MATH 597I, Fall '00 Homework 3 Arnold

1. In proving the convergence of the conjugate gradient method for solving linear systems, a key step is showing that

$$(\star) \qquad \min_{\substack{p\in\mathcal{P}_n\\ p(0)=1}} \max_{x\in[a,b]} |p(x)| = \frac{2}{\left(\frac{1+\sqrt{a/b}}{1-\sqrt{a/b}}\right)^n + \left(\frac{1-\sqrt{a/b}}{1+\sqrt{a/b}}\right)^n}$$

   for $0 < a < b$. In fact, the polynomial for which the minimum is achieved is a scaled Chebyshev polynomial:

$$p(x) = T_n(\hat{x})/T_n\left(\frac{b+a}{b-a}\right), \text{ where } x = \frac{b+a}{2} - \frac{b-a}{2}\hat{x},$$

   and the right hand side of $(\star)$ is just $1/T_n\big((b+a)/(b-a)\big)$. Prove all this.

2. Let $f \in C^1([a,b])$. Prove that the cubic spline interpolant with derivative end conditions minimizes the quantity $\|g''\|_{L^2([a,b])}$ among all C^2 functions on $[a,b]$ which interpolate f at the x_i and f' at a and b. Since the second derivative is a measure of curvature, this says that in a certain sense the cubic spline interpolant is the straightest, or smoothest, function satisfying the interpolation conditions.

3. Let $x_0 < x_1 < \cdots < x_p$ and suppose that s is a cubic spline defined on all of $\mathbb{R}$ with breakpoints at the x_i (only), and for such that $s(x) \equiv 0$ if $x \le x_0$ or $x \ge x_p$. Prove that if $p \le 3$, then $s(x) \equiv 0$. In other words, there does not exist a nonzero cubic spline supported in just 3 intervals.

4. With the same notation as the previous problem show that that such a nonzero cubic spline $s(x)$ does exist if $p = 4$. Show that $s(x)$ is determined uniquely up to a constant multiple. With appropriate normalization $s(x)$ is called the cubic B-spline for the knots $x_0, \ldots x_4$.

5. Give the explicit formula for the cubic B-spline $B(x)$ with knots $x_i = i$, $i = 0, \ldots, 4$, normalized so that $\sum B(i) = 1$. Draw a plot of this function.

6. Let $\Pi_N = \{\, (a_i)_{i=-\infty}^{\infty} \,|\, a_i \in \mathbb{C}, a_{i+N} = a_i \,\}$ denote the space of bi-infinite N-periodic complex sequences. If $\boldsymbol{a}$, $\boldsymbol{b} \in \Pi_N$ we define the *convolution* $\boldsymbol{c} = \boldsymbol{a} * \boldsymbol{b}$ by

$$c_k = \sum_{j=0}^{N-1} a_j b_{k-j}.$$

   Prove that the discrete Fourier transform converts convolution into multiplication: $(\mathcal{F}_N\boldsymbol{c})_k = (\mathcal{F}_N\boldsymbol{a})_k(\mathcal{F}_N\boldsymbol{b})_k$.

7. Let p and q be polynomials of degree less than n, where n is a power of 2. Explain how the coefficients of the product pq can be computed from the coefficients of p and q in $O(n\log_2 n)$ operations.