{i+1/2} = \sum_p (\tilde{G}^{p,\pm}_{i+1/2})^A R^p(U^A_{i+1/2})$ $\diamond\ s = .5\ (n = 1);\ s = 1\ (n \geq 2)$.
4. Else, then (contiguous states are not close: Use 2 Jacobians)

   Compute $L(U^L)$, $L(U^R)$, $R(U^L)$, $R(U^R)$.

   Then $G^{\pm}_{i+1/2} = \sum_p (\tilde{G}^{p,\pm}_{i+1/2})^L R^p(U^L_{i+1/2}) + (\tilde{G}^{p,\pm}_{i+1/2})^R R^p(U^R_{i+1/2})$

Numerical implementation - Wet/Dry fronts

- If $h_i \neq 0$ and $h_{i+1} = 0$: the dry zone is on the right, then
  - Compute $U^L_{i+1/2}$ with ENO interpolation.
  - Set $U^R_{i+1/2} = U^L_{i+1/2}$.
- If $h_i = 0$ and $h_{i+1} \neq 0$: the dry zone is on the left, then
  - Compute $U^R_{i+1/2}$ with ENO interpolation.
  - Set $U^L_{i+1/2} = U^R_{i+1/2}$.

⟶ **No mixed information from wet and dry states**
(no LLF-flux at wet/dry fronts).

⟶ **Non zero reconstruction from dry side**
(useful for C-property verification at wet/dry fronts).

Numerical implementation - Dry bed generation

- ENO iterpolation: If $u_i - u_{i+1} > 2c_i + 2c_{i+1}$ holds:

  $U^L_{i+1/2}$ interpolated considering that right states are vacuum cells.

  $U^R_{i+1/2}$ interpolated considering that left states are vacuum cells.

- Modified-flux computation:

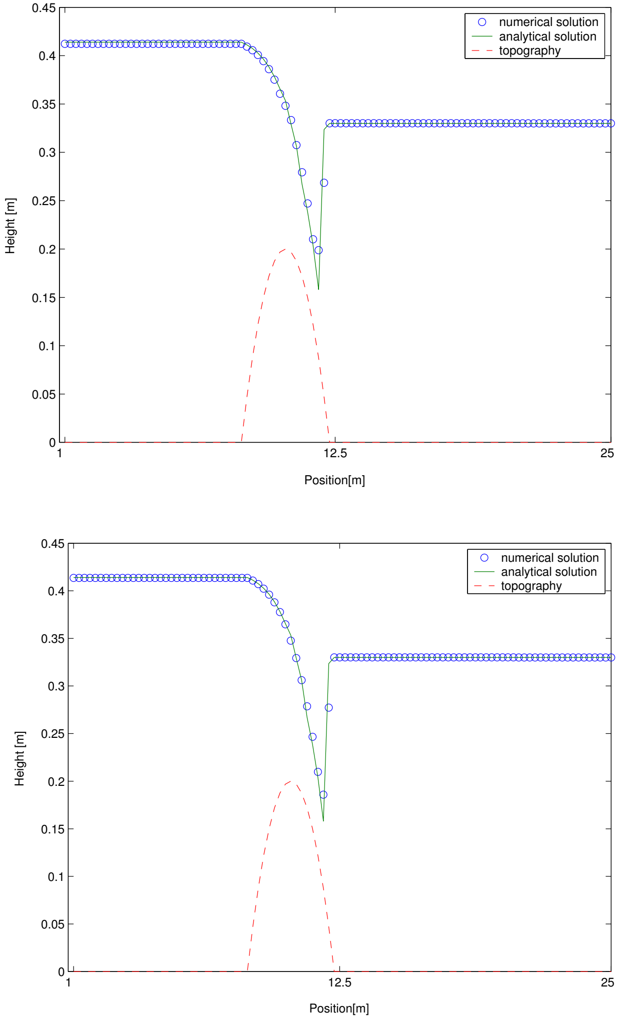
  If $u^R_{i+1/2} - u^L_{i+1/2} > 2c^L_{i+1/2} + 2c^R_{i+1/2}$, then for every p-th characteristic wave:

  We compute $(\tilde{G}^p_{i+1/2})^L$ and $(\tilde{G}^p_{i+1/2})^R$ with the entropy-fix of [Harten-Hyman] adapted to the two-sided characteristic decomposition of Marquina's flux formula, i.e.:

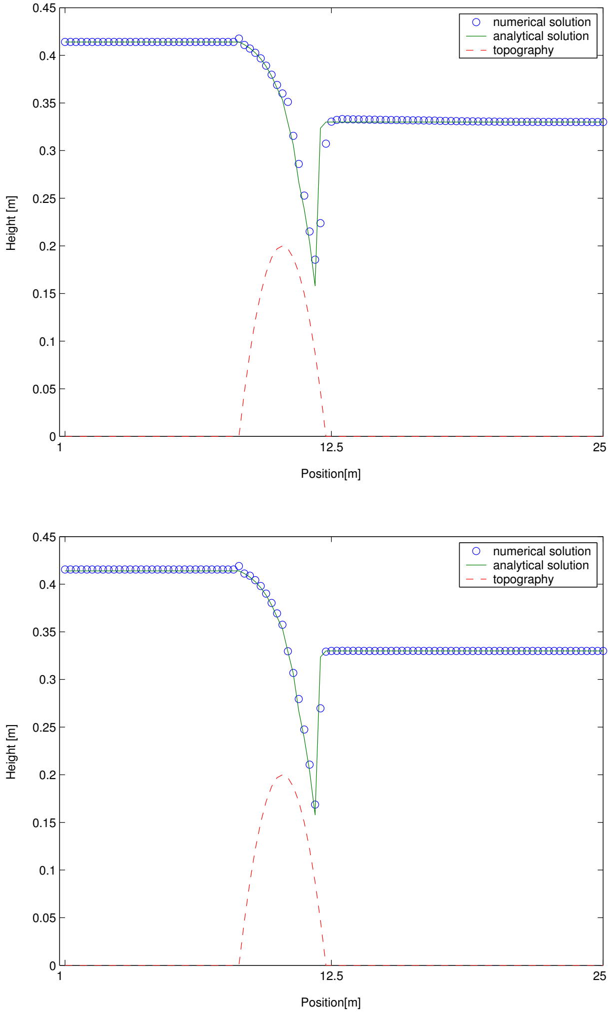
$$\alpha_{i+1/2} = \frac{\hat{\lambda}^p_{i+1/2}(\lambda^p(U^R) + \lambda^p(U^L)) - 2\lambda^p(U^R)\lambda^p(U^L)}{\lambda^p(U^R) - \lambda^p(U^L)}$$

  where $\hat{\lambda}^p_{i+1/2}$: p-th eigenvalue of Jacobian matrix evaluated at Roe's average.

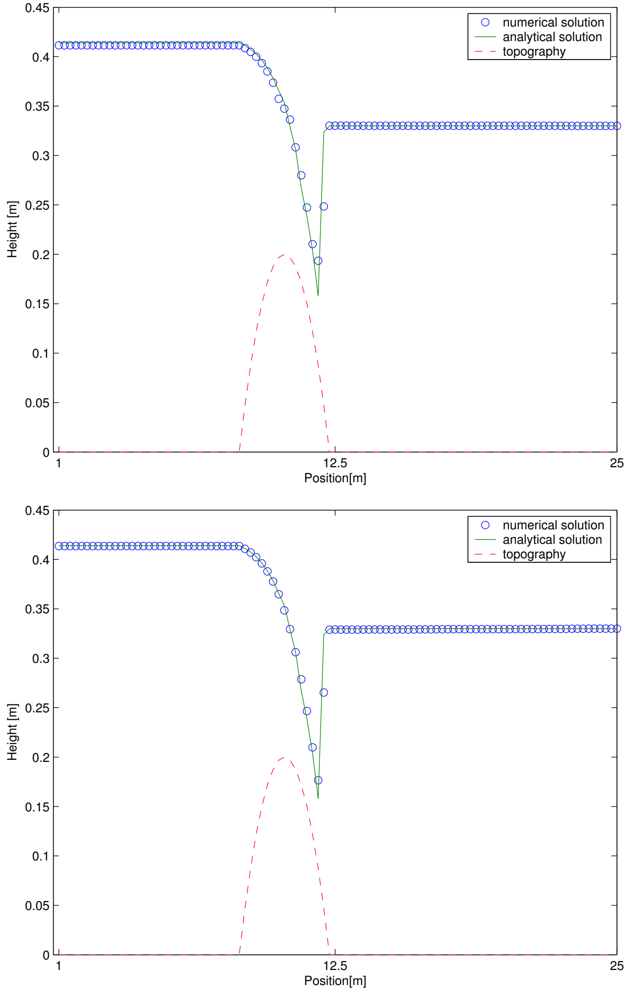
Evaluation - Transcritical flow with shock over a hump



Roe

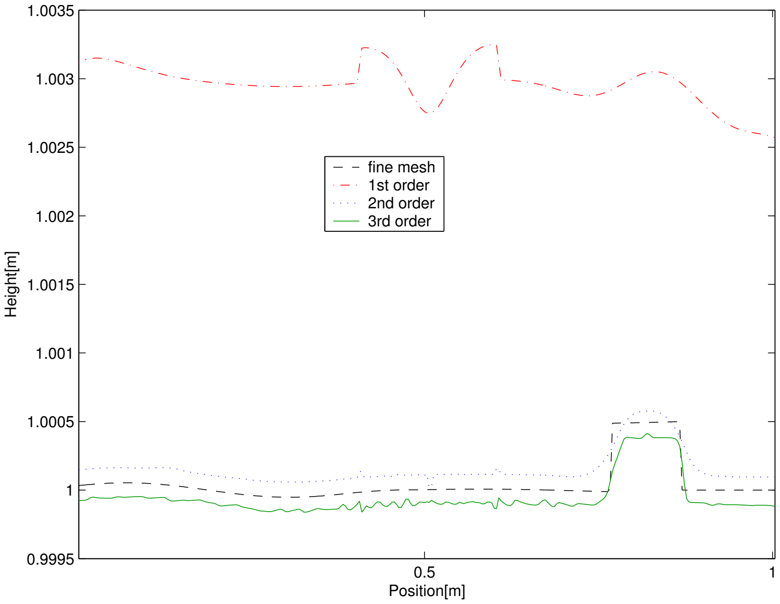


Marquina

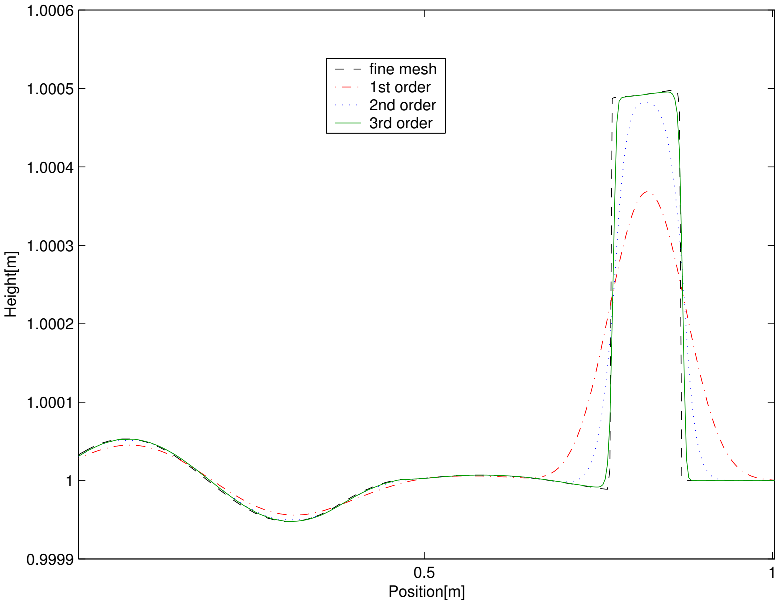


Combined 1J2J

Evaluation - Quasi stationary case

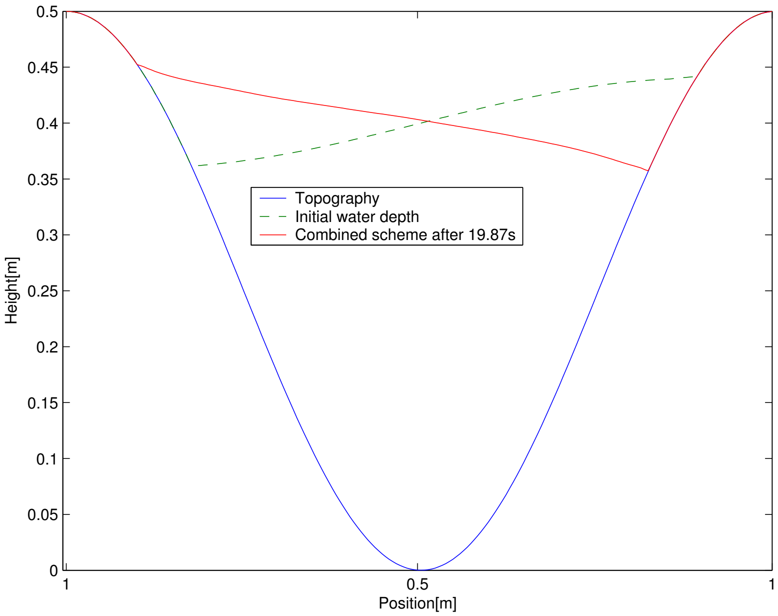


Marquina

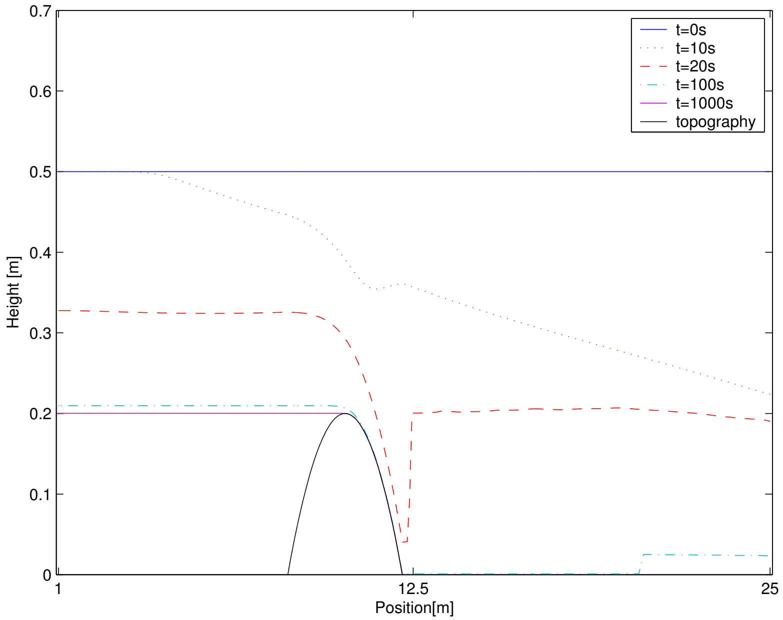


Combined 1J2J

Evaluation - Wet/dry fronts

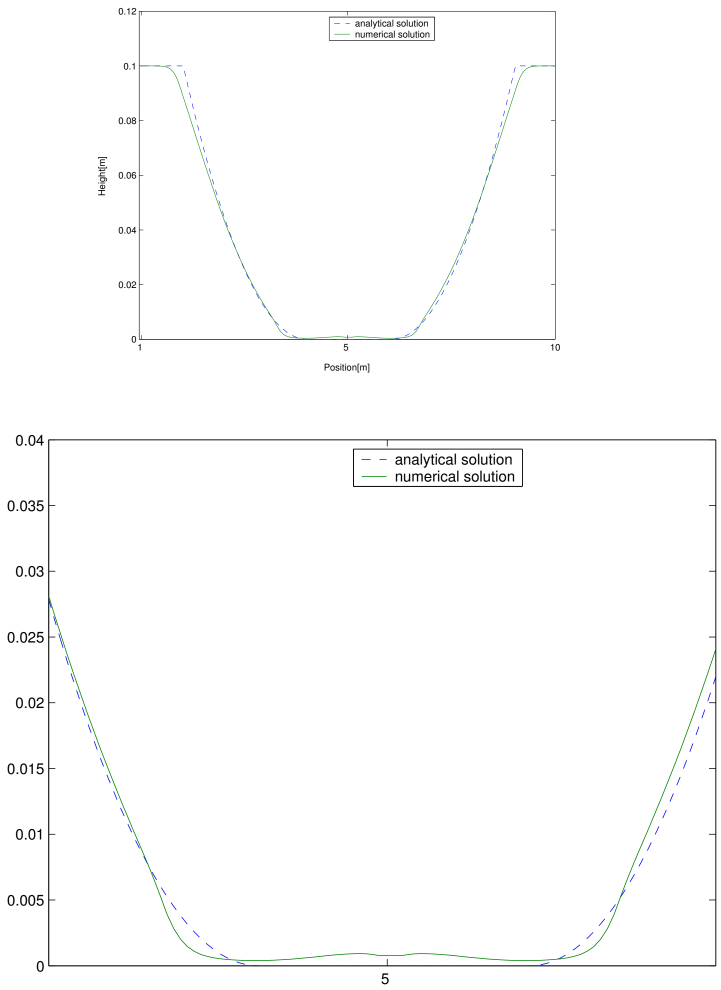


Oscillating lake

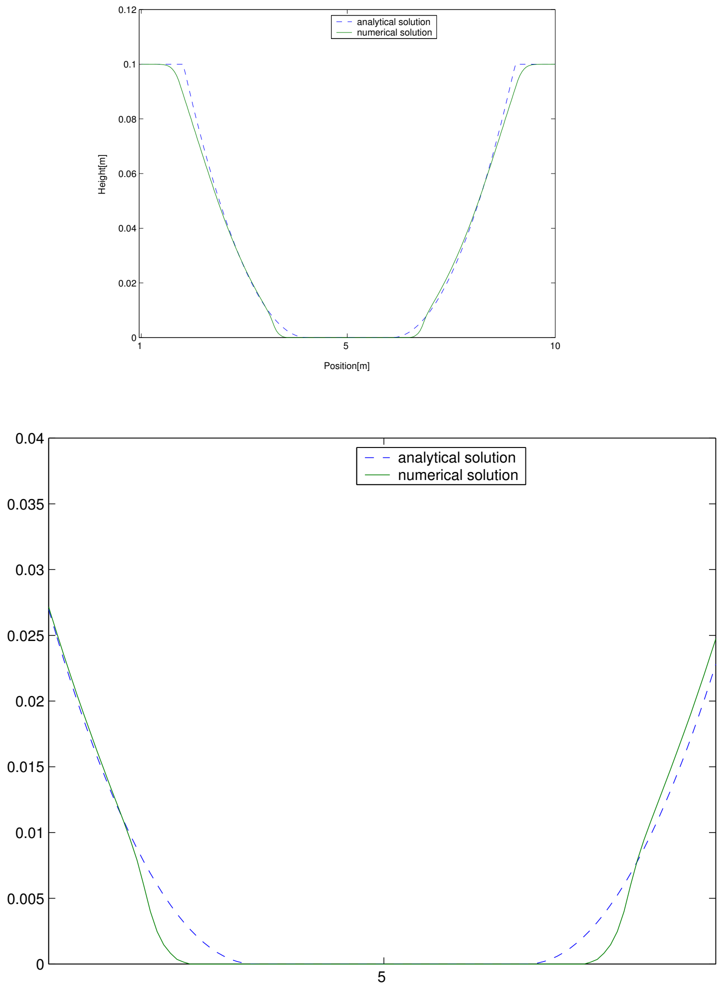


Drain on a
non-flat bottom

Evaluation - Dry bed generation

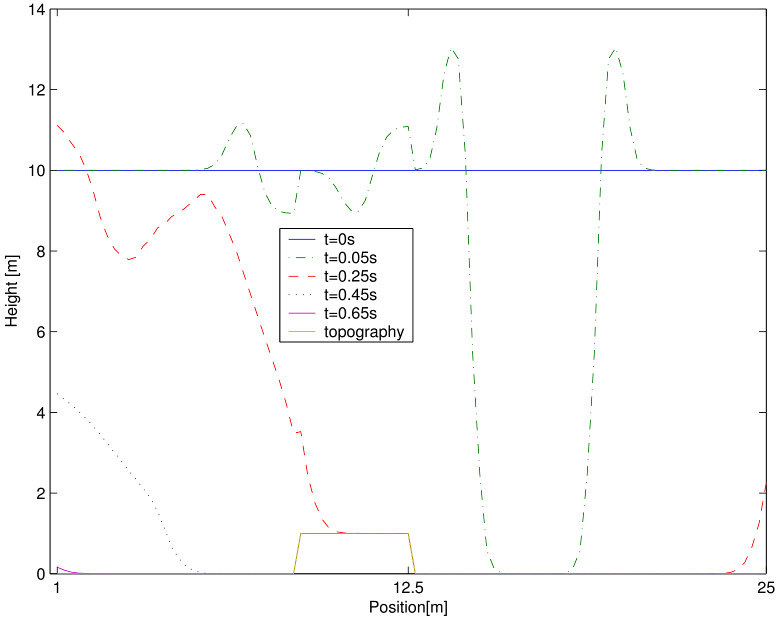


Local Lax-Friedrichs EF

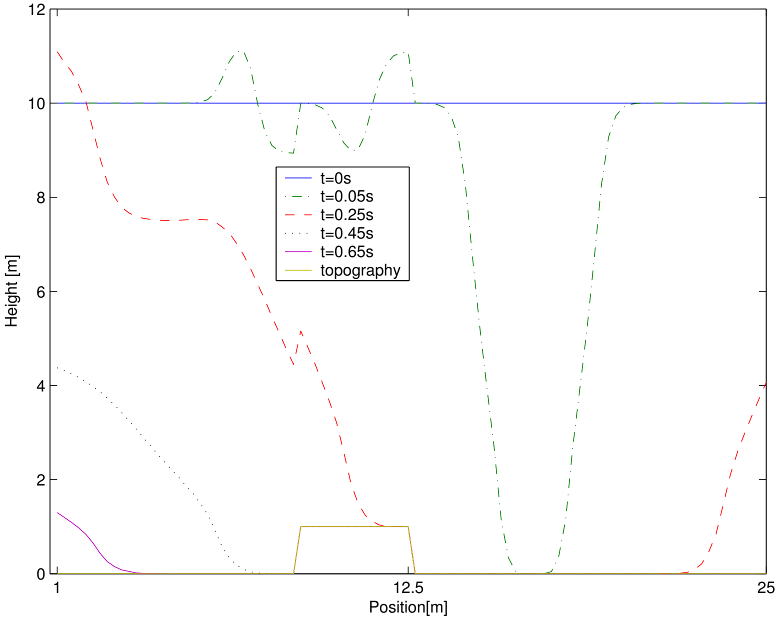


Modified Harten-Hyman EF

Evaluation - Dry bed generation



Modif. Harten-Hyman EF
(orig. ENO)



Modif. Harten-Hyman EF
(modif. ENO)