

## Example on Applying Stokes' Theorem

This example is provided to help you make sure you understand how to apply Stokes' theorem in situations analogous to those in the homework problems. It is the solution to Problem 3, page 414 (Section 6.4).

**Question:** Evaluate the following line integral using Stokes' theorem:

$$\int_C z^2 dx + x^2 dy + y^2 dz$$

where the curve  $C$  is parametrized by:

$$\mathbf{x}(\theta) = (\sin(2\theta), \cos(\theta), \sin(\theta)), \text{ where } 0 \leq \theta \leq 2\pi$$

**Solution:** First let us rewrite the given line integral using notation that is more familiar to you. The vector field that is integrated is:

$$\mathbf{F}(x, y, z) = (z^2, x^2, y^2)$$

So the line integral we're looking at is:

$$\int_C \mathbf{F} \cdot d\mathbf{x}$$

One way to evaluate this integral is to apply directly the definition of the line integral of a vector field. We will not do that, but just for the record, let us recall how that goes:

$$\int_C \mathbf{F} \cdot d\mathbf{x} := \int_0^{2\pi} \mathbf{F}(\mathbf{x}(\theta)) \cdot \mathbf{x}'(\theta) d\theta$$

Instead, we will use Stokes' theorem. Remember that Stokes' theorem allows us to express the *line integral* of a vector field along a closed curve  $C$  as the *surface integral* of a related vector field over a surface  $M$  whose boundary  $\partial M$  coincides with the curve  $C$ .

More precisely, if  $\partial M = C$ , then Stokes' theorem reads:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int \int_M \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$$

provided that that orientations of the surface  $M$  and the curve  $C$  agree; more on that in a little bit. Recall also that the hypothesis of the theorem requires the vector field  $\mathbf{F}$  to be continuously differentiable in a neighborhood of the surface  $M$ . Our given vector field in this question is perfectly nice and smooth all over  $\mathbf{R}^3$ , so that hypothesis is certainly satisfied: no need to worry.

To apply Stokes' theorem, then, we have to first find a surface  $M$  whose boundary is the curve  $C$ . There are many such; there is no point in choosing an exotic one, however: in fact, we should choose the most convenient such surface, keeping in mind that we will have to evaluate a surface integral over it.

One look at the parametrization for the curve  $C$  reveals that the  $(x, y, z)$ -coordinates of points on this curve satisfy a simple equation. Indeed, if we remember the trigonometric identity:

$$\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$$

then we see that if a point  $(x, y, z)$  is on our curve  $C$ , there should hold the relation:

$$x = 2yz$$

In other words, the curve lies on the surface described by the graph of the equation  $x(y, z) = 2yz$ . And in fact, it forms the boundary of a part of that surface. Which part? Consider the projection of our curve to the  $yz$ -plane. There, it traces out a circle of radius 1 centered at the origin. So our curve  $C$  is the boundary of the part of the surface where  $y$  and  $z$  coordinates of points lie in the unit disk. More precisely (and more succinctly), we take:

$$M := \{(x, y, z) \in \mathbf{R}^3 : x = 2yz \text{ and } y^2 + z^2 \leq 1\}$$

To evaluate the surface integral in Stokes' theorem, we need to find a parametrization for our surface  $M$ . Remember that when the surface is described as the graph of some function, it is very easy to write down a parametrization for it. Since our surface is the graph of a function of the variables  $y$  and  $z$ , we can think of  $y$  and  $z$  as the parameters of our parametrization. We'll give them new names,  $s$  and  $t$ , respectively. The parameter domain  $R$  for our parameters is the unit disk, as we noted above (in other words,  $s^2 + t^2 \leq 1$ ). With that, our parametrization is:

$$\mathbf{f}(s, t) := (2st, s, t) \text{ where } s^2 + t^2 \leq 1$$

We will have to check whether the orientation given by this parametrization agrees with the orientation of the given curve  $C$  (a requirement of Stokes' theorem); if not, we'll have to switch it. Next we should compute  $\nabla \times \mathbf{F}$ , since the formula in Stokes' theorem demands it. Let us call the resulting vector field  $\mathbf{G}$ :

$$\mathbf{G} := \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2(y, z, x)$$

Now we recall below the definition of the surface integral of a vector field:

$$\int \int_M \mathbf{G} \cdot \mathbf{n} d\sigma := \int \int_R \mathbf{G}(\mathbf{f}(s, t)) \cdot \left( \frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t} \right) ds dt$$

We therefore have to compute the two terms:

$$\mathbf{G}(\mathbf{f}(s, t)) \text{ and } \left( \frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t} \right)$$

and then find their dot product and then compute the double integral. First, we find:

$$\mathbf{G}(\mathbf{f}(s, t)) = \mathbf{G}(2st, s, t) = 2(s, t, 2st)$$

Then, we compute:

$$\begin{aligned} \frac{\partial \mathbf{f}}{\partial s} &= (2t, 1, 0) \\ \frac{\partial \mathbf{f}}{\partial t} &= (2s, 0, 1) \end{aligned}$$

Taking the cross product of these two vectors:

$$\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 1 & 0 \\ 2s & 0 & 1 \end{vmatrix} = (1, -2t, -2s)$$

From the cross product, we see that the normal to the surface points in the positive  $x$  direction, and hence the orientation of the surface as given by this parametrization agrees with the orientation of the curve  $C$  (check this yourself till you are convinced). Continuing our calculation we get:

$$\mathbf{G}(\mathbf{f}(s, t)) \cdot \left( \frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t} \right) = 2(s, t, 2st) \cdot (1, -2t, -2s) = 2(s - 2t^2 - 2s^2t)$$

And so, in our situation, Stokes' theorem says:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int \int_R 2(s - 2t^2 - 2s^2t) ds dt$$

All that remains to do is to evaluate the integral on the right hand side. Doing that can be simplified by the following observations. First, we can split that integral into three pieces, like so:

$$\int \int_R 2(s - 2t^2 - 2s^2t) ds dt = \int \int_R 2s ds dt - 4 \int \int_R t^2 ds dt - 4 \int \int_R s^2t ds dt$$

The first and the third integral can be seen to be 0 without evaluating them, on symmetry grounds (review Section 5.3, page 307 in this regard). To do the middle one, first note that the domain  $R$  is the unit circle, so why not express things in polar coordinates (review Section 5.7 in this regard): Letting

$$\begin{aligned} s &= r \cos(\theta) \\ t &= r \sin(\theta) \end{aligned}$$

we get:

$$\begin{aligned} -4 \int \int_R t^2 ds dt &= -4 \int_0^{2\pi} \int_0^1 r^2 \sin^2(\theta) r dr d\theta \\ &= - \int_0^{2\pi} r^4 \sin^2(\theta) \Big|_0^1 d\theta \\ &= - \int_0^{2\pi} \sin^2(\theta) d\theta \\ &= - \int_0^{2\pi} \frac{1}{2} (1 - \cos(2\theta)) d\theta \\ &= - \frac{\theta}{2} \Big|_0^{2\pi} \\ &= -\pi \end{aligned}$$