1. Show that for any $1 \le p < q \le \infty$ the L^p and L^q norms on $C(I)$ are not equivalent.

2. Let $f(x) = e^x$. a) For $p = 1$, 2, and ∞ find the best L^p approximation to f in $\mathcal{P}_0(I)$. b) For $p = 2$ and ∞ find the best L^p approximation to f in $\mathcal{P}_1(I)$.

3. Prove or disprove: if $f \in C([-1,1])$ is odd then a best approximation to f by odd polynomials of degree at most n is a best approximation to f among all polynomials of degree n.

4. Prove or disprove: if $f \in C([-1,1])$ has mean value zero, then a best approximation to f by polynomials of degree at most n with mean value zero is a best approximation to f among all polynomials of degree n.

5. State and prove the Jackson theorem in $C^k([a,b])$ paying attention to the dependence of the constant on the interval $[a,b]$. (You can use the Jackson theorems proved in class.)

6. If $f \in C(\mathbb{R})$ and $\delta > 0$ define $R_\delta f \in C(\mathbb{R})$ by

$$R_\delta f(x) = \frac{1}{\delta} \int_{x-\delta/2}^{x+\delta/2} f(t)\,dt.$$

Note that $R_\delta f \in C^1(\mathbb{R})$. Prove that $\|f - R_\delta f\| \le \omega(\delta)$, where ω denotes the modulus of continuity of f (i.e., $\omega(\delta)$ is the supremum of $|f(x) - f(y)|$ over x, y for which $|x - y| \le \delta$).

7. Let $f \in C_{2\pi}$ and let ω denote its modulus of continuity. Using the Jackson theorem in $C^1_{2\pi}$ and the regularization operator of the previous problem, prove that

$$\inf_{p \in \mathcal{T}_n} \|f - p\|_\infty \le c\,\omega\left(\frac{1}{n+1}\right).$$

Give an explicit expression for c.

8. Let $f \in C([-1,1])$ and let ω denote its modulus of continuity. Prove that

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_\infty \le c\,\omega\left(\frac{1}{n+1}\right).$$

Give an explicit expression for c.

9. Suppose that $f \in C([-1,1])$ satisfies the Holder condition $|f(x) - f(y)| \le M|x - y|^\alpha$ where $M, \alpha > 0$. What can you say about the rate of convergence of the best uniform approximation to f by polynomials of increasing degree?