On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities

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Abstract

In this paper we present new versions of the classical Brunn-Minkowski inequality for different classes of measures and sets. We show that the inequality

$$\mu\left(\lambda A + (1 - \lambda)B\right)^{1/n} \geq \lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}$$

holds true for an unconditional product measure $\mu$ with decreasing density and a pair of unconditional convex bodies $A, B \subset \mathbb{R}^n$. We also show that the above inequality is true for any unconditional log-concave measure $\mu$ and unconditional convex bodies $A, B \subset \mathbb{R}^n$. Finally, we prove that the inequality is true for a symmetric log-concave measure $\mu$ and a pair of symmetric convex sets $A, B \subset \mathbb{R}^2$, which, in particular, settles two-dimensional case of the conjecture for Gaussian measure proposed in [13].

In addition, we deduce the $1/n$-concavity of the parallel volume $t \mapsto \mu(A + tB)$, Brunn’s type theorem and certain analogues of Minkowski first inequality.

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1 Introduction

The classical Brunn-Minkowski inequality states that for any two non-empty compact sets $A, B \subset \mathbb{R}^n$ and any $\lambda \in [0, 1]$ we have

$$\text{vol}_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \text{vol}_n(A)^{1/n} + (1 - \lambda)\text{vol}_n(B)^{1/n},$$

with equality if and only if $B = aA + b$, where $a > 0$ and $b \in \mathbb{R}^n$. Here $\text{vol}_n$ stands for the Lebesgue measure on $\mathbb{R}^n$ and

$$A + B = \{a + b : a \in A, b \in B\}$$

is the Minkowski sum of $A$ and $B$. Due to homogeneity of the volume, this inequality is equivalent to

$$\text{vol}_n(A + B)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(B)^{1/n}.$$  

The Brunn-Minkowski inequality turns out to be a powerful tool. In particular, it implies the classical isoperimetric inequality: for any compact set $A \subset \mathbb{R}^n$ we have

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vol_n(A_t) \geq vol_n(B_t), t \geq 0, where B is a Euclidean ball satisfying vol_n(A) = vol_n(B) and A_t stands for the t-enlargement of A, i.e., A_t = A + tB^n_2, where B^n_2 is the unit Euclidean ball, B^n_2 = \{x : |x| = 1\}. To see this it is enough to observe that

\[ \text{vol}_n(A + tB^n_2)_{1/n} \geq \text{vol}_n(A)_{1/n} + \text{vol}_n(tB^n_2)_{1/n} = \text{vol}_n(B)_{1/n} + \text{vol}_n(tB^n_2)_{1/n} = \text{vol}_n(B + tB^n_2)_{1/n}. \]

Taking \( t \to 0^+ \) one gets a more familiar form of isoperimetry: among all sets with fixed volume the surface area

\[ \text{vol}_n^*(\partial A) = \liminf_{t \to 0^+} \frac{\text{vol}_n(A + tB^n_2) - \text{vol}_n(A)}{t} \]

is minimized in the case of the Euclidean ball. We refer to [11] for more information on Brunn-Minkowski-type inequalities.

Using the inequality between means one gets an a priori weaker dimension free form of (1), namely

\[ \text{vol}_n(\lambda A + (1 - \lambda)B) \geq \text{vol}_n(A)^\lambda \text{vol}_n(B)^{1-\lambda}. \]  (2)

In fact (2) and (1) are equivalent. To see this one has to take \( \tilde{A} = A/\text{vol}_n(A)_{1/n}, \tilde{B} = B/\text{vol}_n(B)_{1/n} \) and \( \tilde{\lambda} = \lambda \text{vol}_n(A)_{1/n}/(\lambda \text{vol}_n(A)_{1/n} + (1 - \lambda) \text{vol}_n(B)_{1/n}) \) in (2). This phenomenon is a consequence of homogeneity of the Lebesgue measure.

The above notions can be generalized to the case of the so-called s-concave measures. Here we assume that \( s > 0 \), whereas in general the notion of s-concave measures makes sense for any \( s \in [-\infty, \infty] \). We say that a measure \( \mu \) on \( \mathbb{R}^n \) is s-concave if for any non-empty compact sets \( A, B \subset \mathbb{R}^n \) we have

\[ \mu(\lambda A + (1 - \lambda)B)^s \geq \lambda \mu(A)^s + (1 - \lambda) \mu(B)^s. \]  (3)

Similarly, a measure \( \mu \) is called log-concave (or 0-concave) if for any compact sets \( A, B \subset \mathbb{R}^n \) we have

\[ \mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \]  (4)

We say that the support of measure \( \mu \) is non-degenerate if it is not contained in any affine subspace of \( \mathbb{R}^n \) of dimension less than \( n \). It was proved by Borell (see [2]) that a measure \( \mu \), with non-degenerate support, is log-concave if and only if it has a log-concave density, i.e. a density of the form \( \varphi = e^{-V} \), where \( V \) is convex (and may attain value \( +\infty \)). Moreover, \( \mu \) is s-concave with \( s \in (0, 1/n) \) if and only if it has a density \( \varphi \) such that \( \varphi^{\lambda s} \) is concave. In the case \( s = 1/n \) the density has to satisfy the strongest condition \( \varphi(x + (1 - \lambda)y) \geq \max(\varphi(x), \varphi(y)). \) An example of such measure is the uniform measure on a convex body \( K \subset \mathbb{R}^n. \) Let us also notice that a measure with non-degenerate support cannot be s-concave with \( s > 1/n. \) It can be seen by taking \( \tilde{A} = \varepsilon A \) and \( \tilde{B} = \varepsilon B \) in [3], sending \( \varepsilon \to 0^+ \) and comparing the limit with the Lebesgue measure.

Inequality [2] says that the Lebesgue measure is log-concave, whereas [1] means that it is also 1/n-concave. In general log-concavity does not imply s-concavity for \( s > 0. \) Indeed, consider the standard Gaussian measure \( \gamma_n \) on \( \mathbb{R}^n, \) i.e., the measure with density \((2\pi)^{-n/2}\exp(-|x|^2/2). \) This density is clearly log-concave and therefore \( \gamma_n \) satisfies [4]. To see that \( \gamma_n \) does not satisfy [3] for \( s > 0 \) it suffices to take \( B = \{x\} \) and send \( x \to \infty. \) Then the left hand side converges to 0 while the right hand side stays equal to \( \lambda \mu(A)^s, \) which is strictly positive for \( \lambda > 0 \) and \( \mu(A) > 0. \)

One might therefore ask whether [3] holds true for \( \gamma_n \) if we restrict ourselves to some special class of subsets of \( \mathbb{R}^n. \) In [13] R. Gardner and the fourth named author conjectured (Question 7.1) that

\[ \gamma_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \gamma_n(A)^{1/n} + (1 - \lambda) \gamma_n(B)^{1/n} \]  (5)

holds true for any closed convex sets with \( 0 \in A \cap B \) and \( \lambda \in [0, 1] \) and verified this conjecture in the following cases:
(a) when $A$ and $B$ are products of intervals containing the origin,

(b) when $A = [-a_1, a_2] \times \mathbb{R}^{n-1}$, where $a_1, a_2 > 0$ and $B$ is arbitrary,

(c) when $A = aK$ and $B = bK$ where $a, b > 0$ and $K$ is a convex set, symmetric with respect to the origin.

It is interesting to note that the case (c) is related to the B-conjecture for Gaussian measures proposed by Banaszczyk (see [16]) and solved by Cordero-Erausquin, Fradelizi, and Maurey (see [7]). It states that for any convex symmetric set $K$ the function $t \mapsto \gamma_n(e^t K)$ is log-concave. The B-conjecture is asking the same question for the general class of the even log-concave measures. It was shown in [7] that the conjecture is true for the case of unconditional log-concave measures and unconditional sets (see the definition below). Moreover, the conjecture has an affirmative answer for $n = 2$ due to the works of Livne Bar-on [17] and of Saroglou [28]. In [22] the second named author proved that the assertion of the B-conjecture for a measure $\mu$ with a radially decreasing density and a symmetric convex body $K$ formally implies the $1/n$-concavity of the measure $\mu$ on the set of dilates of $K$.

In [23] T. Tkocz and the third named author showed that in general [5] is false under the assumption $0 \in A \cap B$. For sufficiently small $\varepsilon > 0$ and $\alpha < \pi/2$ sufficiently close to $\pi/2$ the pair of sets

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha - \varepsilon\}$$

serves as a counterexample. The authors however conjectured that [5] should be true for (centrally) symmetric convex bodies $A, B$.

One of the most important Brunn-Minkowski type inequalities for the Gaussian measure is Ehrhard’s inequality, which states that for any two non-empty compact sets $A, B \subset \mathbb{R}^n$ and any $\lambda \in [0, 1]$ we have

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A)) + (1 - \lambda)\Phi^{-1}(\gamma_n(B)),$$

where $\Phi(t) = \gamma_1((\varepsilon, \infty])$. This inequality has been considered for the first time by Ehrhard in [9], where the author proved it assuming that both $A$ and $B$ are convex. Then Latała in [15] generalized Ehrhard’s result to the case of arbitrary $A$ and convex $B$. In its full generality, the inequality (6) has been established by Borell, [4] (see also [11]). Note that [5] is an inequality of the same type, with $\Phi(t)$ replaced with $t^n$, but none of them is a direct consequence of the other. The crucial property of Ehrhard’s inequality is that it (in fact a more general form where $\lambda$ and $1 - \lambda$ are replaced with $\alpha$ and $\beta$, under the conditions $\alpha + \beta \geq 1$ and $|\alpha - \beta| \leq 1$) gives the Gaussian isoperimetry as a simple consequence.

In this paper, $\mathcal{K}$ denotes a family of sets closed under dilations, i.e., $A \in \mathcal{K}$ implies $tA \in \mathcal{K}$ for any $t \geq 0$. In particular, we assume that for any $A \in \mathcal{K}$ we have $0 \in A$. Classical families of such sets include the class of star-shaped bodies, the class of convex bodies containing the origin, the class of symmetric bodies and the class of unconditional bodies.

A general form of the Brunn-Minkowski inequality can be stated as follows.

**Definition 1.** We say that a Borel measure $\mu$ on $\mathbb{R}^n$ satisfies the Brunn-Minkowski inequality in the class of sets $\mathcal{K}$ if for any $A, B \in \mathcal{K}$ and for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}.$$

Before we state our results, we introduce some basic notation and definitions.

**Definition 2.**
1. We say that a function \( f : \mathbb{R}^n \to \mathbb{R} \) is unconditional if for any choice of signs \( \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \) and any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we have \( f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = f(x) \).

2. We say that an unconditional function is decreasing if for any \( 1 \leq i \leq n \) and any real numbers \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \) the function

\[
t \mapsto f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)
\]

is non-increasing on \([0, \infty)\).

3. A set \( A \subseteq \mathbb{R}^n \) is called an ideal if \( 1_A \) is unconditional and decreasing. In other words, a set \( A \subseteq \mathbb{R}^n \) is an ideal if \((x_1, \ldots, x_n) \in A \) implies \((\delta_1 x_1, \ldots, \delta_n x_n) \in A \) for any choice of \( \delta_1, \ldots, \delta_n \in [-1, 1] \).

The class of all ideals (in \( \mathbb{R}^n \)) will be denoted by \( \mathcal{I} \).

4. A set \( A \subseteq \mathbb{R}^n \) is called symmetric if \( A = -A \). The class of all symmetric convex sets in \( \mathbb{R}^n \) will be denoted by \( \mathcal{S} \).

5. A measure \( \mu \) on \( \mathbb{R}^n \) is called unconditional if it has an unconditional density.

We note that the class of ideals contains the class of unconditional convex bodies, but it also contains some non-convex sets. For example, \( B_p^n = \{ x \in \mathbb{R}^n : \sum |x_i|^p \leq 1 \} \) for \( p \in (0, 1) \) are ideals. We also note that if an unconditional measure \( \mu \) on \( \mathbb{R}^n \) is a product measure, i.e. \( \mu = \mu_1 \otimes \ldots \otimes \mu_n \), then the measures \( \mu_i \) are even on \( \mathbb{R} \).

Our first theorem reads as follows.

**Theorem 1.** Let \( \mu \) be an unconditional product measure with decreasing density. Then \( \mu \) satisfies the Brunn-Minkowski inequality in the class \( \mathcal{I} \) of all ideals in \( \mathbb{R}^n \).

In addition, the Examples 1 and 2 at the end of the paper show that neither the assumption that \( \mu \) is a product measure, nor the unconditionality of our sets \( A \) and \( B \) can be dropped.

In the second part of this article we provide a link between the Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality. To state our observation we need two definitions.

**Definition 3.** Let \( \mathcal{K} \) be a class of subsets closed under dilations. We say that a family \( \circ = (\circ_\lambda)_{\lambda \in [0,1]} \) of functions \( \mathcal{K} \times \mathcal{K} \to \mathcal{K} \) is a geometric mean if for any \( A, B \in \mathcal{K} \) the set \( A \circ_\lambda B \) is measurable, satisfies an inclusion \( A \circ_\lambda B \subseteq A A + (1 - \lambda) B \), and \((sA) \circ_\lambda (tB) = s^\lambda t^{1-\lambda}(A \circ_\lambda B)\), for any \( s, t > 0 \).

**Definition 4.** We say that a Borel measure \( \mu \) on \( \mathbb{R}^n \) satisfies the log-Brunn-Minkowski inequality in the class of sets \( \mathcal{K} \) with a geometric mean \( \circ \), if for any sets \( A, B \in \mathcal{K} \) and for any \( \lambda \in [0, 1] \) we have

\[
\mu(A \circ_\lambda B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.
\]

**Remark 1.** We shall use two different geometric means. The first one is the geometric mean \( \circ^S : \mathcal{K}_S \times \mathcal{K}_S \to \mathcal{K}_S \), defined by the formula

\[
A \circ^S_\lambda B = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h^\lambda_A(u) h^{1-\lambda}_B(u), \forall u \in S^{n-1} \}.
\]

Here \( h_A \) is the support function of \( A \), i.e., \( h_A(u) = \sup_{x \in A} \langle x, u \rangle \) (see, [12], [29]).

The second mean \( \circ^I : \mathcal{K}_I \times \mathcal{K}_I \to \mathcal{K}_I \) is defined by

\[
A \circ^I_\lambda B = \bigcup_{x \in A, y \in B} [-|x_1|^\lambda |y_1|^{1-\lambda}, |x_1|^\lambda |y_1|^{1-\lambda}] \times \ldots \times [-|x_n|^\lambda |y_n|^{1-\lambda}, |x_n|^\lambda |y_n|^{1-\lambda}].
\]

It is straightforward to check, with the help of the inequality \( a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b \), \( a, b \geq 0 \), that both means are indeed geometric.
In the Section 3 we prove the following proposition.

**Proposition 1.** Suppose that a Borel measure \( \mu \) with a radially decreasing density \( f \), i.e. density satisfying \( f(tx) \geq f(x) \) for any \( x \in \mathbb{R}^n \) and \( t \in [0,1] \), satisfies the log-Brunn-Minkowski inequality, with a geometric mean \( \circ \), in a certain class of sets \( K \). Then \( \mu \) satisfies the Brunn-Minkowski inequality in the class \( K \).

Böröczky, Lutwak, Yang and Zhang \([5]\), proved the log-Brunn-Minkowski inequality for the Lebesgue measure and symmetric convex bodies on \( \mathbb{R}^2 \) equipped with geometric mean \( \circ^S \). Saroglou \([28]\), generalized the inequality to the case of measures with even log-concave densities on \( \mathbb{R}^2 \) (see Corollary 3.3 therein). Thus, as a consequence of Proposition 1 and Remark 1, we get the following theorem.

**Theorem 2.** Let \( \mu \) be a measure on \( \mathbb{R}^2 \) with an even log-concave density. Then \( \mu \) satisfies the Brunn-Minkowski inequality in the class \( K_S \) of all symmetric convex sets in \( \mathbb{R}^2 \).

Moreover, in \([17]\) (Proposition 8, see also Proposition 4.2 in \([27]\)) the authors proved the following fact.

**Theorem 3.** The log-Brunn-Minkowski inequality holds true with the geometric mean \( \circ^I \) for any measure with unconditional log-concave density in the class \( K_I \) of all ideals in \( \mathbb{R}^n \).

For the sake of completeness, we recall the argument in Section 3. As a consequence, applying our Proposition 1 together with Remark 1 we deduce:

**Theorem 4.** Let \( \mu \) be an unconditional log-concave measure on \( \mathbb{R}^n \). Then \( \mu \) satisfies the Brunn-Minkowski inequality in the class \( K_I \) of all ideals in \( \mathbb{R}^n \).

The rest of this article is organized as follows. In the next section we present the proof of Theorem 1. In Section 3 we prove Proposition 1 and recall the proof of Theorem 3. In Section 4 we present applications of the above results. In the last section we discuss equality cases in Theorem 2 and Theorem 4. We also give examples showing optimality of Theorem 1 and state some open questions.

## 2 Proof of Theorem 1

Our strategy is to prove a certain functional version of \([7]\). A functional version of the classical Brunn-Minkowski inequality is called the Prékopa-Leindler inequality, see \([11]\) for the proof.

**Prékopa-Leindler inequality, \([20]\), \([20]\):** Let \( f, g, m \) be non-negative measurable functions on \( \mathbb{R}^n \) and let \( \lambda \in [0,1] \). If for all \( x, y \in \mathbb{R}^n \) we have \( m(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} \) then

\[
\int m \, dx \geq \left( \int f \, dx \right)^\lambda \left( \int g \, dx \right)^{1-\lambda}.
\]

Here we prove a version of the above inequality under the assumption of unconditionality of functions \( f, g \) and \( m \).

**Proposition 2.** Fix \( \lambda, p \in (0,1) \). Suppose that \( m, f, g \) are unconditional decreasing non-negative functions and let \( \mu \) be an unconditional product measure with decreasing density on \( \mathbb{R}^n \). Assume that for any \( x, y \in \mathbb{R}^n \) we have

\[
m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p}.
\]

Then

\[
\int m \, d\mu \geq \left[ \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1-p} \right)^{1-p} \right]^n \left( \int f \, d\mu \right)^p \left( \int g \, d\mu \right)^{1-p}.
\]
The above proposition allows us to prove the following lemma, which is in fact a reformulation of Theorem 1.

**Lemma 1.** Let $A, B$ be ideals in $\mathbb{R}^n$ and let $\mu$ be an unconditional product measure with decreasing density on $\mathbb{R}^n$. Then for any $\lambda \in [0, 1]$ and $p \in (0, 1)$ we have

$$
\mu(\lambda A + (1 - \lambda)B) \geq \left[ \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1 - p} \right)^{1 - p} \right]^n \mu(A)^p \mu(B)^{1 - p}.
$$

It is worth noticing that the factor on the right hand side of this inequality replaces in some sense the lack of homogeneity of our measure $\mu$. The main idea of the proof is to introduce an additional parameter $p \neq \lambda$ and do the optimization with respect to $p$.

We first show how Lemma 1 implies Theorem 1.

**Proof of Theorem 1.** Without loss of generality we assume that $\lambda \in (0, 1)$. Let us assume for a moment that $\mu(A)\mu(B) > 0$. Then we can use Lemma 1 with

$$
p = \frac{\lambda \mu(A)^{1/n}}{\lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}} \in (0, 1).
$$

Note that

$$
\frac{\lambda}{p} = \frac{\lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}}{\mu(A)^{1/n}}, \quad \frac{1 - \lambda}{1 - p} = \frac{\lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}}{\mu(B)^{1/n}}.
$$

Then

$$
\left[ \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1 - p} \right)^{1 - p} \right]^n \mu(A)^p \mu(B)^{1 - p} = \left( \lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n} \right)^n.
$$

Thus the inequality in Lemma 1 becomes

$$
\mu(\lambda A + (1 - \lambda)B) \geq \left( \lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n} \right)^n.
$$

Now suppose that, say, $\mu(B) = 0$. Since $B$ is a non-empty ideal, we have $0 \in B$. Therefore, $\lambda A \subseteq \lambda A + (1 - \lambda)B$. Let $\varphi$ be the unconditional decreasing density of $\mu$. Hence,

$$
\mu(\lambda A + (1 - \lambda)B) \geq \mu(\lambda A) = \int_{\lambda A} \varphi(x) \, dx = \lambda^n \int_A \varphi(\lambda y) \, dy = \lambda^n \int_A \varphi(|y_1|, \ldots, |y_n|) \, dy = \lambda^n \mu(A).
$$

Therefore,

$$
\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} = \lambda \mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}.
$$

Next we show that Proposition 2 implies Lemma 1.

**Proof of Lemma 2.** We can assume that $\lambda \in (0, 1)$. Let us take $m(x) = \mathbf{1}_{\lambda A + (1 - \lambda)B}(x)$, $f(x) = \mathbf{1}_A(x)$, $g(x) = \mathbf{1}_B(x)$. Clearly, $f, g$ and $m$ are unconditional and decreasing, and verify $m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1 - p}$ for any $p \in (0, 1)$. Our assertion follows from Proposition 2. 

\[ \square \]
For the proof of Proposition 2 we need a one dimensional Brunn-Minkowski inequality for unconditional measures.

**Lemma 2.** Let $A, B$ be two symmetric intervals and let $\mu$ be an unconditional measure with decreasing density on $\mathbb{R}$. Then for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \lambda \mu(A) + (1 - \lambda)\mu(B).$$

**Proof.** We can assume that $A = [-a, a]$ and $B = [-b, b]$ for some $a, b > 0$. Let $\varphi$ be the density of $\mu$. Then our assertion is equivalent to

$$\int_0^{\lambda a + (1 - \lambda)b} \varphi(x) \, dx \geq \lambda \int_0^a \varphi(x) \, dx + (1 - \lambda) \int_0^b \varphi(x) \, dx.$$

In other words, the function $t \mapsto \int_0^t \varphi(x) \, dx$ should be concave on $[0, \infty)$. This is equivalent to $t \mapsto \varphi(t)$ being non-increasing on $[0, \infty)$.

**Proof of Proposition 2.** We proceed by induction on $n$. Let us begin with the case $n = 1$. We can assume that $\|f\|_\infty, \|g\|_\infty > 0$. If we multiply the functions $m, f, g$ by positive numbers $c_m, c_f, c_g$ satisfying $c_m = c_f c_g^{1-p}$, the hypothesis and the assertion do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$, $c_m = \|f\|_\infty^{-p}\|g\|_\infty^{-(1-p)}$ we can assume that $\|f\|_\infty = \|g\|_\infty = 1$. Then the sets $\{f > t\}$ and $\{g > t\}$ are non-empty for $t \in (0, 1)$. Moreover, $\lambda\{f > t\} + (1 - \lambda)\{g > t\} \subseteq \{m > t\}$. Indeed, if $x \in \{f > t\}$ and $y \in \{g > t\}$ then $m(x) + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p} > t^p t^{1-p} = t$. Thus, $\lambda x + (1 - \lambda)y \in \{m > t\}$. Therefore, using Lemma 2 we get

$$\int m \, d\mu = \int_0^\infty \mu(\{m > t\}) \, dt \geq \int_0^1 \mu(\{f > t\} + (1 - \lambda)\{g > t\}) \, dt$$

$$\geq \lambda \int_0^1 \mu(\{f > t\}) \, dt + (1 - \lambda) \int_0^1 \mu(\{g > t\}) \, dt$$

$$= \lambda \int f \, d\mu + (1 - \lambda) \int g \, d\mu.$$

Now, using the inequality $pa + (1 - p)b \geq a^p b^{1-p}$, $a, b \geq 0$, we get

$$\lambda \int f \, d\mu + (1 - \lambda) \int g \, d\mu = \frac{\lambda}{p} \int f \, d\mu + (1 - \lambda) \int g \, d\mu$$

$$\geq \left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p} \left(\int f \, d\mu\right)^p \left(\int g \, d\mu\right)^{1-p}.$$

(9)

Next, we do the induction step. Let us assume that the assertion is true in dimension $n - 1$. Let $m, f, g : \mathbb{R}^n \to [0, \infty)$ be unconditional decreasing. For $x_0, y_0, z_0 \in \mathbb{R}$ we define functions $m_{z_0}, f_{x_0}, g_{y_0}$ by

$$m_{z_0}(x) = m(z_0, x), \quad f_{x_0}(x) = f(x_0, x), \quad g_{y_0}(x) = g(y_0, x).$$

Clearly, these functions are also unconditional. Moreover, due to our assumptions on $m, f, g$ we have

$$m_{\lambda x_0 + (1 - \lambda)y_0}(\lambda x + (1 - \lambda)y) = m(\lambda x_0 + (1 - \lambda)y_0, \lambda x + (1 - \lambda)y)$$

$$\geq f(x_0, x)^p g(y_0, y)^{1-p} = f_{x_0}(x)^p g_{y_0}(y)^{1-p}.$$

Let us decompose $\mu$ in the form $\mu = \mu_1 \times \mu$, where $\mu_1$ is a measure on $\mathbb{R}$. Note that $\mu_1$ and $\bar{\mu}$ are unconditional and $\bar{\mu}$ is a product measure on $\mathbb{R}^{n-1}$. Thus, by our induction assumption we have

$$\int m_{\lambda x_0 + (1 - \lambda)y_0} \, d\bar{\mu} \geq \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p}\right]^{n-1} \left(\int f_{x_0} \, d\mu\right)^p \left(\int g_{y_0} \, d\mu\right)^{1-p}.$$ 

(11)
Now we define the functions

\[
M(z_0) = \left[ \frac{\lambda}{p} \left( \frac{1 - \lambda}{1 - p} \right) \right]^{-(n-1)} \int m_{z_0}(\xi) \, d\bar{\mu}(\xi),
\]

(12)

\[
F(x_0) = \int f_{x_0}(\xi) \, d\bar{\mu}(\xi), \quad G(y_0) = \int g_{y_0}(\xi) \, d\bar{\mu}(\xi).
\]

(13)

Using inequality (11) we immediately get that

\[
M(\lambda x_0 + (1 - \lambda)y_0) \geq F(x_0)^p G(y_0)^{1-p}.
\]

Moreover, it is easy to see that \(M, F, G\) are unconditional decreasing on \(\mathbb{R}\). Thus, using Lemma 2 (the one-dimensional case), we get

\[
\int M(z_0) \, d\mu_1(z_0) \geq \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1 - p} \right)^{1-p} \left( \int F(x_0) \, d\mu_1(x_0) \right)^p \left( \int G(y_0) \, d\mu_1(y_0) \right)^{1-p}.
\]

(14)

Observe that

\[
\int M(z_0) \, d\mu_1(z_0) = \left[ \frac{\lambda}{p} \left( \frac{1 - \lambda}{1 - p} \right) \right]^{-(n-1)} \int \int m_{z_0}(\xi) \, d\mu_{n-1}(\xi) \, d\mu_1(z_0)
\]

\[
= \left[ \frac{\lambda}{p} \left( \frac{1 - \lambda}{1 - p} \right) \right]^{-(n-1)} \int m \, d\mu.
\]

Similarly,

\[
\int F(x_0) \, d\mu_1(x_0) = \int f \, d\mu, \quad \int G(y_0) \, d\mu_1(y_0) = \int g \, d\mu.
\]

Our assertion follows.

\[\square\]

3 Proof of Proposition 1

In this section we first prove Proposition 1. The argument has a flavour of our previous proof.

Proof of Proposition 1. Let us first assume that \(\mu(A)\mu(B) > 0\). From the definition of geometric mean we have \(A \odot_p B \subseteq pA + (1-p)B\), for any \(p \in (0, 1)\). Thus,

\[
\mu(\lambda A + (1 - \lambda)B) = \mu \left( p \cdot \frac{\lambda}{p} A + (1 - p) \cdot \frac{1 - \lambda}{1 - p} B \right) \geq \mu \left( \left( \frac{\lambda}{p} A \right) \odot_p \left( \frac{1 - \lambda}{1 - p} B \right) \right)
\]

\[
= \mu \left( \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1 - p} \right)^{1-p} A \odot_p B \right).
\]

Let \(t = \left( \frac{\lambda}{p} \right)^p \left( \frac{1 - \lambda}{1 - p} \right)^{1-p}\) and \(C = A \odot_p B\). From the concavity of the logarithm it follows that \(0 \leq t \leq 1\). We have

\[
\mu(tC) = \int_{tC} f(x) \, dx = t^n \int_C f(tx) \, dx \geq t^n \int_C f(x) \, dx = t^n \mu(C).
\]

(15)
Therefore,
\[
\mu(\lambda A + (1 - \lambda)B) \geq t^n \mu(A \odot_p B) \geq t^n \mu(A)^p \mu(B)^{1-p} = \left[ \left( \frac{\lambda}{p} \right)^p \left( \frac{1-\lambda}{1-p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p}.
\]

Taking
\[
p = \frac{\lambda \mu(A)^{1/n}}{\lambda \mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}}
\]
gives
\[
\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}.
\]

If, say, \( \mu(B) = 0 \) then by (15), applied for \( C \) replaced with \( A \), and the fact that \( 0 \in B \) we get
\[
\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \mu(\lambda A)^{1/n} \geq \lambda \mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}.
\]

\[\square\]

We now sketch the proof of Theorem 3.

Proof. Let \( A,B \in \mathcal{K}_I \) and let us take \( f,g,m: [0,+\infty)^n \to [0,+\infty) \) given by \( f = 1_{A \cap [0,\infty)^n} \), \( g = 1_{B \cap [0,\infty)^n} \) and \( m = 1_{(A \odot I) \cap [0,\infty)^n} \). Let \( \varphi \) be the unconditional log-concave density of \( \mu \). We define
\[
F(x) = f(e^{x_1},...,e^{x_n}) \varphi(e^{x_1},...,e^{x_n}) e^{x_1+\cdots+x_n}, \quad G(x) = g(e^{x_1},...,e^{x_n}) \varphi(e^{x_1},...,e^{x_n}) e^{x_1+\cdots+x_n},
\]
\[
M(x) = m(e^{x_1},...,e^{x_n}) \varphi(e^{x_1},...,e^{x_n}) e^{x_1+\cdots+x_n}.
\]

One can easily check, using the definition of \( \mathcal{K}_I \) and the definition of the geometric mean \( \odot^f \), as well as the inequalities
\[
\varphi(e^{\lambda x_1 + (1-\lambda)y_1},...,e^{\lambda x_n + (1-\lambda)y_n}) \geq \varphi(\lambda e^{x_1} + (1-\lambda)e^{y_1},...,\lambda e^{x_n} + (1-\lambda)e^{y_n}) \geq \varphi(e^{x_1},...,e^{x_n})^{\lambda} \varphi(e^{y_1},...,e^{y_n})^{1-\lambda},
\]
that the functions \( F,G,M \) satisfy the assumptions of the Prékopa-Leindler inequality. As a consequence, we get \( \mu((A \odot^f I B) \cap [0,\infty)^n) \geq \mu((A \cap [0,\infty)^n)^{\lambda} \mu(B \cap [0,\infty)^n)^{1-\lambda} \). The assertion follows from unconditionality of our measure \( \mu \) and the fact that \( A,B \) and \( A \odot^f I B \) are ideals. \[\square\]

4 Applications

Let us describe some corollaries of the Brunn-Minkowski type inequality we established, which are analogues to well-known offsprings of the Brunn-Minkowski inequality for the volume. In what follows a pair \( (\mathcal{K},\mu) \) is called nice if one of the following three cases holds.

(a) \( \mathcal{K} = \mathcal{K}_I \) and \( \mu \) is an unconditional, product measure with decreasing density on \( \mathbb{R}^n \),

(b) \( \mathcal{K} = \mathcal{K}_I \) and \( \mu \) is an unconditional log-concave measure on \( \mathbb{R}^n \),

(c) \( \mathcal{K} = \mathcal{K}_S \) and \( \mu \) is an even log-concave measure on \( \mathbb{R}^2 \).

Corollary 1. Suppose that a pair \( (\mathcal{K},\mu) \) is nice. Let \( A,B \subset \mathcal{K} \) be convex. Then the function \( t \mapsto \mu(A + tB)^{1/n} \) is concave on \( [0,\infty) \).
Indeed, for any \( \lambda \in [0, 1] \) and \( t_1, t_2 \geq 0 \) we have

\[
\mu(A + (\lambda t_1 + (1 - \lambda)t_2)B)^{1/n} = \mu(\lambda(A + t_1 B) + (1 - \lambda)(A + t_2 B))^{1/n} \\
\geq \lambda \mu(A + t_1 B)^{1/n} + (1 - \lambda)\mu(A + t_2 B)^{1/n}.
\]

Note that in the first line we have used the convexity of \( A \) and \( B \). If \( B = B^n_2 \) is the unit Euclidean ball, the expression \( \mu(A + tB) \) is called the parallel volume and has been studied in the case of the Lebesgue measure by Costa and Cover in [3] as an analogue of concavity of entropy power in Information theory. The authors conjectured that for any measurable set \( A \) the parallel volume is \( 1/n \)-concave. In [10], M. Fradelizi and the second named author proved that this conjecture is true for any measurable set in dimension 1 and for any connected set in dimension 2. However, the authors proved that this conjecture fails for arbitrary sets in dimension \( n \geq 2 \). In a recent paper [21] the second named author investigated the parallel volume \( \mu(A + tB^n_2) \) in the context of \( s \)-concave measures as well as functional versions. Our Corollary 1 gives the Costa-Cover conjecture for any convex set \( A \in K \), where \( (K, \mu) \) is a nice pair. Moreover, \( B^n_2 \) can be replaced with any convex set \( B \in K \).

Second, we state the following analogue of Brunn’s theorem on volumes of sections of convex bodies (see [11], [12] and [29] for the volume case).

**Corollary 2.** Suppose that a pair \( (K, \mu) \) is nice. Let \( A \in K \) be a convex set and let \( \varphi \) be the density of \( \mu \). Then the function \( t \mapsto \mu_{n-1}(A \cap \{x_1 = t\}) \) is \( \frac{1}{n-1} \)-concave on its support, where

\[
\mu_{n-1}(A \cap \{x_1 = t\}) = \int_{(t,x_2,\ldots,x_n) \in A} \varphi(t,x_2,\ldots,x_n) \, dx_2 \ldots dx_n.
\]

Indeed, let us denote \( A_{\{x_1 = t\}} = A \cap \{x_1 = t\} \). By convexity of \( A \) we get

\[
\lambda A_{\{x_1 = t_1\}} + (1 - \lambda)A_{\{x_1 = t_2\}} \subseteq A_{\{x_1 = \lambda t_1 + (1 - \lambda)t_2\}}.
\]

Thus, using (7), for any \( \lambda \in [0, 1] \) and \( t_1, t_2 \in \mathbb{R} \) such that \( A_{\{x_1 = t_1\}} \) and \( A_{\{x_1 = t_2\}} \) are both non-empty, we get

\[
\mu_{n-1}(A_{\{x_1 = \lambda t_1 + (1 - \lambda)t_2\}}) \geq \mu_{n-1}(\lambda A_{\{x_1 = t_1\}} + (1 - \lambda)A_{\{x_1 = t_2\}}) \geq \lambda \mu_{n-1}(A_{\{x_1 = t_1\}}) + (1 - \lambda)\mu_{n-1}(A_{\{x_1 = t_2\}}).
\]

Third, let us mention the relation of our result to the Gaussian isoperimetric inequality and the S-inequality. The Gaussian isoperimetric inequality (established by Sudakov and Tsirelson, [30], and independently by Borell, [32]), states that for any measurable set \( A \subset \mathbb{R}^n \) and any \( t > 0 \), the quantity \( \gamma_n(A_t) \) is minimized, among all sets with prescribed measure, for the half spaces \( H_{a, \theta} = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq a\} \), with \( a \in \mathbb{R} \) and \( \theta \in S^{n-1} \). Infinitesimally, it says that among all sets with prescribed measure the half spaces are those with the smallest Gaussian surface area, i.e., the quantity

\[
\gamma_n^+(\partial A) = \liminf_{t \to 0^+} \frac{\gamma_n(A + tB^n_2) - \gamma_n(A)}{t}.
\]

The S-inequality of Latała and Oleszkiewicz, see [18], states that for any \( t > 1 \) and any symmetric convex body \( A \) the quantity \( \gamma_n(tA) \) is minimized, among all subsets with prescribed measure, for the strip of the form \( S_L = \{x \in \mathbb{R}^n : |x_1| \leq L\} \). This result admits an equivalent infinitesimal version, namely, among all symmetric convex bodies \( A \) with prescribed Gaussian measure the strip \( S_L \) minimizes the quantity \( \frac{d}{dt} \gamma_n(tA)\big|_{t=1} \), which is equivalent to maximizing

\[
M_{\gamma_n}(A) = \int_A |x|^2 \, d\gamma_n(x),
\]
see [14] or [25]. For a general measure \( \mu \) with a density \( e^{-\psi} \), one can show that the infinitesimal version of S-inequality is an issue of maximizing the quantity

\[
M_\mu(A) = \int_A (x, \nabla \psi(x)) \, d\mu(x),
\]

see equation (22) below. Not much is known about an analogue of S-inequality in the case of general measure. In the unconditional case it has been solved for some particular product measures like products of Gaussian and Weibull distributions, see [24]. It turns out that inequality (5) implies a certain mixture of Gaussian isoperimetry and reverse S-inequality. Namely, we have the following corollary.

**Corollary 3.** Let \( A \) be an ideal in \( \mathbb{R}^n \) (or a general symmetric convex set in \( \mathbb{R}^2 \)) and let \( r > 0 \). Then we have

\[
r_\gamma^+(\partial A) + M_\gamma(A) \geq n_\gamma(rB_2^n) \frac{1}{r} \gamma_\mu(A)^{1-\frac{1}{n}}
\]

with equality for \( A = rB_2^n \).

Let us note that

\[
\gamma_\mu(rB_2^n + \varepsilon B_2^n) = (2\pi)^{-n/2}(r + \varepsilon)^n \int_{B_2^n} e^{-\frac{|(r+\varepsilon)x|^2}{2}} \, dx
\]

\[
= (2\pi)^{-n/2}(r^n + nr^{n-1}\varepsilon + o(\varepsilon)) \int_{B_2^n} e^{-\frac{|rx|^2}{2}} (1 - \varepsilon|x|^2 + o(\varepsilon)) \, dx
\]

\[
= \gamma_\mu(rB_2^n) + \frac{\varepsilon}{r} (n\gamma_\mu(rB_2^n) - M_\gamma(rB_2^n)) + o(\varepsilon).
\]

Thus,

\[
r_\gamma^+(\partial(rB_2^n)) = n\gamma_\mu(rB_2^n) - M_\gamma(rB_2^n).
\]

Hence, if \( \gamma_\mu(A) = \gamma_\mu(rB_2^n) \) in Corollary 3, then we get

\[
r_\gamma^+(\partial A) + M_\gamma(A) \geq r_\gamma^+(\partial(rB_2^n)) + M_\gamma(rB_2^n).
\]

In other words, Euclidean balls minimize the quantity \( r_\gamma^+(\partial A) + M_\gamma(A) \) among ideals in \( \mathbb{R}^n \) (or symmetric convex sets in \( \mathbb{R}^2 \)) with prescribed measure.

It is known that among all symmetric convex sets (in fact among all measurable sets) with prescribed Gaussian measure, the quantity \( M_\gamma(A) \) is minimized by Euclidean balls \( rB_2^n \) (this fact can be seen as a reverse S-inequality). Indeed, suppose that \( \gamma_\mu(A) = \gamma_\mu(rB_2^n) \). Then

\[
M_\gamma(A) - M_\gamma(rB_2^n) = \int_{A \setminus (rB_2^n)} |x|^2 \, d\gamma_\mu(x) - \int_{(rB_2^n) \setminus A} |x|^2 \, d\gamma_\mu(x)
\]

\[
\geq r^2 (\gamma_\mu(A \setminus (rB_2^n)) - \gamma_\mu((rB_2^n) \setminus A)) = 0.
\]

However, in general the quantity \( r_\gamma^+(\partial A) \) is not minimized by Euclidean balls, e.g., one can check that for large values of \( \gamma_2(A) \) the symmetric strip has smaller Gaussian surface area than the Euclidean ball, see [19] Lemma 3. Hence, inequality (18) is a new isoperimetric-type inequality that links the Gaussian isoperimetry and reverse S-inequality.

Let us state and prove a more general version of Corollary 3. Let \( \mu^+(\partial A) \) be the \( \mu \) surface area of \( A \), i.e.,

\[
\mu^+(\partial A) = \liminf_{t \to 0^+} \frac{\mu(A + tB_2^n) - \mu(A)}{t}.
\]

Let

\[
V^n_\mu(A, B) = \frac{1}{n} \liminf_{t \to 0^+} \frac{\mu(A + tB) - \mu(A)}{t}
\]

be the first mixed volume of arbitrary sets \( A \) and \( B \), with respect to measure \( \mu \). Clearly, \( \mu^+(\partial A) = nV^n_\mu(A, B_2^n) \).
Corollary 4. Let $A, B \in \mathcal{K}$ and suppose that $(\mathcal{K}, \mu)$ is a nice pair. Then we have
\[ V_1^\mu(A, B) + \frac{1}{n} M_\mu(A) \geq \mu(B)\frac{1}{n} \mu(A)^{1-1/n}. \] (19)

In particular,
\[ r \mu^+(\partial A) + M_\mu(A) \geq n \mu(r B^n_2)^{1/n} \mu(A)^{1-1/n}. \] (20)

To prove this we note that for any sets $A, B \in \mathcal{K}$ and any $\varepsilon \in [0, 1)$ we have
\[ \mu(A + \varepsilon B)^{1/n} \geq (1 - \varepsilon)\mu \left( \frac{A}{1 - \varepsilon} \right)^{1/n} + \varepsilon \mu(B)^{1/n}. \] (21)

Indeed, it suffices to use Theorem 1 with $\lambda = 1 - \varepsilon$ and $\tilde{A} = A/(1 - \varepsilon)$, $\tilde{B} = B$. Note that for $\varepsilon = 0$ we have equality. Thus, differentiating (21) at $\varepsilon = 0$ we get
\[ \frac{1}{n} \mu(A)^{1/n - 1} \cdot n V_1^\mu(A, B) \geq \mu(B)^{1/n} - \mu(A)^{1/n} + \frac{1}{n} \mu(A)^{1/n - 1} \frac{d}{dt} \mu(tA) \bigg|_{t=1}. \]

By changing variables we obtain
\[ \frac{d}{dt} \mu(tA) \bigg|_{t=1} = \frac{d}{dt} \int_A e^{-\psi(tx)} t^n \, dx \bigg|_{t=1} = n \mu(A) - \int_A \langle x, \nabla \psi(x) \rangle \, d\mu(x) = n \mu(A) - M_\mu(A). \] (22)

Thus,
\[ \mu(A)^{1/n - 1} V_1^\mu(A, B) \geq \mu(B)^{1/n} - \frac{1}{n} \mu(A)^{1/n - 1} M_\mu(A), \]

which is exactly (19). To get (20) one has to take $B = r B^n_2$ in (19).

The above inequalities can be seen as an analogue of the so-called Minkowski first inequality for the Lebesgue measure (see [11], [12] and [29]), which says that for any two convex bodies $A, B$ in $\mathbb{R}^n$ we have
\[ V_1^{\text{vol}_n}(A, B) \geq \text{vol}_n(A)^{1-\frac{1}{n}} \text{vol}_n(B)^{\frac{1}{n}}. \]

5 Examples and open problems

We first discuss equality cases in Theorem 2 and Theorem 4.

Remark 2. The equality in Theorem 2 and Theorem 4 is achieved only if $A$ is a dilation of $B$. Indeed, in the proof of Proposition 1 we use the inclusion $\tilde{A} \odot_p \tilde{B} \subseteq p\tilde{A} + (1 - p)\tilde{B}$, where $\tilde{A} = \frac{1}{p} A$ and $\tilde{B} = \frac{1-p}{p} B$, with $p$ given by (16). To have equality in (7) we need to have, in particular, equality in the above inclusion (with this particular choice of $p$). Notice that $a^p b^{1-p} = pa + (1 - p)b$, $a, b \geq 0$, if and only if $a = b$. Thus, $\tilde{A} \odot_p \tilde{B} = p\tilde{A} + (1 - p)\tilde{B}$ if and only if $\tilde{A} = \tilde{B}$ (by using the fact that $h_{\tilde{A}} = h_{\tilde{B}}$ if and only if $\tilde{A} = \tilde{B}$). Similarly, one has $\tilde{A} \odot_p \tilde{B} = p\tilde{A} + (1 - p)\tilde{B}$ if and only if $\tilde{A} = \tilde{B}$. This means that $A$ is a dilation of $B$.

In general one cannot hope to have equality cases only if $A = B$. Let us illustrate this in the case of the Lebesgue measure. Indeed, then we have equality in (7) if $A = aK$ and $B = bK$, where $K$ is some fixed convex set. In this case the equality $\tilde{A} = \tilde{B}$ leads to the condition $\frac{\lambda}{p} a = \frac{1 - \lambda}{1 - p} b$, which is equivalent to choosing $p = \frac{\lambda a}{\lambda a + (1 - \lambda)b}$. This coincides with (16).

However, one can get $A = B$ as the only case of equality if one assumes that the density of $\mu$ is strictly decreasing. To see this it suffices to observe that for the equality in (7) we have to have $t = 1$ in the proof of Proposition 1, which leads to $\mu(A) = \mu(B)$. Together with the fact that $A$ is a dilation of $B$ we get $A = B$. 

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We also show that the assumptions of Theorem 1 are necessary. Namely, as long as we work with decreasing densities, which may not be log-concave, one has to assume that the measure is product and the sets are unconditional.

**Example 1.** The assumption, that our measure $\mu$ in Theorem 1 is a product, is important. Indeed, let us take the square $C = \{x, |y| \leq 1\} \subset \mathbb{R}^2$ and take the measure with density $\varphi(x) = \frac{1}{2} 1_{2C}(x) + \frac{1}{2} 1_{C}(x)$. This density is unconditional, however it is not a product. Let us define $\psi(a) = \sqrt{\mu(aC)}$. The assertion of Theorem 1 implies that $\psi$ is concave. However, we have $\psi(a) = \sqrt{2a^2 + 2}$ for $a \in [1, 2]$, which is strictly convex. Thus, $\mu$ does not satisfy (7).

**Example 2.** In general, under the assumption that our measure $\mu$ is unconditional and a product, one cannot prove that Theorem 1 holds true for arbitrary symmetric convex sets. To see this, let us take the product measure $\mu = \mu_0 \otimes \mu_0$ on $\mathbb{R}^2$, where $\mu_0$ has an unconditional density $\varphi(x) = p + (1-p) 1_{[-\sqrt{2}, \sqrt{2}]^2}(x)$ for some $p \in [0, 1]$.

To simplify the computation let us rotate the whole picture by angle $\pi/4$. Then consider the rectangle $R = [-1, 1] \times [-\lambda, \lambda]$ for $0 < \lambda \leq 1/2$. As in the previous example, it is enough to show that the function $\psi(a) = \sqrt{\mu(aR)}$ is not concave. Let us consider this function only on the interval $[1/\lambda, \infty)$. The condition $\lambda \leq 1/2$ ensures that the point $(a, \lambda a)$ lies in the region with density $p^2$. Let us introduce lengths $l_1, l_2, l_3$ (see the picture below).

Note that $l_1 = \sqrt{2}\lambda a$, $l_2 = \sqrt{2}(\lambda a - 1)$ and $l_3 = a - (1 + \lambda a)$. Let $\omega(a) = \mu(aR)$. We have

$$\omega(a) = 2 + 4\sqrt{2} p \cdot \frac{l_1 + l_2}{2} + p^2 l_1^2 + p^2 l_2^2 + 4p^2 l_3 \lambda a$$

$$= 2 + 4p(2\lambda a - 1) + 2p^2 \lambda^2 a^2 + 2p^2(\lambda a - 1)^2 + 4p^2 \lambda a(a - 1 - \lambda a)$$

$$= 2(1 - p)^2 + 4p\lambda a(pa + 2 - 2p) = d_0 + d_1 a + d_2 a^2,$$

where $d_0 = 2(1 - p)^2$, $d_1 = 8p(1 - p)\lambda$, $d_2 = 4p^2 \lambda$. We show that $\psi$ is strictly convex for $p \in (0, 1)$ and $0 < \lambda < 1/2$. Indeed, $\psi'' > 0$ is equivalent to $2\omega \omega'' > (\omega')^2$. But

$$2\omega(a)\omega''(a) - (\omega'(a))^2 = 4d_2(d_0 + d_1 a + d_2 a^2) - (2d_2 a + d_1)^2 = 4d_2 d_0 - d_1^2$$

$$= 32\lambda p^2(1 - p)^2 - 64\lambda^2 p^2(1 - p)^2 = 32\lambda p^2(1 - p)^2(1 - 2\lambda) > 0.$$

We would like to finish the paper with a list of open questions that arose during our study.

**Question.** Let us assume that the measure $\mu$ has an even log-concave density (not-necessarily product).

- Does the assertion of Theorem 1 holds true for arbitrary symmetric sets $A$ and $B$?
- If not, is it true under additional assumption that the measure is product?
- In particular, can one remove the assumption of unconditionality in the Gaussian Brunn-Minkowski inequality?
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