MATH 597I, Fall '00 Homework 2 Arnold

Divided differences and Newton's formula

Notation: Let f be a real-valued function on an interval, and let $x_0, \dots, x_n$ denote $n+1$ distinct points in J. Let $p_k \in \mathcal{P}_k$ denote the Lagrange polynomial interpolating f at the first $k+1$ points $x_0, \dots, x_k$.

1. Prove that $p_n(x) - p_{n-1}(x) = c(x - x_0) \cdots (x - x_{n-1})$ for some constant c. We use the notation $f[x_0, \dots, x_n]$ to denote this constant and call it the *nth divided difference of f at the x_i*. Use Lagrange's formula for the interpolating polynomial to derive an expression for $f[x_0, \dots, x_n]$ in terms of x_i and $f(x_i)$.

2. Prove that $f[x_0, \dots, x_n]$ is a symmetric function of its $n+1$ arguments.

3. Prove the recursion relation

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0},$$

where, by convention, $f[x] := f(x)$. (This explains the terminology "divided difference".)

4. Give explicit formulas for $f[a]$, $f[a, b]$, $f[a, b, c]$, and $f[x, x+h, x+2h, \dots, x+nh]$.

5. Prove Newton's formula for the interpolating polynomial

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \cdots + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_{n-1}),$$

and the error formula

$$f(x) - p_n(x) = f[x_0, \dots, x_n, x](x - x_0) \cdots (x - x_n).$$

6. Prove that if $f \in C^n(J)$, then there exists a point ξ in the interior of J such that

$$f[x_0, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi).$$

7. Assuming that $f \in C^n(J)$, use the recursion defining the divided differences to establish the Hermite-Gennochi formula

$$f[x_0, \dots, x_n] = \int_{S_n} f^{(n)}(t_0 x_0 + t_1 x_1 + \cdots + t_n x_n)\, d\boldsymbol{t},$$

where

$$S_N = \left\{ \boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n \,|\, t_i \geq 0, \sum_1^n t_i \leq 1 \right\},$$

and $t_0 = 1 - \sum_1^n t_i$.

8. The Hermite-Gennochi formula shows that as a function of $n+1$ variables the nth divided difference extends to a function on all of J^{n+1} (the arguments need not be distinct). Find simple closed form expressions for $f[a, a]$, $f[a, a, b]$, and $f[a, a, b, b]$.