BOUNDARY INTEGRAL EQUATIONS OF THE FIRST KIND FOR THE HEAT EQUATION

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INTRODUCTION

Boundary element methods are being applied with increasing frequency to time dependent problems, especially to boundary value problems for parabolic differential equations. Here we shall consider the heat equation as the prototype of such equations. Various types of integral equations arise when solving boundary value problems for the heat equation. An important one is the single layer heat potential operator equation, i.e., the Volterra integral equation of the first kind with the fundamental solution as kernel. This equation is not well understood. The fundamental questions of existence and uniqueness of solutions and continuous dependence of the solution on the data have thus far not been answered. Such an investigation is basic. It must precede any rigorous analysis of the convergence of numerical methods for the equation. In this paper we shall set out the proper mathematical framework and establish the well-posedness of the single layer heat potential operator equation.

We begin by recalling how this equation arises. The direct method for deriving an integral equation formulation for transient heat conduction begins with a representation of the temperature at any point in the spatial domain $\Omega \subset \mathbb{R}^3$ and any positive time in terms of the temperature and flux on the boundary for all previous times and the initial temperature. Let $u(x,t)$ denote the temperature at a point x in Ω, the spatial domain, and a time $t \geq 0$, and assume that the thermal diffusivity is scaled to unity, so that u satisfies the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \Delta\, u(x,t) = 0, \qquad x \in \Omega,\ t > 0.$$

Denote by $K(x,y)$ the fundamental solution for the heat equation,

$$K(x,t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)^{3/2}}, & x \in \mathbb{R}^3,\ t > 0, \\ 0, & x \in \mathbb{R}^3,\ t \leq 0. \end{cases} \tag{1}$$

Let Γ denote the boundary of Ω (which we assume for convenience to be smooth) and $n = n_y$ the unit outward normal to Ω at a point $y \in \Gamma$. The representation in question can then be written

$$\begin{aligned} &u(x,t) \\ &= \int_0^t \int_\Gamma \left[\frac{\partial u}{\partial n}(y,s) K(x-y,t-s) - u(y,s) \frac{\partial K}{\partial n_y}(x-y,t-s) \right] d\sigma_y ds \\ &\qquad + \int_\Omega u(y,0) K(x-y,t)\, dy, \qquad x \in \Omega,\ t > 0. \end{aligned} \tag{2}$$

This can be verified simply by using Green's theorem. See, e.g., Pina and Fernandez.[1] The quantity

$$U(x,t) := \int_0^t \int_\Gamma q(y,s) K(x-y,t-s)\, d\sigma_y ds, \qquad x \in \mathbb{R}^3,\ t \geq 0, \tag{3}$$

which occurs in Equation (2) (with $q = \partial u / \partial n$) is called the *single layer heat potential* with density q. Assuming that the density is continuous on Γ, the single layer heat potential defines a continuous function for all $x \in \mathbb{R}^3$ and all $t \geq 0$ which satisfies the heat equation everywhere except for x in Γ and which vanishes when $t = 0$. The derivatives of the single layer potential, however, are discontinuous on Γ. In fact for any $x \in \Gamma$ the classical *jump relation* states that

$$\lim_{z \to x} \frac{\partial U}{\partial n}(z,t) = \pm \frac{1}{2} q(x,t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial n_x}(x-y,t-s)\, d\sigma_y ds. \tag{4}$$

The plus sign holds in Equation (4) if z tends to x from within Ω (nontangentially to Γ), while the minus sign holds if z tends to x from outside Ω. (The coefficient $1/2$ depends on the fact that the boundary Γ is smooth. If x were an edge or a conical point on the boundary this coefficient would assume another value.) A proof of the jump relation can be found, for example, in Friedman,[2] Chapter 5. The *double layer heat potential*

$$V(x,t) := \int_0^t \int_\Gamma q(y,s) \frac{\partial K}{\partial n_y}(x-y,t-s)\, d\sigma_y ds, \qquad x \in \mathbb{R}^3,\ t \geq 0,$$

satisfies a similar jump relation:

$$\lim_{z \to x} V(z,t) = \mp \frac{1}{2} q(x,t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial n_y}(x-y,t-s)\, d\sigma_y ds, \tag{5}$$

where now the plus sign holds for the limit from the exterior. Now let the point x in Equation (2) tend to a point on the boundary. Using Equation (5), we find that

$$\frac{1}{2}u(x,t) = \int_0^t \int_\Gamma \left[\frac{\partial u}{\partial n}(y,s)K(x-y,t-s) - u(y,s)\frac{\partial K}{\partial n_y}(x-y,t-s)\right] d\sigma_y ds + \int_\Omega u(y,0)K(x-y,t)\,dy, \qquad x \in \Gamma,\ t > 0. \quad (6)$$

The boundary integral equation Equation (6) relates the temperature u and flux $\partial u/\partial n$ on the lateral boundary $\Gamma \times [0,T]$ of the space-time cylinder and the initial data $u(\cdot,0)$. Now, for the standard initial-boundary value problems the initial data is known and at each point of the lateral boundary either the temperature or the flux is known. (Or, in the case of a Robin problem, one may be expressed in terms of the other). Then Equation (6) may be used to determine the unknown quantity, either the boundary temperature or flux. Once both the boundary temperature and flux are known the integral representation in Equation (2) determines the temperature everywhere.

If the flux is known (i.e., if we are dealing a Neumann problem), then Equation (6) is a Volterra integral equation of the *second* kind with the double layer heat potential kernel, $\partial K/\partial n$, for the unknown value of the temperature u on Γ. The theory for such second kind equations is quite well developed. In particular it is known that the equation admits a unique solution which can be written as a convergent Neumann series. See, for example, Chapter 5 of Friedman[2] or Chapter 13 of Pogorzelski.[3] Similar considerations are valid for the Robin problem, for which the flux is given in terms of the boundary temperature. This leads again to a second kind equation with only a slightly more complicated kernel. The arguments involved in establishing the well-posedness of the second kind equations can also be applied to the study of collocation and Galerkin methods for their numerical solution. Such an analysis has been carried out in some generality by Costabel, Onishi, and Wendland.[4]

For the Dirichlet problem, in which the temperature is given on the boundary, Equation (6) becomes a Volterra integral equation of the *first* kind for the unknown flux with the kernel K. The theory of first kind integral equations is much less straightforward than that for second kind equations. The main result of this paper is to show that in fact this equation has a unique solution, and to determine norms for the solution and data for which continuous dependence can be shown.

Before entering into the technical detail that will be needed to accomplish this goal, we shall describe the nature of the result informally. Since the time of Hadamard a problem has been called *well-posed* if for all data in some reasonable class there exists one and only one solution in some other class, and if the map which associates the solution to the data is continuous. To formalize this notion for a particular problem, we must specify the classes in which the data and solution are to lie, and, most importantly, we must specify in what sense the solution operator is continuous. We shall use a Hilbert space setting. That is, we shall define two Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, say, whose elements are functions (or distributions) on $\Gamma \times [0,T]$. We shall define an operator, L, which associates to each function q in $\mathcal{H}_1$ a function Lq in $\mathcal{H}_2$ in a continuous fashion. This operator L will be the single layer heat potential operator in the sense that if q is a nice function on $\Gamma \times [0,T]$, then Lq will be the restriction to $\Gamma \times [0,T]$ of the function U in Equation (3). In this operator notation, our problem may be stated thus: given a function g in $\mathcal{H}_2$ find a function q in $\mathcal{H}_1$ such that

$$Lq = g. \tag{7}$$

We shall show that for every function g Equation (7) has one and only one solution u. Further we shall have that the solution operator which send g to u, namely the operator L^{-1}, is continuous. In other words we shall show that L is an *isomorphism* from the Hilbert space $\mathcal{H}_1$ onto the Hilbert space $\mathcal{H}_2$.

THE SINGLE LAYER POTENTIAL

The standard single layer potential plays the same role for electrostatic problems as the single layer heat potential for heat conduction problems. In this context, too, the question arises: in what sense is the single layer potential operator equation well-posed? This question has been answered by several authors. See Hsiao and McCamy[5] and Nedelec and Planchard.[6] The answer and the arguments used to derive it have been essential to establishing the basic convergence theory for boundary element methods for potential problems. Here we shall briefly describe a simplified version of the argument in Nedelec and Planchard, since it allows us to introduce in simple setting some (but not all!) of the essential ideas for the single layer heat potential.

For $\phi \in C_0^\infty(\mathbb{R}^3)$ we define $\|\phi\|_W = \|\nabla \phi\|_{L^2(\mathbb{R}^3)}$ and let W denote the closure of $C_0^\infty(\mathbb{R}^3)$ in this norm. Then W is a Hilbert space. Since $C_0^\infty(\mathbb{R}^3)$ is dense in W, the dual space, W^*, of W may be identified with a space of distributions on $\mathbb{R}^3$. It can be shown using Hardy's inequality (Hardy, Littlewood, and Polya,[7] Theorem 328) that

$$\int_{\mathbb{R}^3} \frac{|\phi(x)|^2}{|x|^2}\, dx \leq 4\|\phi\|_W^2 \tag{8}$$

for all $\phi \in W$. Note that the space W is *not* contained in $L^2(\mathbb{R}^3)$ and correspondingly W^* does not contain $L^2(\mathbb{R}^3)$. However, it follows from the inequality (8) that all functions that are square integrable with respect to the measure $|x|^2 dx$ are contained in W^*, and in particular $C_0^\infty(\mathbb{R}^3) \subset W^*$. For $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\phi \in W$ the duality pairing $\langle \phi, \psi \rangle_{W \times W^*}$ and the integral $\int_{\mathbb{R}^3} \phi\psi$ coincide. We shall follow convention in using the integral notation for such duality pairings even when the distribution is not a locally integrable function.

The usual Sobolev space $H^1(\mathbb{R}^3)$ is a dense subspace of W with continuous inclusion. These two spaces differ only in the permitted behavior near infinity. Intuitively, if $\phi \in H^1(\mathbb{R}^3)$, then—since ϕ is square integrable—$\phi(x)$ decays at least as fast as $|x|^{-3/2}$ as $|x| \to \infty$. However functions in W—which need only have square integrable gradient—may decay as slowly as $|x|^{-1/2}$. For any bounded smooth region K, though, the set of restrictions of functions in W to K coincides with $H^1(K)$. Consequently we may carry over the trace theory for Sobolev spaces to the space W. It is well known that for any bounded smooth surface Γ, the trace operator, which extends the restriction operator from $C_0^\infty(\mathbb{R}^3)$, maps $H^1(\mathbb{R}^3)$ boundedly onto $H^{1/2}(\Gamma)$. (For example, see Lions and Magenes,[8] Theorem 8.3.) Consequently, the trace operator also defines a continous linear operator of W onto $H^{1/2}(\Gamma)$.

Let Ω denote a smoothly bounded region in $\mathbb{R}^3$ with boundary Γ and denote by γ the trace operator on W. We use the notation δ_q for $\gamma^* q$, the image of q under the adjoint operator, so

$$\langle \delta_q, v \rangle = \langle q, \gamma v \rangle = \int_\Gamma qv\, d\sigma_x \qquad \text{for all} \quad q \in H^{-1/2}(\Gamma),\ v \in W.$$

Since $\gamma : W \to H^{1/2}(\Gamma)$ is surjective, its adjoint maps $H^{-1/2}(\Gamma)$ isomorphically onto a closed subspace of W^*, i.e., there is a constant C depending only on Γ such that

$$C^{-1}\|\delta_q\|_{W^*} \leq \|q\|_{H^{-1/2}(\Gamma)} \leq C\|\delta_q\|_{W^*} \qquad \text{for all} \quad q \in H^{-1/2}(\Gamma).$$

Let u_q be the unique solution to the variational problem

$$\int_{\mathbb{R}^3} \nabla u_q \nabla v \, dx = \langle \delta_q, v \rangle \qquad \text{for all} \quad v \in W, \tag{9}$$

and define $L : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by $Lq = \gamma u_q$. It follows directly from this definition that L is a continuous self-adjoint linear operator. Since Equation (9) is the variational formulation of Poisson's problem on $\mathbb{R}^3$, the solution may be written as a convolution with the fundamental solution of the Laplacian, i.e.,

$$u_q(x) = \langle N_x, \delta_q \rangle,$$

where $N_x(y) = 1/(4\pi|x-y|)$. In particular, if q is a smooth function, then

$$Lq(x) = \frac{1}{4\pi} \int_\Gamma \frac{q(y)}{|x-y|} \, d\sigma_y. \tag{10}$$

Choosing $v = u_q$ in Equation (9) we get

$$\langle q, Lq \rangle = \|u_q\|_W^2 = \|\delta_q\|_{W^*}^2 \geq C^{-1} \|q\|_H^{-1/2}(\Gamma).$$

Consequently the mapping $q \mapsto \langle q, Lq \rangle$ is an innerproduct on $H^{-1/2}(\Gamma)$ which gives rise to an equivalent norm. This implies that L maps $H^{-1/2}(\Gamma)$ isomorphically onto $H^{1/2}(\Gamma)$. Summarizing these considerations, we have proven the following theorem.

THEOREM 1. *The single layer potential operator defined in Equation* (10) *extends to a bounded self-adjoint linear operator L from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ which is one-to-one and onto. Moreover the associated bilinear form $q \mapsto \langle q, Lq \rangle$ is innerproduct on $H^{-1/2}(\Gamma)$ equivalent to the usual one.*

THE INITIAL VALUE PROBLEM

In order to develop a Hilbert space theory for the single layer heat potential, we shall need to know the basic results of the Hilbert space theory for the heat operator itself. For a more complete version of the theory, including the case of nonzero initial data, see Lions and Magenes,[8] Chapter 3.

The Hilbert space approach to the heat equation may be based on the following variant of the projection lemma due to Lions.[9]

Lemma 2. *Let H be a Hilbert space, Φ a subspace of H, and $\Lambda : \Phi \to H^*$ a linear operator. (It is not assumed that Φ is closed nor that Λ is continuous). Assume that there exists $\epsilon > 0$ such that*

$$\langle \Lambda\phi, \phi\rangle \geq \epsilon \|\phi\|_H^2 \qquad \text{for all} \quad \phi \in \Phi. \tag{11}$$

Then for all $F \in H^*$ there exists $u \in H$ such that

$$\langle \Lambda\phi, u\rangle = \langle F, \phi\rangle \qquad \text{for all} \quad \phi \in \Phi.$$

Moreover $\|u\|_H \leq \epsilon^{-1}\|F\|_{H^*}$.

In applying this formulation to the heat equation we shall require several norms and spaces of functions defined on $\mathbb{R}^3 \times (0,T)$. We define

$$S = L^2(0,T;W), \qquad V = \left\{\, u \in L^2(0,T;W) : \partial u/\partial t \in L^2(0,T;W^*) \,\right\},$$

with the associated norms

$$\|u\|_S = \left(\int_0^T \|u\|_W^2 \, dt\right)^{1/2} = \left(\int_0^T \int_{\mathbb{R}^3} |\nabla u(x)|^2 \, dx dt\right)^{1/2}$$

and

$$\|u\|_V = \left(\int_0^T (\|u\|_W^2 + \|\frac{\partial u}{\partial t}\|_{W^*}^2) \, dt\right)^{1/2}.$$

It can be shown that every $u \in V$ maps $[0,T]$ continuously into $L^2(\mathbb{R}^3)$, with

$$\sup_{t\in([0,T]} \|u(.,t)\|_{L^2(\mathbb{R}^3)} \leq C\|u\|_W.$$

(See Theorem 3.1 in Chapter 1 of Lions and Magenes[8] for the proof of a similar result.) Consequently, for each $t \in [0,T]$, we may define the closed subspace

$$V^{(t)} = \{\, u \in V : u(\cdot,t) = 0 \,\}.$$

We have $V^{(t)} \subset S \subset L^2(\mathbb{R}^3 \times [0,T])$ with continuous inclusions, each space being dense in the next. Identifying $L^2(\mathbb{R}^3 \times [0,T])$ with its dual, we then have $L^2(\mathbb{R}^3 \times [0,T]) \subset S^* \subset V^{(t)*}$.

THEOREM 3. *For all $f \in S^*$ there exists a unique $u \in V$ such that*

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{on } \mathbb{R}^3 \times (0,T), \tag{12}$$

$$u = 0 \quad \text{on } \mathbb{R}^3 \times \{0\}. \tag{13}$$

Moreover, there exists a constant C such that

$$\|u\|_V \le C\|f\|_{S^*}.$$

Proof. We begin by applying Lemma 2 with $H = S$, $\Phi = V^{(T)}$, and

$$\langle \Lambda v, u\rangle = \int_0^T \int_{\mathbb{R}^3} (-u\frac{\partial v}{\partial t} + \nabla u \nabla v)\, dxdt \qquad \text{for all} \quad v \in V^{(T)},\ u \in S.$$

Note that

$$\langle \Lambda v, v\rangle = \frac{1}{2}\|v(\cdot,0)\|^2_{L^2(\mathbb{R}^3)} + \|v\|^2_S,$$

so the inequality (11) holds with $\epsilon = 1$. We conclude that for all $F \in S^*$ there exists $u \in S$ with $\|u\|_S \le \|f\|_{S^*}$ satisfying

$$\int_0^T \int_{\mathbb{R}^3} (-u\frac{\partial v}{\partial t} + \nabla u \nabla v)\, dxdt = \langle f, v\rangle \qquad \text{for all} \quad v \in V^{(T)}. \tag{14}$$

Now Equation (14) implies that $\partial u/\partial t$ exists in the sense of distributions and equals $\Delta u + f \in S^*$, whence $u \in V$ and $\|\partial u/\partial t\|_{S^*} \le 2\|f\|_{S^*}$. Then Equation (14) also implies that $u(\cdot,0) = 0$. Thus we have shown existence of the solution and the desired *a priori* estimate. To show uniqueness of the solution, it suffices to show that if $\partial u/\partial t - \Delta u = 0$ for some u in $V^{(0)}$, then $u = 0$. Pairing the differential equation with u gives $\|u(T)\|^2_{L^2(\mathbb{R}^3)} + \|u\|^2_S = 0$, so u indeed vanishes. This completes the proof.

Define a bilinear form $B : V^{(0)} \times S \to \mathbb{R}$ by

$$B(u,v) = \int_0^T \int_{\mathbb{R}^3} (\frac{\partial u}{\partial t}v + \nabla u \nabla v)\, dxdt.$$

Then Theorem 3 shows that the correspondence $u \mapsto B(u,\cdot)$ is a linear isomorphism of $V^{(0)}$ onto S^*. In the exactly the same way, we can show that the bilinear form

$$B'(v,u) = \int_0^T \int_{\mathbb{R}^3} (-u\frac{\partial v}{\partial t} + \nabla u \nabla v)\, dxdt,$$

defines an isomorphism, $v \mapsto B'(v,\cdot)$, of $V^{(T)}$ onto S^*. It follows that the adjoint operator is an isomorphism of S onto $V^{(T)*}$. This adjoint operator is given by the correspondence $u \mapsto B'(\cdot,u)$. Thus we have the following result.

THEOREM 4. *For all $f \in V^{(T)*}$ there exists a unique $u \in S$ such that Equation* (14) *holds. Moreover, there exists a constant C such that*

$$\|u\|_S \leq C\|f\|_{V^{(T)*}}.$$

Theorem 4 shows that even when f is only in $V^{(T)*}$, a sense can be given to the initial value problem for which it is well-posed. If $f \in S^*$, then the solution $u \in V^{(0)}$ to Equations (12)-(13), whose existence is guaranteed by Theorem 3, satisfies Equation (14), and so coincides with the solution of Theorem 4. In short, the heat operator defines an isomorphism of S onto $V^{(T)*}$. Its restriction to $V^{(0)}$ defines an isomorphism of this spaces onto S^*.

THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

To discuss the heat equation with Dirichlet boundary conditions in a variational setting, we must give a sense to the the boundary values of functions in S and V. Since such functions are not continous, it is not immediately obvious how to do this. Fortunately it is well understood how to define such traces. The spaces S and V differ from the spaces $L^2(0,T;H^1(\mathbb{R}^3))$ and $L^2(0,T;H^1(\mathbb{R}^3))\cap H^1(0,T;H^{-1}(\mathbb{R}^3))$, respectively, only in the behavior near infinity. In light of this we can easily extend the known theory of traces for the latter spaces to the former. Let Γ be a smooth bounded surface in $\mathbb{R}^3$. Recall from the last section that the trace operator maps W boundedly onto $H^{1/2}(\Gamma)$. Consequently, if we define $\mathcal{S} = L^2(0,T;H^{1/2}(\Gamma))$, then the restriction operator from $C_0^\infty(\mathbb{R}^3\times[0,T])$ to $C^\infty(\Gamma\times[0,T])$ extends to a bounded linear operator of S onto $\mathcal{S}$. Define $\mathcal{V} = L^2(0,T;H^{1/2}(\Gamma))\cap H^{1/4}(0,T;L^2(\Gamma))$. Then it can be shown that for any t, this trace operator maps $V^{(t)}$ boundedly onto $\mathcal{V}$. This follows from Lions and Magenes,[10] Chapter 4, §15.5. We denote by S_Γ, V_Γ, $V_\Gamma^{(t)}$ the subspaces of functions in S, V, and $V^{(t)}$, respectively, which vanish on Γ, i.e., which are in the kernel of the trace operator.

We continue to denote by Ω a smoothly bounded region in $\mathbb{R}^3$ with boundary Γ. We shall simultaneously consider the Dirichlet problems for the heat equation in Ω and in the exterior domain $\mathbb{R}^3\setminus\bar{\Omega}$. Thus, given functions f on $\mathbb{R}^3\times(0,T)$ and g on $\Gamma\times(0,T)$, we wish to find a function u on $\mathbb{R}^3\times[0,T]$ which satisfies

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{on } (\mathbb{R}^3\setminus\Gamma)\times(0,T), \tag{15}$$

$$u = g \quad \text{on } \Gamma\times(0,T), \tag{16}$$

$$u = 0 \quad \text{on } \mathbb{R}^3\times\{0\}. \tag{17}$$

We first consider the case of zero Dirichlet data. As above we can show the well-posedness of two distinct formulations of the problem.

THEOREM 5. *For all $f \in S_\Gamma^*$ there exists a unique $u \in V$ such that*

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta\, u = f \quad & \text{on } (\mathbb{R}^3 \setminus \Gamma) \times (0, T), \\ u = 0 \quad & \text{on } \Gamma \times (0, T), \\ u = 0 \quad & \text{on } \mathbb{R}^3 \times \{0\}. \end{aligned}$$

Moreover, there exists a constant C such that

$$\|u\|_V^{(0)} \leq C \|f\|_{S_\Gamma^*}.$$

The proof is entirely analogous to that of Theorem 3. In this case Lemma 2 is applied with $H = S_\Gamma$, $\Phi = V_\Gamma^{(T)}$.

Arguing as in the proof of Theorem 4 we can also prove well-posedness of a weaker formulation of the Dirichlet problem.

THEOREM 6. *For all $f \in V_\Gamma^{(T)*}$ there exists a unique $u \in S_\Gamma$ such that*

$$\int_0^T \int_{\mathbb{R}^3} (-u \frac{\partial v}{\partial t} + \nabla\, u\, \nabla\, v)\, dx dt = \langle f, v \rangle \qquad \text{for all} \quad v \in V_\Gamma^{(T)}.$$

Moreover, there exists a constant C such that

$$\|u\|_S \leq C \|f\|_{V_\Gamma^{(T)*}}.$$

It is an easy matter to extend this result to inhomogeneous Dirichlet data. Given $g \in \mathcal{V}$ we may find $u_1 \in V^{(0)}$ such that $u_1 = g$ on Γ. Defining u_2 as the unique solution of the heat equation

$$\frac{\partial u_2}{\partial t} - \Delta\, u_2 = f - \frac{\partial u_1}{\partial t} + \Delta\, u_1$$

with homogeneous Dirichlet boundary conditions and homogeneous initial condition, we have a solution $u = u_1 + u_2 \in V$ to Equations (15)–(17). That this solution is unique follows immediately from Theorem 5. In a similar way we may extend Theorem 6 to the case of inhomogeneous Dirichlet data. These two results are stated in the following theorems.

THEOREM 7. *For all* $f \in S_\Gamma^*$, $g \in \mathcal{V}$ *there exists a unique* $u \in V$ *such that Equations* (15)–(17) *hold. Moreover, there exists a constant* C *such that*

$$\|u\|_V \leq C(\|f\|_{S_\Gamma^*} + \|g\|_{\mathcal{V}}). \tag{18}$$

THEOREM 8. *For all* $f \in V_\Gamma^{(T)*}$, $g \in \mathcal{S}$, *there exists a unique* $u \in S$ *such that the Dirichlet condition* (16) *is satisfied and*

$$\int_0^T \int_{\mathbb{R}^3} (-u\frac{\partial v}{\partial t} + \nabla u \nabla v)\, dxdt = \langle f, v\rangle \qquad \textit{for all} \quad v \in V_\Gamma^{(T)}.$$

Moreover, there exists a constant C *such that*

$$\|u\|_S \leq C(\|f\|_{V_\Gamma^{(T)*}} + \|g\|_{\mathcal{S}}).$$

As in the case of the initial value problem, we may regard Theorem 7 as a regularity theorem for the weak solution guaranteed by Theorem 6.

THEOREM 9. *If* $f \in S_\Gamma^*$ *and* $g \in \mathcal{V}$, *then the function* u *of Theorem 8 belongs to* V *and satisfies Equations* (15)–(17) *and the inequality* (18).

THE SINGLE LAYER HEAT POTENTIAL

Let

$$u_q(x,t) = \int_0^t \int_\Gamma q(y,s) K(x-y, t-s)\, d\sigma_y dt, \qquad x \in \mathbb{R}^3,\ t > 0,$$

denote the single layer heat potential with density q. (Recall that K denotes the fundamental solution, given explicitly in Equation (1)). We define the boundary integral operator

$$Lq(x,t) = \int_0^t \int_\Gamma K(x-y, t-s) q(y,s)\, d\sigma_y dt, \qquad x \in \Gamma,\ t \in (0,T). \tag{19}$$

Thus Lq is just the restriction of u_q to $\Gamma \times [0,T]$. In this section we will show that the boundary integral equation

$$Lq(x,t) = g(x,t) \qquad \text{for all} \quad x \in \Gamma,\ t \in (0,T)$$

admits a unique solution for a wide class of functions g. More precisely, we shall prove the following theorem, which is the analogue of Theorem 1 for the single layer heat potential, and which constitutes the main result of this paper.

THEOREM 10. *The single layer heat potential operator L defined in Equation* (19) *extends to a bounded linear operator which maps $\mathcal{S}^*$ isomorphically onto $\mathcal{V}$ and $\mathcal{V}^*$ isomorphically onto $\mathcal{S}$. Moreover, there is a constant $\epsilon > 0$ such that*

$$\langle q, Lq \rangle \geq \epsilon \|q\|_{\mathcal{V}^*}^2 \qquad \textit{for all} \quad q \in \mathcal{S}^*. \tag{20}$$

Now the distributions in $\mathcal{V}^*$ are not necessarily functions, and the integral in Equation (19) need not make sense. So our first task is to give a sense to Lq for $q \in \mathcal{V}^*$. Now when q is smooth, u_q satisfies the heat equation on Ω and vanishes for $t = 0$. Therefore for any smooth function v on $\mathbb{R}^3 \times [0, T]$,

$$\begin{aligned} 0 = \int_0^T \int_\Omega (\frac{\partial u_q}{\partial t} - \Delta\, u) v \, dx dt = \int_0^T \int_\Omega (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \, \nabla\, v) \, dx dt \\ + \int_\Omega u(x, T) v(x, T) \, dx - \int_0^T \int_\Gamma \frac{\partial u_q}{\partial n_-} v \, d\sigma_x dt. \end{aligned} \tag{21}$$

Here $\partial u_q / \partial n_-$ denotes the outward normal derivative of $u_q|_\Omega$. Supposing that v has compact support, or at least decays sufficiently rapidly near infinity, we may perform a similar integration by parts on the complementary domain $\Omega^c = \mathbb{R}^3 \setminus \bar{\Omega}$ to get

$$\begin{aligned} 0 = \int_0^T \int_{\Omega^c} (\frac{\partial u_q}{\partial t} - \Delta\, u) v \, dx dt = \int_0^T \int_{\Omega^c} (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \, \nabla\, v) \, dx dt \\ + \int_{\Omega^c} u(x, T) v(x, T) \, dx + \int_0^T \int_\Gamma \frac{\partial u_q}{\partial n_+} v \, d\sigma_x dt, \end{aligned} \tag{22}$$

where $\partial u_q / \partial n_+$ denotes the outward normal derivative of $u_q|_{\Omega^c}$. Now, from Equation (4),

$$q = [\partial u / \partial n] = \frac{\partial u_q}{\partial n_-} - \frac{\partial u_q}{\partial n_+}.$$

Thus, adding Equations (21) and (22), we see that

$$\int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \, \nabla\, v) \, dx dt = \int_0^T \int_\Gamma q v \, d\sigma_x dt - \int_{\mathbb{R}^3} u(x, T) v(x, T) \, dx$$

If we further assume that v vanishes for $t = T$ we have

$$\int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \, \nabla\, v) \, dx dt = \int_0^T \int_\Gamma q v \, d\sigma_x dt. \tag{23}$$

Although we have derived Equation (23) under the assumption that v is smooth and vanishes when $t = T$, it is not hard to show that it holds for all $v \in V^{(T)}$.

Now suppose only that $q \in \mathcal{V}^*$ and define $\delta_q \in V^{(T)*}$ by $\langle \delta_q, v\rangle = \langle q, \gamma v\rangle$ where $\gamma : V \to \mathcal{V}$ is the trace operator. Then $C^{-1}\|\delta_q\|_{V^{(T)*}} \leq \|q\|_{\mathcal{V}^*} \leq C\|\delta_q\|_{V^{(T)*}}$, for some constant C depending only on Γ and T. We define $u_q \in S$ by

$$\int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \,\nabla\, v)\, dxdt = \langle \delta_q, v\rangle \qquad \text{for all} \quad v \in V^{(T)}. \quad (24)$$

By Theorem 4, u_q is uniquely determined. Moreover $C^{-1}\|u_q\|_S \leq \|q\|_{\mathcal{V}^*} \leq C\|u_q\|_S$. Note that Equation (24) agrees with Equation (23) if q is smooth. Thus we set $Lq = \gamma u_q$. By construction L defines a bounded linear operator from $\mathcal{V}^*$ to $\mathcal{S}$. Note that if $q \in \mathcal{S}^* \subset \mathcal{V}^*$, then $\delta_q \in S^*$, $u_q \in V^{(0)}$, and $Lq \in \mathcal{V}$. In this case

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial u_q}{\partial t} v + \nabla\, u_q \,\nabla\, v)\, dxdt = \langle \delta_q, v\rangle \qquad \text{for all} \quad v \in S,$$

and, taking $v = u_q$ we get

$$\begin{aligned}\langle q, Lq\rangle &= \int_0^T \int_{\mathbb{R}^3} (\frac{\partial u_q}{\partial t} u_q + |\,\nabla\, u_q|^2)\, dxdt \\ &= \frac{1}{2}\|u_q(\cdot, T)\|^2_{L^2(\mathbb{R}^3)} + \|u_q\|^2_S \geq C^{-1}\|q\|^2_{\mathcal{V}^*},\end{aligned}$$

i.e., the inequality (20) holds. We summarize these considerations in the following theorem.

THEOREM 11. *The single layer heat potential operator defined by Equation* (19) *for smooth q extends to a bounded linear operator $L : \mathcal{V}^* \to \mathcal{S}$ which also maps $\mathcal{S}^*$ boundedly into $\mathcal{V}$. Moreover there is a constant $\epsilon > 0$ such that the inequality* (20) *holds.*

We may now apply Lemma 2 with $H = \mathcal{V}^*$, $\Phi = \mathcal{S}^*$, and $\Lambda = L$. In light of Theorem 11, the hypotheses of the lemma are fulfilled. We deduce that for all $g \in \mathcal{V}$ there exists $p \in \mathcal{V}^*$ such that

$$\langle p, Lq\rangle = \langle g, q\rangle \qquad \text{for all} \quad q \in \mathcal{S}^*.$$

We require the analogous result for the adjoint operator also. Namely, for $p \in \mathcal{V}^*$ we define $L'p \in \mathcal{S}$ as follows. Define $v_p \in S$ by the equation

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial u}{\partial t} v_p + \nabla u \nabla v_p)\, dxdt = \langle u, \delta_p \rangle \qquad \text{for all} \quad u \in V^{(0)},$$

and set $L'p = \gamma v_p$. If $p \in \mathcal{S}^*$ then $v_p \in V^{(T)}$, $L'p \in \mathcal{V}$, and

$$\int_0^T \int_{\mathbb{R}^3} (-u \frac{\partial v_p}{\partial t} + \nabla u \nabla v_p)\, dxdt = \langle u, \delta_p \rangle \qquad \text{for all} \quad u \in S. \tag{25}$$

Thus, like L, L' defines a bounded linear operator from $\mathcal{V}^*$ to $\mathcal{S}$ which maps $\mathcal{S}^*$ into $\mathcal{V}$. Applying Lemma 2 as before we conclude that for all $h \in \mathcal{V}$ there exists $q \in \mathcal{V}^*$ such that

$$\langle L'p, q \rangle = \langle p, h \rangle \qquad \text{for all} \quad p \in \mathcal{S}^*. \tag{26}$$

Now the operators $L : \mathcal{V}^* \to \mathcal{S}$ and $L' : \mathcal{S}^* \to \mathcal{V}$ are indeed adjoint to each other. In fact, if $q \in \mathcal{V}^*$ and $p \in \mathcal{S}^*$, then

$$\begin{aligned}
\langle Lq, p \rangle &= \langle \gamma u_q, p \rangle && \text{(definition of } Lq) \\
&= \langle u_q, \delta_p \rangle && \text{(definition of } \delta_p) \\
&= \int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v_p}{\partial t} + \nabla u_q \nabla v_p)\, dxdt && \text{(by Equation (25))} \\
&= \langle \delta_q, v_p \rangle && \text{(by Equation (24))} \\
&= \langle q, L'p \rangle. && \text{(definition of } L'p)
\end{aligned}$$

We may therefore reinterpret Equation (26) as saying that for all $h \in \mathcal{V}$ there exists $q \in \mathcal{V}^*$ such that $\langle p, Lq - h \rangle = 0$ for all $p \in \mathcal{S}^*$. Since $\mathcal{S}^*$ is a dense subspace of $\mathcal{V}^*$ this means that $Lq = h$. This shows that the range, $L(\mathcal{V}^*)$, of L contains all of $\mathcal{V}$. In fact, L even maps the smaller space $\mathcal{S}^*$ onto $\mathcal{V}$.

THEOREM 12. *If $q \in \mathcal{V}^*$ and $Lq \in \mathcal{V}$, then $q \in \mathcal{S}^*$ and $\|q\|_{\mathcal{S}^*} \leq C\|Lq\|_{\mathcal{V}}$.*

Proof. Let $q \in \mathcal{V}^*$ and suppose that $Lq \in \mathcal{V}$. Recalling that $V_\Gamma^{(T)}$ denotes the subspace of functions in $V^{(T)}$ which vanish on $\Gamma \times [0, T]$, it it follows immediately from Equation (24) that

$$\int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v}{\partial t} + \nabla u_q \nabla v)\, dxdt = 0 \qquad \text{for all} \quad v \in V_\Gamma^{(T)}.$$

Thus u_q coincides with the function u given in Theorem 8 (with $f = 0$, $g = Lq$), and from Theorem 9 we know that $u_q \in V$ and that the *a priori* estimate

$$\|u_q\|_V \leq C\|Lq\|_{\mathcal{V}} \tag{27}$$

holds. Now let v be a nonzero smooth function on $\Gamma \times [0,T]$ which vanishes on $\Gamma \times T$. Then

$$\begin{aligned}
\frac{|\langle q, \gamma v\rangle|}{\|v\|_S} &= \frac{1}{\|v\|_S}\int_0^T \int_{\mathbb{R}^3} (-u_q \frac{\partial v}{\partial t} + \nabla\, u_q \,\nabla\, v)\, dxdt \\
&= \frac{1}{\|v\|_S}\int_0^T \int_{\mathbb{R}^3} (\frac{\partial u_q}{\partial t} v + \nabla\, u_q \,\nabla\, v)\, dxdt \\
&\leq C\|u_q\|_V.
\end{aligned}$$

Taking the supremum over v, and noting that $V^{(T)}$ is dense in S, we infer that in fact $q \in \mathcal{S}^*$ and

$$\|q\|_{\mathcal{S}^*} \leq C\|u_q\|_V.$$

Together with the estimate (27) this proves the theorem.

It is now a simple matter to complete the proof of Theorem 10. The coercivity inequality (20) has already been shown in Theorem 11. Moreover, we have shown that $L(\mathcal{V}^*) \supset \mathcal{V}$. In view of Theorem 12, we conclude that $L(\mathcal{S}^*) \supset \mathcal{V}$. Thus L maps $\mathcal{S}^*$ onto $\mathcal{V}$. From the inequality (20) we see that L is an injection. Thus L is a continuous, one-to-one linear operator from $\mathcal{S}^*$ onto $\mathcal{V}$. Now Banach's theorem tells us that under these conditions, the inverse L^{-1} is also continuous. Thus L is an isomorphism of $\mathcal{S}^*$ onto $\mathcal{V}$. Finally, we may reverse the roles of L and L' and then take adjoints to deduce that L maps $\mathcal{V}^*$ isomorphically onto $\mathcal{S}$ as well.

NOTATION

Basic notations

$\mathbb{R}^3$	three dimensional space
$N(x)$	the fundamental solution of the Laplace equation in $\mathbb{R}^3$
$K(x,t)$	the fundamental solution of the heat equation in $\mathbb{R}^3$
Ω	the spatial domain in $\mathbb{R}^3$
Γ	the boundary of Ω
$[0,T]$	finite time interval
X^*	for any space X, its dual space

Spaces of functions depending on the space variable alone

C_0^∞	infinitely differentiable functions with compact support
L^2	square integrable functions
H^s	the Sobolev spaces of order s
W	the closure of $C_0^\infty(\mathbb{R}^3)$ in the norm $\int_{\mathbb{R}^3} \vert \nabla \phi \vert^2$

Spaces of functions depending on space and time

S	$L^2(0,T;W)$
V	$\{\, u \in S : \partial u/\partial t \in W^* \,\}$
$V^{(t)}$	$\{\, u \in V : u(x,t) = 0 \ \forall x \,\}$
S_Γ	$\{\, u \in S : u(x,t) = 0 \ \forall x \in \Gamma,\ \forall t \,\}$
V_Γ	$\{\, u \in V : u(x,t) = 0 \ \forall x \in \Gamma,\ \forall t \,\}$
$\mathcal{S}$	the set of traces on $\Gamma \times [0,T]$ of functions in S
$\mathcal{V}$	the set of traces on $\Gamma \times [0,T]$ of functions in V

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