Log improvement of the Prodi-Serrin criteria for Navier-Stokes equations

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Abstract: This article is devoted to a Log improvement of Prodi-Serrin criterion for global regularity to solutions to Navier-Stokes equations in dimension 3. It is shown that the global regularity holds under the condition that $|u|^5/(\log(1+|u|))$ is integrable in space time variables.

keywords: Navier-Stokes, regularity criterion, a priori estimates

MSC: 35B65, 76D03, 76D05

1 Introduction

In this article, we consider the Navier-Stokes equation on $\mathbb{R}^3$, given by

$$\partial_t u - \triangle u + \mathrm{div}(u \otimes u) + \nabla p = 0, \tag{1}$$

$$\mathrm{div}(u) = 0, \tag{2}$$

where u is a vector-valued function representing the velocity of the fluid, and p is the pressure. Note that the pressure depends in a non local way on the velocity u. It can be seen as a Lagrange multiplier associated to the incompressible condition (2). The initial value problem of the above equation is endowed with the condition that $u(0,\cdot) = u_0 \in L^2(\mathbb{R}^3)$. Leray [11] and Hopf [6] had already established the existence of global weak solutions for the Navier-Stokes equation. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equation, and proved that, for every given initial datum $u_0 \in L^2(\mathbb{R}^3)$, there exists a global weak solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ verifying the Navier-Stokes equation in the sense of distribution. From that time on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equation. Different Criteria for regularity of the weak solutions have been proposed. The Prodi-Serrin conditions (see Serrin [16], Prodi [14], and [17]) states that any weak Leray-Hopf solution verifying $u \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with $2/p + 3/q = 1$, $2 \leq p < \infty$, is regular on $(0,\infty) \times \mathbb{R}^3$. Notice that if $p = q$, this corresponds to $u \in L^5((0,\infty) \times \mathbb{R}^3)$. The limit case of $L^\infty(0,\infty; L^3(\mathbb{R}^3))$ has been solved very recently by L. Escauriaza,

G. Seregin, and V. Sverak (see [7]). Other criterions have been later introduced, dealing with some derivatives of the velocity. Beale Kato and Majda [1] showed the global regularity under the condition that the vorticity $\omega = \text{curl } u$ lies in $L^\infty(0,\infty; L^1(\mathbb{R}^3))$ (see Kozono and Taniuchi for improvement of this result [9]). Beirão da Veiga show in [2] that the boundedness of ∇u in $L^p(0,\infty; L^q(\mathbb{R}^3))$ for $2/p + 3/q = 2$, $1 < p < \infty$ ensures the global regularity. In [3], Constantin and Fefferman gave a condition involving only the direction of the vorticity. Let us also cite a condition involving the lower bound of the pressure introduced by Seregin and Sverak in [15], and conditions involving only one of the component of u (see Penel and Pokorny [13], He [5], and Zhou [19]).
This article is devoted to the following log improvement of the Prodi-Serrin criterion corresponding to $p = q = 5$:

Theorem 1. *Suppose that u is a weak Leray-Hopf solution of the Navier-Stokes equation (1) (2) satisfying*

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{\log(1+|u|)} dx\, ds < \infty,$$

then, $u \in C^\infty((0,\infty) \times \mathbb{R}^3)$.

Montgomery-Smith introduced the following criterium in [12]:

$$\int_0^\infty \frac{\|u(t)\|^p_{L^q(\mathbb{R}^3)}}{1 + \log^+ \|u(t)\|_{L^q(\mathbb{R}^3)}}\, dt < \infty.$$

Notice that the log improvement is, here, in time only. This can be seen as a natural Gronwall type extension of the Prodi-Serrin conditions. So we can see it as a one dimension ODE type extension.
The goal of our result is to extend this log improvement also in x. For this purpose we focused on the homogeneous case $p = q = 5$, even though extension to the Prodi-Serrin range $2 \le p < \infty$ should be doable.
The proof of Theorem 1 is split into two parts. The first point is to show that for any time $t > \lambda$, the L^∞ norm of u in x can be bounded in a affine way by

$$\int_0^t \int_{\mathbb{R}^3} |u|^6\, dx\, dt.$$

More precisely, we will show the following Proposition:

Proposition 1.1. *For every λ satisfying $0 < \lambda < 2$, there exists some universal constant $A_\lambda > 0$, depending only on λ, such that, for any solution u of the Navier-Stokes equation on $(0,\infty) \times \mathbb{R}^3$, we have $\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A_\lambda\{1 + \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\}$, for any $T > \lambda$.*

Then Theorem 1 follows from a Gronwall argument on $\|u(t)\|_{L^\infty(\mathbb{R}^3)}$, since:

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \le A_\lambda$$
$$+A_\lambda \int_\lambda^t \|u(s)\|_{L^\infty(\mathbb{R}^3)} \log(1 + \|u(s)\|_{L^\infty(\mathbb{R}^3)}) \left(\int_{\mathbb{R}^3} \frac{|u(s)|^5}{\log(1+|u(s)|)}\, dx \right) ds$$

and the Hypothesis gives that $\int_{\mathbb{R}^3} \frac{|u(s)|^5}{\log(1+|u(s)|)}\, dx$ lies in $L^1(0,\infty)$.
Notice that the inequality of Proposition 1.1 needs to be invariant by the scaling of the Navier-Stokes equation:

$$u_\varepsilon(t,x) = \varepsilon u(t_0 + \varepsilon^2 t, x_0 + \varepsilon x). \tag{3}$$

This is why the L^6 norm pops up, since it has the same scaling as that of the L^∞ norm. Taking advantage of the scaling (3), Proposition 1.1 will follow from the following rescaled Proposition:

Proposition 1.2. *There exists a universal positive constant C^*, such that for any solution u of the Navier-Stokes equation on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(\mathbb{R}^3\times[-1,1])} \leqslant C^*$, we have $|u| \leqslant 1$ almost everywhere on $[-\frac{1}{2},1]\times\mathbb{R}^3$.*

The proof of proposition 1.2 is in the same spirit as the proof given by A. Vasseur [18]. It relies on a method first introduced by De Giorgi to show regularity of solutions to elliptic equations with rough diffusion coefficients [4]. In this paper, the proof of proposition 1.2 is established though sections 2, 3, 4 and 5. In section 6, we will deduce proposition 1.1 from proposition 1.2. Finally, in the last section of this paper, we will use the conclusion of proposition 1.1, together with the fundamental result of Serrin [16], to obtain the result of Theorem 1.

2 Basic setting of the whole paper

In order to prove proposition 1.2, we would like to introduce some notation first. Then, we will state two lemmas and one proposition which are related to the proof of proposition 1.2. So, let us fix our notation as follow.

- for each $k \geqslant 0$, let $Q_k = [T_k, 1]\times\mathbb{R}^3$, in which $T_k = -\frac{1}{2}(1+\frac{1}{2^k})$.
- for each $k \geqslant 0$, let $v_k = \{|u| - (1-\frac{1}{2^k})\}_+$.
- for each $k \geqslant 0$, let $d_k = \frac{(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|\geqslant(1-\frac{1}{2^k})\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2$.
- for each $k \geqslant 0$, let $U_k = \frac{1}{2}\|v_k\|^2_{L^\infty(T_k,1;L^2(\mathbb{R}^3))} + \int_{T_k}^1\int_{\mathbb{R}^3} d_k^2 dx\, dt$.

With the above setting, we are now ready to state the lemmas and proposition which are related to proposition 1.2 as follow.

Lemma 2.1. *For any solution u of the Navier-Stokes equation on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)} \leqslant 1$, we have $U_1 \leqslant A\|u\|^6_{L^6(Q_0)}$, in which A is some universal constant strictly greater than 1.*

Proposition 2.1. *There exists some universal constants B, $\beta > 1$, such that for any solution u of the Navier-Stokes equation on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)} \leqslant \frac{1}{A^{\frac{1}{6}}}$, we have $U_k \leqslant B^k U_{k-1}^\beta$, for all $k \geqslant 1$. Here, A is the universal constant appearing in Lemma 2.1 .*

Let us first show that Lemma 2.1 and Proposition 2.1 provide the result of Proposition 1.2.First we show that the sequence U_k converges to 0 when k goes to infinity. We can use for instance the following easy lemma (see [18]):

Lemma 2.2. *For any given constants* B, $\beta > 1$, *there exists some constant* C_0^* *such that for any sequence* $\{a_k\}_{k\geqslant 1}$ *satisfying* $0 < a_1 \leq C_0^*$ *and* $a_k \leqslant B^k a_{k-1}^{\beta}$, *for any* $k \geqslant 1$, *we have* $lim_{k\to\infty} a_k = 0$.

Indeed, let B, $\beta > 1$ be the constants occurring in proposition 2.1, and let C_0^* be the constant associated to B, β in the sense of lemma 2.2. Now, take $C^* = min\{\frac{1}{A^{\frac{1}{6}}}, (\frac{C_0^*}{A})^{\frac{1}{6}}\}$, in which A is the universal constant appearing in Lemma 2.1. Then, for any solution u of the Navier-Stokes system on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)} \leqslant C^*$, we have $\|u\|_{L^6(Q_0)} \leqslant (\frac{1}{A})^{\frac{1}{6}}$. Hence, proposition 2.1 tells us that $U_k \leqslant B^k U_{k-1}^{\beta}$, for all $k \geqslant 1$ must be valid. On the other hand, Since $\|u\|_{L^6(Q_0)} \leqslant C^* \leqslant 1$, Lemma 2.1 also implies that $U_1 \leqslant A\|u\|_{L^6(Q_0)}^6 \leqslant C_0^*$. Hence, it follows from Lemma 2.2 that $lim_{k\to\infty} U_k = 0$. However, since we have the inequality

$$\frac{1}{2}\int_{\mathbb{R}^3}\{|u(t,x)|-1\}_+^2 dx \leqslant \frac{1}{2} sup_{t\in[-\frac{1}{2},1]}\int_{\mathbb{R}^3} v_k^2 dx \leqslant U_k,$$

for every $t \in [-\frac{1}{2},1]$. As a result, $lim_{k\to\infty} U_k = 0$ immediately implies that $|u| \leqslant 1$ almost everywhere on $[-\frac{1}{2},1]\times\mathbb{R}^3$. This gives the result of Proposition 1.2.

3 proof of lemma 2.1

In this section, we will devote our effort in proving Lemma 2.1. Let us recall that the Navier-Stokes equation on $(-\infty,\infty)\times\mathbb{R}^3$ is

$$\partial_t u - \triangle u + div(u\otimes u) + \nabla P = 0,$$

together with the divergence free condition $div(u) = 0$. Now, by multiplying the above equation by the term $\frac{v_1}{|u|}u$, we yield the following inequality, which is valid in the sense of distribution.

$$\partial_t(\frac{1}{2}v_1^2) + d_1^2 - \triangle(\frac{1}{2}v_1^2) + div(\frac{v_1^2}{2}u) + \frac{v_1}{|u|}u\nabla P \leqslant 0.$$

Consider now the variables σ, t with $T_0 \leqslant \sigma \leqslant T_1 \leqslant t \leqslant 1$, where $T_0 = -1$, and $T_1 = -\frac{1}{2}(1+\frac{1}{2})$. We mention that we have the following , which is valid in the sense of distribution.

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{1}{2}v_1^2) dx\, ds = \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx - \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx.$
- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_1^2}{2}u) - \triangle(\frac{v_1^2}{2}) dx\, ds = 0.$

Hence, by taking the integral over $[\sigma, t] \times \mathbb{R}^3$ to the above inequality, we yield the following estimation.

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx + \int_\sigma^t \int_{\mathbb{R}^3} d_k^2 dx\, ds &\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \int_\sigma^t |\int_{\mathbb{R}^3} \frac{v_1}{|u|} u\nabla P dx| ds \\
&= \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \int_\sigma^t |\int_{\mathbb{R}^3} P\nabla(\frac{v_1}{|u|} u) dx| ds \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + 3\int_\sigma^t \int_{\mathbb{R}^3} d_1 |P| \chi_{\{v_1>0\}} dx\, ds \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \frac{3}{2}\int_\sigma^t \int_{\mathbb{R}^3} \alpha^2 d_1^2 dx\, ds \\
&+ \frac{3}{2}\int_\sigma^t \int_{\mathbb{R}^3} \frac{|P|^2}{\alpha^2}\chi_{\{v_1>0\}} dx\, ds,
\end{aligned}
$$

in which α can be any positive constant (In the third step of the above deduction, we have used the nontrival fact that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$, whose justification will be given in the last part of Section 4). Hence we yield the following inequality which is valid for any $\alpha > 0$.

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{v_1^2(t,x)}{2} dx + \int_\sigma^t \int_{\mathbb{R}^3} \frac{(2-3\alpha^2)d_1^2}{2} dx\, ds \leqslant \int_{\mathbb{R}^3} \frac{v_1^2(\sigma,x)}{2} dx \\
+ \int_\sigma^t \int_{\mathbb{R}^3} \frac{3|P|^2 \chi_{\{v_1>0\}}}{2\alpha^2} dx\, ds.
\end{aligned}
$$

If we choose $\alpha = (\frac{1}{2})^{\frac{1}{2}}$, then the inequality shown as above becomes

$$
\frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx + \frac{1}{4}\int_\sigma^t \int_{\mathbb{R}^3} d_1^2 dx\, ds \leqslant \int_{\mathbb{R}^3} \frac{v_1^2(\sigma,x)}{2} dx + 3\int_\sigma^t \int_{\mathbb{R}^3} |P|^2 \chi_{\{v_1>0\}} dx\, ds.
$$

By taking average over $\sigma \in [T_0, T_1]$, we can carry out the following estiamtion

$$
\int_{\mathbb{R}^3} \frac{v_1^2(t,x)}{2} dx + \int_{T_1}^t \int_{\mathbb{R}^3} \frac{d_1^2}{4} dx\, ds \leqslant \frac{4}{2}\int_{-1}^{T_1} \int_{\mathbb{R}^3} v_1^2(\sigma,x) + 3\int_{-1}^t \int_{\mathbb{R}^3} |P|^2 \chi_{\{v_1>0\}} dx\, ds.
$$

Notice that, in the above inequality, the integer 4 appears in the first term of the right hand side because $\frac{1}{T_k - T_{k-1}} = 2^2 = 4$. Now, by taking the L^∞-norm over $t \in [T_1, 1]$, we yield

$$
\frac{1}{4}U_1 \leqslant 2\int_{-1}^{T_1} \int_{\mathbb{R}^3} v_1^2 dx\, ds + 3\int_{Q_0} |P|^2 \chi_{\{v_1>0\}}.
$$

But, we notice that

$$\int_{-1}^{T_1}\int_{\mathbb{R}^3} v_1^2 dx\, ds \leqslant \int_{Q_0} v_1^2 \chi_{\{v_1>0\}}$$
$$\leqslant (\int_{Q_0} v_1^6)^{\frac{1}{3}} (\int_{Q_0} \chi_{\{v_1>0\}})^{\frac{2}{3}}$$
$$\leqslant \|u\|^2_{L^6(Q_0)} (\int_{Q_0} \chi_{\{v_0>\frac{1}{2}\}})^{\frac{2}{3}}$$
$$\leqslant \|u\|^2_{L^6(Q_0)} (2^6 \int_{Q_0} v_0^6)^{\frac{2}{3}} = 2^4 \|u\|^6_{L^6(Q_0)}.$$

On the other hand, since the pressure P satisfies the equation $-\triangle P = \sum \partial_i \partial_j (u_i u_j)$. So, by the Riesz theorem in the theory of singular integral, we have $\|P\|_{L^3(Q_0)} \leqslant C_3 \|u\|^2_{L^6(Q_0)}$, in which C_3 is some universal constant . Hence, it follows that

$$\int_{Q_0} |P|^2 \chi_{\{v_1>0\}} \leqslant \|P\|^2_{L^3(Q_0)} \|\chi_{\{v_1>0\}}\|_{L^3(Q_0)}$$
$$\leqslant C_3^2 \|u\|^4_{L^6(Q_0)} \|\chi_{\{v_0>\frac{1}{2}\}}\|_{L^3(Q_0)}$$
$$\leqslant C_3^2 \|u\|^4_{L^6(Q_0)} (2^6 \int_{Q_0} v_0^6)^{\frac{1}{3}}$$
$$= 4C_3^2 \|u\|^6_{L^6(Q_0)}.$$

Hence it follows that

$$\frac{1}{4} U_1 \leqslant 2 \int_{Q_0} v_1^2 + 3 \int_{Q_0} |P|^2 \chi_{\{v_1>0\}}$$
$$\leqslant 2^5 \|u\|^6_{L^6(Q_0)} + 12 C_3^2 \|u\|^6_{L^6(Q_0)}.$$

As a result, by taking $A = 2^7 + 48C_3^2$, we can at once deduce that

$$U_1 \leqslant A \|u\|^6_{L^6(Q_0)}.$$

So, we are done in establishing Lemma 2.1

4 Preliminaries for the proof of proposition 2.1

Lemma 4.1. *There exists some constant $C > 0$, such that for any $k \geqslant 1$, and any $F \in L^\infty(T_k, 1; L^2(\mathbb{R}^3))$ with $\nabla F \in L^2(Q_k)$, we have $\|F\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C \|F\|^{\frac{2}{5}}_{L^\infty(T_k,1;L^2(\mathbb{R}^3))} \|\nabla F\|^{\frac{3}{5}}_{L^2(Q_k)}$.*

Proof. By Sobolev-embedding Theorem, there is a constant C, depending only on the dimension of $\mathbb{R}^3$, such that

$$(\int_{\mathbb{R}^3} |F(t,x)|^6 dx)^{\frac{1}{6}} \leqslant C (\int_{\mathbb{R}^3} |\nabla F(t,x)|^2 dx)^{\frac{1}{2}}.$$

for any $t \in [T_k, 1]$, where $k \geqslant 1$, and F is some function which verifies $F \in L^{\infty}(T_k, 1; L^2(\mathbb{R}^3))$, and $\nabla F \in L^2(Q_k)$. By taking the power 2 on both sides of the above inequality and then taking integration along the variable $t \in [T_k, 1]$, we yield

$$\int_{T_k}^{1} (\int_{\mathbb{R}^3} |F|^6 dx)^{\frac{1}{3}} dt \leqslant C^2 \int_{T_k}^{1} \int_{\mathbb{R}^3} |\nabla F|^2 dx\, dt.$$

On the other hand, by Holder's inequality, we have

$$\begin{aligned} \|F\|_{L^{\frac{10}{3}}(Q_k)}^{\frac{10}{3}} = \int_{T_k}^{1} \int_{\mathbb{R}^3} |F|^2 |F|^{\frac{4}{3}} dx\, dt & \\ & \leqslant \int_{T_k}^{1} (\int_{\mathbb{R}^3} |F|^6 dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |F|^2 dx)^{\frac{2}{3}} dt \\ & \leqslant \|F\|_{L^{\infty}(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|F\|_{L^2(T_k,1;L^6(\mathbb{R}^3))}^2. \end{aligned}$$

By taking the advantage that $\|F\|_{L^2(T_k,1;L^6(\mathbb{R}^3))} \leqslant C \|\nabla F\|_{L^2(Q_k)}$, we yield

$$\|F\|_{L(Q_k)^{\frac{10}{3}}}^{\frac{10}{3}} \leqslant C^2 \|F\|_{L^{\infty}(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|\nabla F\|_{L^2(Q_k)}^2.$$

Hence, we have

$$\|F\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C \|F\|_{L^{\infty}(T_k,1;L^2(\mathbb{R}^3))}^{\frac{2}{5}} \|\nabla F\|_{L^2(Q_k)}^{\frac{3}{5}}.$$

so, we are done □

Lemma 4.2. *For any $1 < q < \infty$, we have $\|\chi_{\{v_k > 0\}}\|_{L^q(Q_{k-1})} \leqslant 2^{\frac{10k}{3q}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$.*

Proof. First, we have to notice that $\{v_k > 0\}$ is a subset of $\{v_{k-1} > \frac{1}{2^k}\}$, hence we have

$$\int_{Q_{k-1}} \chi_{\{v_k > 0\}} \leqslant \int_{Q_{k-1}} \chi_{\{v_{k-1} > \frac{1}{2^k}\}} \leqslant 2^{\frac{10k}{3}} \int_{Q_{k-1}} |v_{k-1}|^{\frac{10}{3}}.$$

By our previous Lemma, we have

$$\begin{aligned} \|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})}^{\frac{10}{3}} \leqslant & \\ & C^2 \|v_{k-1}\|_{L^{\infty}(T_{k-1},1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|\nabla v_{k-1}\|_{L^2(Q_{k-1})}^2 \\ & \leqslant C^2 (U_{k-1}^{\frac{1}{2}})^{\frac{4}{3}} \|d_{k-1}\|_{L^2(Q_{k-1})}^2 \\ & \leqslant C^2 U_{k-1}^{\frac{2}{3}} U_{k-1} \\ & = C^2 U_{k-1}^{\frac{5}{3}}. \end{aligned}$$

So, it follows that $\int_{Q_{k-1}} \chi_{\{v_k > 0\}} \leqslant 2^{\frac{10k}{3}} C^2 U_{k-1}^{\frac{5k}{3}}$, and hence we have $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leqslant 2^{\frac{10k}{3q}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$, where C is some universal constant. So, we are done. □

In the proof of Lemma 4.2, we have used the fact that $|v_k| \leqslant d_k$, whose justification will be given immediately in the following paragraph.
Before we leave this section, we also want to list out some inequalities which will often be used in the proof of proposition 1.1 as follow:

- $|(1-\frac{v_k}{|u|})u| \leqslant 1-\frac{1}{2^k}$.
- $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$.
- $\chi_{\{v_k>0\}}|\nabla|u|| \leqslant d_k$.
- $|\nabla v_k| \leqslant d_k$.
- $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

Now, we first want to justify the validity of $|(1-\frac{v_k}{|u|})u| \leqslant 1-\frac{1}{2^k}$. In the case in which the point (t,x) satisfies $|u(t,x)| < 1-\frac{1}{2^k}$, we have $v_k(t,x) = 0$, and hence it follows that

$$|\{1-\frac{v_k(t,x)}{|u(t,x)|}\}u(t,x)| = |u(t,x)| < 1-\frac{1}{2^k}.$$

In the case in which (t,x) satisfies $|u(t,x)| \geqslant 1-\frac{1}{2^k}$, we have $v_k(t,x) = |u(t,x)| - (1-\frac{1}{2^k})$, and hence it follows that

$$|\{1-\frac{v_k}{|u|}\}u(t,x)| = |1-\frac{|u|-(1-\frac{1}{2^k})}{|u|}||u| = 1-\frac{1}{2^k}.$$

So, no matter in which case, we always have the conclusion that $|(1-\frac{v_k}{|u|})u| \leqslant 1-\frac{1}{2^k}$.
Next, according to the definition of d_k^2, we can carry out the following estimation

$$d_k^2 \geqslant \frac{v_k}{|u|}|\nabla u|^2 \geqslant \{\frac{v_k}{|u|}|\nabla u|\}^2.$$

Hence, by taking square root, it follows at once that $d_k \geqslant \frac{v_k}{|u|}|\nabla u|$.
We now turn our attention to the inequality $\chi_{\{|u|\geqslant(1-\frac{1}{2^k})\}}|\nabla|u|| \leqslant d_k$. To justify it, we recall that $|\nabla u| \geqslant |\nabla|u||$. Hence, it follows from the definition of d_k^2 that

$$d_k^2 \geqslant \frac{1-\frac{1}{2^k}}{|u|}\chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||^2 + \{1-\frac{1-\frac{1}{2^k}}{|u|}\}\chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||^2.$$

So, by simplifying the right-hand side of the above inequality, we can deduce that $d_k^2 \geqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||^2$. Hence, we have $d_k \geqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||$. In addition, since it is obvious to see that $\nabla v_k = \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}\nabla|u|$, we also have the result that $|\nabla v_k| \leqslant d_k$.
Finally, we want to justify the inequality that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$. So, we notice that, by applying the product rule, we have

$$\nabla(\frac{v_k}{|u|}u) = \nabla(v_k)\frac{u}{|u|} + \frac{v_k}{|u|}\nabla u - \frac{v_k}{|u|^2}u\nabla|u|.$$

However, since $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$, and $|\frac{v_k}{|u|^2}u\nabla|u|| \leqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u|| \leqslant d_k$, it follows at once from the above expression that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

5 proof of proposition 2.1

To begin the argument, we recall that, by multiplying the equation $\partial_t u - \triangle u + div(u\otimes u) + \nabla P = 0$ on $(-\infty,\infty)\times\mathbb{R}^3$, we yield the following inequality formally, which is indeed valid in the sense of distribution

$$\partial_t(\frac{v_k^2}{2}) + d_k^2 - \triangle(\frac{v_k^2}{2}) + div(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u\nabla P \leqslant 0.$$

Next, let us consider the variables σ , t verifying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. Then, we have

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{v_k^2}{2})dx\,ds = \int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx - \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx.$
- $\int_\sigma^t \int_{\mathbb{R}^3} \triangle(\frac{v_k^2}{2})dx\,ds = 0.$
- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_k^2}{2}u)dx\,ds = 0.$

So, it is straightforward to see that

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_\sigma^t \int_{\mathbb{R}^3} d_k^2 dx\,ds \leqslant \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx + \int_\sigma^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds,$$

for any σ, t satisfying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. By taking the average over the variable σ, we yield

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_{T_k}^t \int_{\mathbb{R}^3} d_k^2 dx\,ds \leqslant 2^k \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^3} v_k^2(s,x)dx\,ds + \int_{T_{k-1}}^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds.$$

By taking the sup over $t \in [T_k, 1]$. the above inequality will give the following

$$U_k \leqslant 2^k \int_{Q_{k-1}} v_k^2 + \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|}u\nabla P dx|ds.$$

But, from Lemma 4.2 and Holder's inequality, we have

$$
\begin{aligned}
\int_{Q_{k-1}} v_k^2 = \int_{Q_{k-1}} v_k^2 \chi_{\{v_k>0\}} \\
&\leqslant (\int_{Q_{k-1}} v_k^{\frac{10}{3}})^{\frac{3}{5}} \|\chi_{\{v_k>0\}}\|_{L^{\frac{5}{2}}(Q_{k-1})} \\
&\leqslant \|v_k\|^2_{L^{\frac{10}{3}}(Q_{k-1})} 2^{\frac{4k}{3}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant \|v_{k-1}\|^2_{L^{\frac{10}{3}}(Q_{k-1})} 2^{\frac{4k}{3}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant C U_{k-1}^{\frac{5}{3}} 2^{\frac{4k}{3}}.
\end{aligned}
$$

As a result, we have the following conclusion

$$
U_k \leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla p dx| ds. \tag{4}
$$

Now, in order to estimate the term $\int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| ds$, we would like to carry out the following computation

$$
\begin{aligned}
-\triangle P = \sum \partial_i \partial_j (u_i u_j) \\
&= \sum \partial_i \partial_j \{(1 - \frac{v_k}{|u|}) u_i (1 - \frac{v_k}{|u|}) u_j + 2(1 - \frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j\} \\
&+ \sum \partial_i \partial_j \{\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j\}.
\end{aligned}
$$

This motivates us to decompose P as $P = P_{k1} + P_{k2}$, in which

$$
-\triangle P_{k1} = \sum \partial_i \partial_j \{(1 - \frac{v_k}{|u|}) u_i (1 - \frac{v_k}{|u|}) u_j + 2(1 - \frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j\},
$$

and that

$$
-\triangle P_{k2} = \sum \partial_i \partial_j \{\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j\}.
$$

First, we have to notice that:

$$
\begin{aligned}
|(1 - \frac{v_k}{|u|})^2 u_i u_j + 2(1 - \frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j| \\
&\leqslant (1 - \frac{1}{2^k}) \{(1 - \frac{v_k}{|u|}) |u_j| + 2 \frac{v_k}{|u|} |u_j|\} \\
&\leqslant (1 - \frac{v_k}{|u|}) |u_j| + 2 \frac{v_k}{|u|} |u_j| \\
&\leqslant 3|u_j| \leqslant 3|u|.
\end{aligned}
$$

So, by Riesz's Theorem in the theroy of singular operator, we yield

$$\|P_{k1}\|_{L^6(Q_{k-1})} \leqslant C_6\|3u\|_{L^6(Q_{k-1})} \leqslant 3C_6(\frac{1}{A})^{\frac{1}{6}} \leqslant 3C_6.$$

So, we have

$$\begin{aligned}
&\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k1} dx| ds \\
&\qquad = \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} P_{k1}\nabla(\frac{v_k}{|u|}u) dx| ds \\
&\qquad \leqslant 3\int_{T_{k-1}}^1 \int_{\mathbb{R}^3} d_k |P_{k1}| \chi_{\{v_k>0\}} dx\, ds \\
&\qquad \leqslant 3\|d_k\|_{L^2(Q_{k-1})}\|P_{k1}\|_{L^6(Q_{k-1})}\|\chi_{\{v_k>0\}}\|_{L^3(Q_{k-1})} \\
&\qquad \leqslant 3(2^{\frac{1}{2}})\|d_{k-1}\|_{L^2(Q_{k-1})} 3C_6 2^{\frac{10k}{9}} C^{\frac{1}{3}} U_{k-1}^{\frac{5}{9}} \\
&\qquad \leqslant 9(2^{\frac{1}{2}}) C_6 C^{\frac{1}{3}} U_{k-1}^{\frac{1}{2}} 2^{\frac{10k}{9}} U_{k-1}^{\frac{5}{9}}.
\end{aligned}$$

That is, we have the following conclusion that

$$\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k1} dx| ds \leqslant C 2^{\frac{10k}{9}} U_{k-1}^{\frac{19}{18}}. \tag{5}$$

Next, we would like to estimate the term $\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k2} dx| ds$. First, we recall that, by the very definition of P_{k2} ,we have

$$P_{k2} = \sum R_i R_j \{\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j\}.$$

, in which R_i , R_j etc are the Riesz's Transforms. Hence, we have

$$\nabla P_{k2} = \sum R_i R_j \{\nabla(\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j)\}.$$

Now, we notice that

$$\begin{aligned}
|\nabla(\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j)| &\leqslant |\nabla(\frac{v_k}{|u|} u_i)||\frac{v_k}{|u|} u_j| + \frac{v_k}{|u|}|u_i||\nabla(\frac{v_k}{|u|} u_j)| \\
&\leqslant 3d_k v_k + v_k(3d_k) \\
&= 6 v_k d_k.
\end{aligned}$$

So, by applying the Riesz's Theorem in the theory of Singular integral, we have

$$
\begin{aligned}
\|\nabla P_{k2}\|_{L^{\frac{3}{2}}(Q_{k-1})} \leqslant C_{\frac{3}{2}} \|v_k d_k\|_{L^{\frac{3}{2}}(Q_{k-1})} &\\
&\leqslant C_{\frac{3}{2}} \{ (\int_{Q_{k-1}} v_k^6)^{\frac{1}{4}} (\int_{Q_{k-1}} d_k^2)^{\frac{3}{4}} \}^{\frac{2}{3}} \\
&= C_{\frac{3}{2}} (\int_{Q_{k-1}} v_k^6)^{\frac{1}{6}} (\int_{Q_{k-1}} d_k^2)^{\frac{1}{2}} \\
&\leqslant C_{\frac{3}{2}} \|u\|_{L^6(Q_0)} \|d_k\|_{L^2(Q_{k-1})} \\
&\leqslant C_{\frac{3}{2}} (\frac{1}{A})^{\frac{1}{6}} (2)^{\frac{1}{2}} \|d_{k-1}\|_{L^2(Q_{k-1})} \\
&\leqslant 2^{\frac{1}{2}} C_{\frac{3}{2}} U_{k-1}^{\frac{1}{2}}.
\end{aligned}
$$

So, by applying the generalized Holder's inequality with exponents $\frac{10}{3}$, 30, $\frac{3}{2}$ to the terms v_k, $\chi_{\{v_k>0\}}$, ∇P_{k2} respectively, we yield

$$
\begin{aligned}
\int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P_{k2} dx | dt &\\
&\leqslant \int_{Q_{k-1}} v_k \chi_{\{v_k>0\}} |\nabla P_{k2}| dx\, dt \\
&\leqslant \|v_k\|_{L^{\frac{10}{3}}(Q_{k-1})} \|\chi_{\{v_k>0\}}\|_{L^{30}(Q_{k-1})} \|\nabla P_{k2}\|_{L^{\frac{3}{2}}(Q_{k-1})} \\
&\leqslant U_{k-1}^{\frac{1}{2}} 2^{\frac{k}{9}} C^{\frac{1}{30}} U_{k-1}^{\frac{5}{90}} 2^{\frac{1}{2}} C_{\frac{3}{2}} U_{k-1}^{\frac{1}{2}} \\
&= C 2^{\frac{k}{9}} U_{k-1}^{1+\frac{5}{90}} \\
&= C 2^{\frac{k}{9}} U_{k-1}^{\frac{19}{18}}.
\end{aligned}
$$

That is, we have

$$
\int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P_{k2} dx | dt \leqslant C 2^{\frac{k}{9}} U_{k-1}^{\frac{19}{18}}. \tag{6}
$$

So, by combining inequalities , we yield

$$
\begin{aligned}
U_k \leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | dt &\\
&\leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + C 2^{\frac{10k}{9}} U_{k-1}^{\frac{19}{18}} + C 2^{\frac{k}{9}} U_{k-1}^{\frac{19}{18}} \\
&\leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{19}{18}}.
\end{aligned}
$$

That is, we will have the result that

$$
U_k \leqslant C 2^{\frac{7k}{3}} U_{k-1}^{\frac{19}{18}},
$$

for any $k \geqslant 1$.

6 Proof of proposition 1.1

Now, we would like to establish proposition 1.1 on the foundation of proposition 1.2. To begin, let C^* be the positive universal constant occuring in proposition 1.2. First, let show the proposition in the special case $\lambda = 2$. We chose T to be an arbitrary chosen positive number greater than 2, and let u be a solution of the Navier-Stokes equation on $(0,\infty) \times \mathbb{R}^3$. In the case in which u satisfies the condition that $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6$, we define the function u^* by $u^*(s,x) = u(s+(T-1),x)$,which can be regarded to be another solution of the Navier-Stokes equation on $[-1,1] \times \mathbb{R}^3$ satsifying

$$\int_{-1}^{1} \int_{\mathbb{R}^3} |u^*|^6 dx\, ds = \int_{T-2}^{T} \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6.$$

Hence, we have $\|u^*\|_{L^6([-1,1]\times\mathbb{R}^3)} \leqslant C^*$. So, it follows from the conclusion of proposition 1.2 that $\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} = \|u^*(1,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$.
So, the above argument shows that

- we have $\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$, if $T > 2$, and u is a solution of the Navier-Stokes equation satisfying $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6$.

Next, we also need to deal with the case in which the solution u satisfies the condition that $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds > (C^*)^6$. In this case, let us consider the function u_ε defined by $u_\varepsilon(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x)$, in which $\varepsilon > 0$ is arbitrary. Then, by applying the change of variable formula, it is easy to see that

$$\int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx\, ds = \varepsilon \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds.$$

So, by taking $\varepsilon = (C^*)^6 \cdot \{2 \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\}^{-1}$, we yield

$$\int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx\, ds = \frac{(C^*)^6}{2} < (C^*)^6.$$

The last inequality signifies that the solution u_ε falls back to the first case in this discussion. Hence, it follows directly form the conclusion we made for the frist case that u_ε must satisfies $\|u_\varepsilon(\frac{T}{\varepsilon^2},\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$. So, we eventually have

$$\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} = \frac{1}{\varepsilon}\|u_\varepsilon(\frac{T}{\varepsilon^2},\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{1}{\varepsilon} = \frac{2}{(C^*)^6} \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds.$$

As a result, by all the discussion we made as above, we conclude that, no matter in which case, we always have the following inequality to be valid for any $T > 2$, and any solution u of the Navier-Stokes equation on $(0,\infty) \times \mathbb{R}^3$

$$\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\},$$

where A is the universal constant defined by $A = max\{1, \frac{2}{(C^*)^6}\}$. This gives the proof of Proposition 1.1 in the special case $\lambda = 2$.
Next, let λ be a fixed positive number satisfying $0 < \lambda < 2$. As usual, let u be a solution of the Navier-Stokes equation on $(0,\infty)\times\mathbb{R}^3$. Now, let us consider the function w which is defined by

$$w(t,x) = (\frac{\lambda}{2})^{\frac{1}{2}} u(\frac{\lambda}{2}t, (\frac{\lambda}{2})^{\frac{1}{2}}x).$$

Then, by applying the above case to w, we have the following estimation, which is valid for any $T > \lambda$.

$$\begin{aligned}\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant (\frac{2}{\lambda})^{\frac{1}{2}}\|w(\frac{2T}{\lambda},\cdot)\|_{L^\infty(\mathbb{R}^3)} \\ \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + \int_0^{\frac{2T}{\lambda}} \int_{\mathbb{R}^3} |w|^6 dx\, ds\} \\ \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + (\frac{\lambda}{2})^{\frac{1}{2}} \int_0^{T} \int_{\mathbb{R}^3} |u|^6 dx\, ds\} \\ \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + \int_0^{T} \int_{\mathbb{R}^3} |u|^6 dx\, ds\}.\end{aligned}$$

This gives proposition 1.1, where the universal constant A_λ is chosen to be $A_\lambda = (\frac{2}{\lambda})^{\frac{1}{2}} A$.

7 establishment of Theorem 1

Finally, we are now ready to establish the conclusion of Theorem 1 on the foundation of proposition 1.1. We make use of the following result due to Kato [8] (see also the book of Lemarié-Rieusset [10]).

Theorem 2. *Let $p > 3$. Then, for any given initial datum $u_0 \in L^p(\mathbb{R}^3)$ satisfying $div(u_0) = 0$, there exists a positive T^* and a unique weak solution $u \in C([0,T^*); L^p(\mathbb{R}^3))$ for the Navier-Stokes equation on $(0,T^*)\times\mathbb{R}^3$ so that $u(0,\cdot) = u_0$. This solution is then smooth on $(0,T^*)\times\mathbb{R}^3$. In addition, such a unique solution will also satisfies the extra condition that $u(t,\cdot) \in C_0(\mathbb{R}^3)$, for all $t \in (0,T^*)$.*

To begin, let u be a weak solution of the Navier-Stokes equation on $(0,\infty)\times\mathbb{R}^3$ satisfying the condition that $\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds < \infty$. Then, by using the elementary inequality $log(1+t) \leqslant t$, which is valid for all $t \geqslant 0$, we can deduce at once that

$$\int_0^\infty \int_{\mathbb{R}^3} |u|^4 dx\, ds \leqslant \int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds < \infty.$$

Now, let $\lambda \in (0,2)$ to be arbitrary chosen and fixed. Since $\int_0^\infty \int_{\mathbb{R}^3} |u|^4 dx\, ds < \infty$, it follows that the quantity $\int_{\mathbb{R}^3} |u(t,x)|^4 dx$ must be finite for almost every $t \in (0,\infty)$. So, with respect to λ, we can choose some τ_0 with $0 < \tau_0 < \lambda$ in such a way that $\int_{\mathbb{R}^3} |u(\tau_0,x)|^4 dx < \infty$, or equivalently $u(\tau_0,\cdot) \in L^4(\mathbb{R}^3)$. So, by using a simple shifting technique, we may apply the Kato's Theorem quoted as above to deduce that there exists some positive constant $T^* > \tau_0$ so that our weak solution u is smooth on $(\tau_0, T^*) \times \mathbb{R}^3$, and that $u(t,\cdot) \in C_0(\mathbb{R}^3)$, for every t with $\tau_0 < t < T^*$. Hence, we know, in particular, that our weak solution u must be lying in the space $L^\infty_{loc}(\tau_0, T^*; L^\infty(\mathbb{R}^3))$. Now, for some technical purpose, we would like to pick up two numbers τ_1 and τ_2 which verify the condition that $\tau_0 < \tau_1 < \tau_2 < min\{\lambda, T^*\}$. Once τ_1 and τ_2 are chosen, they will be fixed. Now, from our original weak solution u, we can construct another weak solution v by requiring that $v(t,x) = u(t+\tau_1, x)$. Now, by applying the conclusion of proposition 1.1 to the weak solution v and the number $\tau_2 - \tau_1$, we can at once deduce that we have the following inequality

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_0^t \int_{\mathbb{R}^3} |v|^6 dx\, ds\},$$

to be valid for all $t > \tau_2 - \tau_1$, in which A is some universal constant depending only on $\tau_2 - \tau_1$. However, this means the same as saying that we have the following inequality

$$\|u(t+\tau_1,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_{\tau_1}^{t+\tau_1} \int_{\mathbb{R}^3} |u|^6 dx\, ds\},$$

which is valid for all $t > \tau_2 - \tau_1$. Hence , it follows that we can make the following conclusion

- for every $t > \tau_2$, we have $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_{\tau_1}^t \int_{\mathbb{R}^3} |u|^6 dx\, ds\}$, in which A is some universal constant depending only on $\tau_2 - \tau_1$.

At this stage, we are ready to apply the Gronwall's argument in the theory of ordinary differential equations as follow. For this purpose, we take $\psi(t) = t \cdot log(1+t)$, which is a strictly increasing positive valued function on $(0,\infty)$ satisfying the condition that

$$\int_1^\infty \frac{1}{\psi(t)} dt = \infty.$$

Then, it follows from our last inequality that

$$\begin{aligned} &\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \\ &\qquad \leqslant A\{1 + \int_{\tau_1}^t \int_{\mathbb{R}^3} \psi(|u|) \frac{|u|^5}{log(1+|u|)} dx\, ds\} \\ &\qquad \leqslant A\{1 + \int_{\tau_1}^t \psi(\|u\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds\}, \end{aligned}$$

which is valid for all $t > \tau_2$.
Next ,we put $F(t) = \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}$. Then, the above inequality can be rewritten as

$$F(t) \leqslant A\{1 + \int_{\tau_1}^{t} \psi(F(s))G(s)ds\}, \qquad (7)$$

for all $t > \tau_2$, where G is the function defined by $G(s) = \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx$. Furthermore, we notice that by the hypothesis of Theorem 1, the function G must satisfies the condition that

$$\int_0^\infty G(s)ds = \int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds < \infty.$$

Here, for the sake of convenience, we define

$$H(t) = A\{1 + \int_{\tau_1}^{t} \psi(F(s))G(s)ds\},$$

for all $t > \tau_1$. Then, our last inequality can be rewritten as

- $F(t) \leqslant H(t)$, for all $t > \tau_2$.

Since ψ is a strictly increasing positive valued function on $(0, \infty)$, it follows at once that

$$\frac{dH}{dt} = A\psi(F(t))G(t) \leqslant A\psi(H(t))G(t),$$

which is valid for all $t > \tau_2$. That is, we have the fact that

- for every $t > \tau_2$, we have $\frac{dH}{dt} \leqslant A\psi(H(t))G(t)$.

As a result, by taking integration in time over the interval (τ_2, t), for $t > \tau_2$, it follows at once that

$$\Psi(H(t)) - \Psi(H(\tau_2)) \leqslant A \int_{\tau_2}^{t} G(s)ds,$$

for all $t > \tau_2$, in which Ψ is the function defined by $\Psi(y) = \int_A^y \frac{1}{\psi(y)} dy$. Hence, we can deduce that

- for every $t > \tau_2$, we have $\Psi(H(t)) \leqslant \Psi(H(\tau_2)) + A \int_{\tau_2}^{t} G(s)ds$.

At this stage, in order to complete the Gronwall's argument successfully, we definitely need to show that $H(\tau_2)$ is finite. To achive this, let us recall that we have already used the Kato's Theorem to deduce that our original weak solution u must satisfies $u \in L^\infty_{loc}(\tau_0, T^*; L^\infty(\mathbb{R}^3))$, and this at once tells us that $\|u\|_{L^\infty([\tau_1,\tau_2]\times\mathbb{R}^3)} = sup_{t\in[\tau_1,\tau_2]} F(t) < +\infty$, because of the fact that $0 < \tau_0 < \tau_1 < \tau_2 < min\{\lambda, T^*\}$. Hence, it follows immediately that

$$H(\tau_2) \leqslant A\{1+\psi(\|u\|_{L^\infty([\tau_1,\tau_2]\times\mathbb{R}^3)})\int_{\tau_1}^{\tau_2} G(s)ds\} < +\infty.$$

So, we can now combine $H(\tau_2) < \infty$, and $\int_0^\infty G(s)ds < \infty$ to deduce that

- for every $t > \tau_2$, $\Psi(H(t)) \leqslant \Psi(H(\tau_2)) + \int_{\tau_2}^t G(s)ds < \infty$.

That is, we now know that $\Psi(H(t))$ must be finite, for every $t > \tau_2$. Since $\int_A^{+\infty} \frac{1}{\psi(y)}dy = +\infty$, this will force us to admit that $H(t) < \infty$, for all $t > \tau_2$. Hence, we eventually have the conclusion

- for every $t > \tau_2$, we have $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} = F(t) \leqslant H(t) < \infty$. So, in particular, we now know also that $u \in L^\infty_{loc}(\tau_2, \infty; L^\infty(\mathbb{R}^3))$.

Since our weak solution u now satisfies the condition $u \in L^\infty_{loc}(\tau_2, \infty; L^\infty(\mathbb{R}^3))$, by applying the famous result of Serrin [16] that we mentioned in the introduction with the case in which $p = q = \infty$, $u \in L^\infty_{loc}((\tau_2, \infty) \times \mathbb{R}^3)$ immediately implies that $u \in C^\infty((\tau_2, \infty) \times \mathbb{R}^3)$, and hence we have the conclusion that u must be smooth on $(\lambda, \infty) \times \mathbb{R}^3$ (notice that $\tau_2 < \lambda$). Since $\lambda \in (0, 2)$ is arbitrary chosen in the above argument, we can finally deduce that any weak solution u satisfying the hypothesis of Theorem 1 must be smooth on $(0, \infty) \times \mathbb{R}^3$.

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