1. Consider the function $f_\alpha(x) = x^\alpha$ on $(0,1)$. For $\alpha > -1$ this function is integrable. Using the computer investigate the rate of convergence of the composite midpoint rule with equal subintervals for computing $\int_0^1 f_\alpha$ for various values of α. Let r_α denote the rate of convergence, i.e., the largest real number so that the error can be bounded by Ch^{r_α} for some constant C independent of h. Based on your experiments conjecture an expression for r_α in terms of α valid for all $\alpha > -1$. State your result precisely and prove it that it is indeed true.

2. Give a thorough analysis of Simpson's rule using the Peano kernel theorem. More specifically, there are four separate Peano kernel representations for the error in Simpson's rule depending on the degree of smoothness we assume of the integrand. Give all four in the case of the simple rule on the interval $[-1,1]$. Give explicit expressions for all four kernels and plot them (indicate the scale on the y axis). Apply this to analyze the error for the composite Simpson's rule on an arbitrary interval using equal subintervals under the assumptions that $f^{(i)}$ is bounded or just integrable for $i = 1, 2, 3$, or 4. For the case $f \in C^4$ also give the result for the composite rule without assuming equal subintervals.

3. Suppose that J_h is an approximation of a desired quantity I for which the asymptotic expansion $I \sim J(h) + c_1 h^{r_1} + c_2 h^{r_2} + \cdots$ holds as $h \to 0$. Here $0 < r_1 < r_2 < \cdots$ and the c_i are independent of h. Imagine that we have computed $J(h)$, $J(h/2)$, $J(h/4)$. Show how Richardson extrapolation can be used to the maximum extent to combine these values to get a higher order approximation to I. What is the order of this approximation?

4. Find the 1- and 2-point Gaussian quadrature rules for the weight function $\log(1/x)$ on $[0,1]$. Find expression for the errors.

5. The n-point Gauss-Lobatto quadrature rule ($n > 1$) is the rule $\int_{-1}^1 f \approx \sum_{i=1}^n w_i f(x_i)$ where the $x_1 = -1$, $x_n = 1$, and the other nodes and the weights are chosen so that the degree of precision is as high as possible. Determine the rule for $n = 2$, 3, and 4. Explain how, for general n, the points relate to the orthogonal polynomials with weight $1 - x^2$. Give a formula for the weights.