Measurability of the Recursive Tree Process for the Mean-field Random Assignment Problem

Antar Bandyopadhyay

(http://www.stat.berkeley.edu/~antar)

Department of Statistics
University of California, Berkeley

July 31, 2002

Random Assignment Problem

Problem :

- There are n machines and n jobs.
- $C_{i,j}$:= cost of performing j^{th} job in i^{th} machine.
- Find the optimal assignment, that is, to find a permutation π, which will solve the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^{n} C_{i,\pi(i)}.$$

- Simplest probabilistic model can be obtained by taking $C_{i,j}$'s as *i.i.d* Unif$[0,1]$, then the main interest is on the objective function A_n.

History :

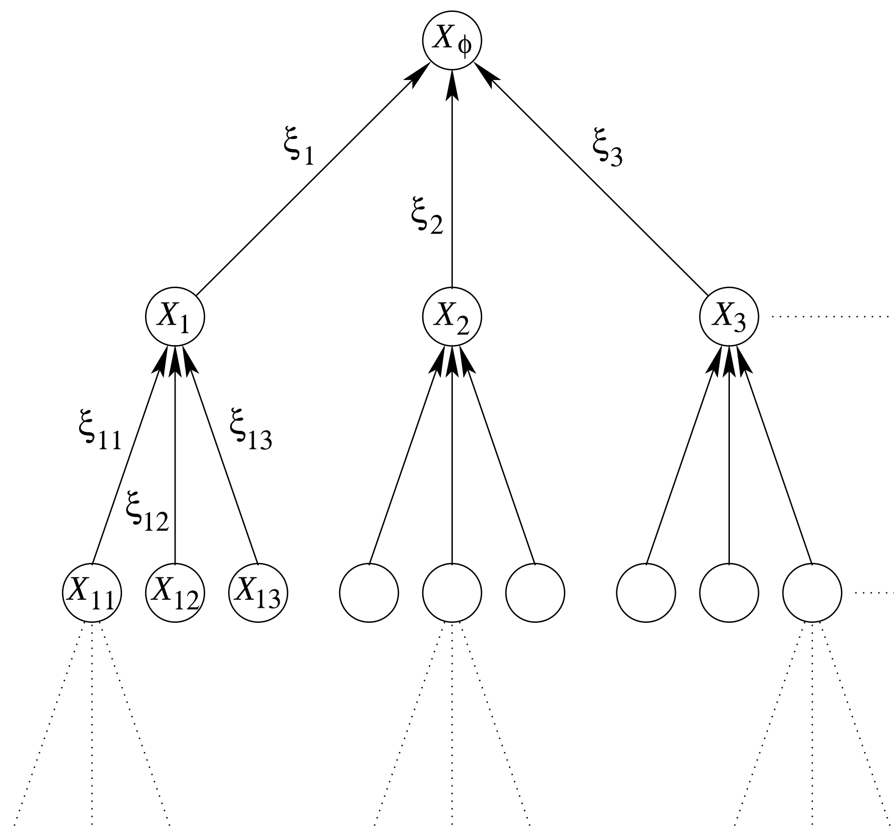
- Well studied in Probability as well as in Statistical Mechanics literature.
- Mézard and Parisi (1987) showed using *replica method* that $\lim_{n\to\infty} \mathbf{E}[A_n] = \pi^2/6$.
- Parisi (1998) conjectured that $\mathbf{E}[A_n] = \sum_{k=1}^{n} k^{-2}$.

Theorem 1 (Aldous, 2001)

$$\lim_{n\to\infty} \mathbf{E}[A_n] = \frac{\pi^2}{6} = \zeta(2).$$

- Aldous identified the limit constant $\zeta(2)$ in terms of *optimal matching* problem on a limit infinite tree with random edge weights, called **P***oisson* **W***eighted* **I***nfinite* **T***ree (PWIT)*.

Definition 1 (PWIT) *Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$,be the canonical infinite tree with $\mathcal{V} := \cup_{m=0}^{\infty} \mathbb{N}^m$ ($\mathbb{N}^0 = \{\emptyset\}$), and $\mathcal{E} := \{e = (\mathbf{i}, \mathbf{i}j) \mid \mathbf{i} \in \mathcal{V}, j \in \mathbb{N}\}$. Consider $\emptyset$ as the root. For every vertex $\mathbf{i} \in \mathcal{V}$, let $\left(\xi_{\mathrm{i}j}\right)_{j \geq 1}$ be points of independent Poisson point processes of rate 1 on $(0, \infty)$. Define the weight of the edge $e = (\mathbf{i}, \mathbf{i}j)$ as $\xi_{\mathrm{i}j}$. The tree $\mathcal{T}$ with the random edge weights is called PWIT.*



Poisson Weighted Infinite Tree (PWIT)

Logistic RDE

One of key step in Aldous' calculation for the optimal matching problem on PWIT is the following *recursive distributional equation (RDE)* :

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right),$$

where $(\xi_j)_{j \geq 1}$ are points of a Poisson point process of rate 1 on $(0, \infty)$, and are independent of $(X_j)_{j \geq 1}$, which are *i.i.d* with same law as of X.

Logistic RTP

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right)$$

- (Aldous, 2001) The above RDE has *unique* solution as *Logistic distribution*, given by

$$\mathbf{P}\left(X \leq x\right) = \frac{1}{1 + e^{-x}}, \;\; x \in \mathbb{R}.$$

- Using *Kolmogorov's consistency* one can construct random variables, say, $X_{\mathbf{i}}$ at each vertex $\mathbf{i}$ of the PWIT such that
  - ◇ $X_{\mathbf{i}} = \min_{j \geq 1} \left(\xi_{\mathbf{i}j} - X_{\mathbf{i}j} \right), \;\; \forall\, \mathbf{i} \in \mathcal{V}$;
  - ◇ $X_{\mathbf{i}} \sim$ Logistic distribution, $\;\forall\, \mathbf{i} \in \mathcal{V}$;
  - ◇ $X_{\mathbf{i}}$ is independent of $\left\{ \left(\xi_{\mathbf{i}'j} \right)_{j \geq 1} \;\middle|\; |\mathbf{i}'| < |\mathbf{i}| \right\}$, for all $\mathbf{i} \in \mathcal{V} \setminus \{\emptyset\}$, where $|\mathbf{i}| = d$ if $\mathbf{i} \in \mathbb{N}^d$.
- Thus we get an invariant *recursive tree process (RTP)* associated with the *Logistic RDE* on PWIT.

Question : Let $\mathcal{G}$ be the σ-field generated by the edge weights $\left\{ \left(\xi_{\mathbf{i}j}\right)_{j \geq 1} \mid \mathbf{i} \in \mathcal{V} \right\}$, then is $X_\emptyset$ $\mathcal{G}$-measurable ?

In other words, is the invariant RTP associated with the Logistic RDE *endogenous* ?

Theorem 2 (Aldous and B.) *$X_\emptyset$ is $\mathcal{G}$-measurable if and only if the bivariate RDE*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min\limits_{j \geq 1} (\xi_j - X_j) \\ \min\limits_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}$$

has unique solution as $X = Y$ a.s. with Logistic marginals, where $\left(\xi_j\right)_{j \geq 1}$ are points of a Poisson point process of rate 1 *on* $(0, \infty)$*, and are independent of $\left\{(X_j, Y_j)\right\}_{j \geq 1}$, which are* i.i.d. *having same distribution as of (X, Y)*

A straight forward approach :

- Let $\mathcal{G}_n$ be the σ-field generated by $\xi_{\mathbf{i}j}$'s in the first n generations, that is for $|\mathbf{i}| \leq n$. Naturally, $\mathcal{G}_n \uparrow \mathcal{G}$.

- By *martingale convergence* theorem

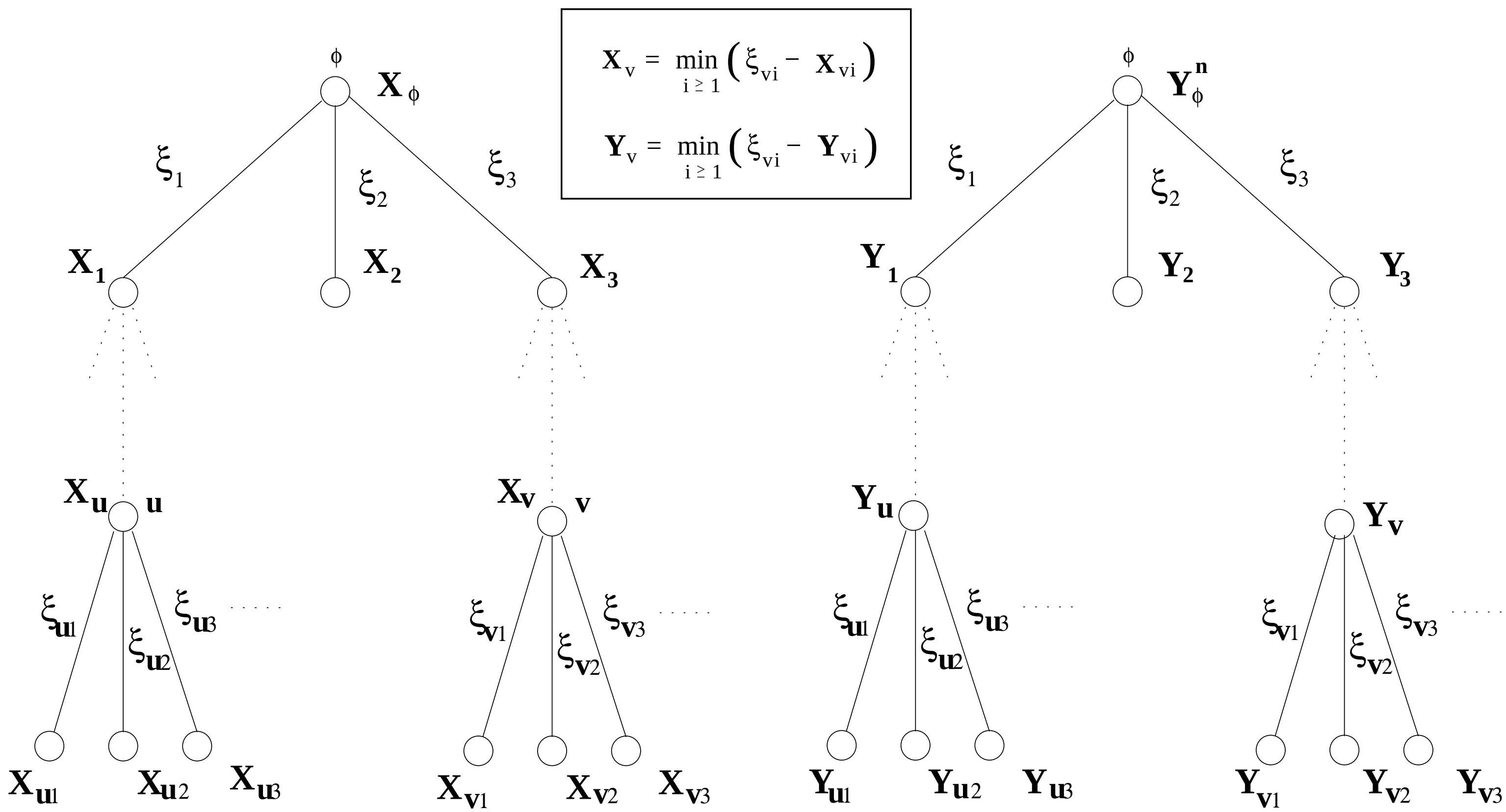
$$\mathbf{E}\left[X_\emptyset \,|\, \mathcal{G}_n\right] \underset{\mathcal{L}_2}{\overset{\text{a.s.}}{\longrightarrow}} \mathbf{E}\left[X_\emptyset \,|\, \mathcal{G}\right].$$

- So $X_\emptyset \in \mathcal{G} \iff \mathbf{E}\left[X_\emptyset - \mathbf{E}\left[X_\emptyset \,|\, \mathcal{G}_n\right]\right]^2 \to 0$ as $n \to \infty$.

- Notice that

$$\sigma_n^2 := \mathbf{E}\left[X_\emptyset - \mathbf{E}\left[X_\emptyset \,|\, \mathcal{G}_n\right]\right]^2 = \frac{1}{2}\mathbf{E}\left[\left(X_\emptyset - Y_\emptyset^n\right)^2\right]$$

where $X_\emptyset$ and $Y_\emptyset^n$ are independent and identical given $\mathcal{G}_n$.



$$X_v = \min_{i \geq 1} \left(\xi_{vi} - X_{vi} \right)$$

$$Y_v = \min_{i \geq 1} \left(\xi_{vi} - Y_{vi} \right)$$

At level n

- Further observe that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n$, and so the sequence $\left(X_\emptyset, Y_\emptyset^n\right)_{n\geq 1}$ is tight.

- It then follows that $\left(X_\emptyset, Y_\emptyset^n\right) \xrightarrow{d} (X, Y)$ for some random variables (X, Y).

- The distribution of (X, Y) then satisfies the following equation

$$\left(\begin{array}{c} X \\ Y \end{array} \right) \stackrel{d}{=} \left(\begin{array}{c} \min\limits_{j\geq 1}(\xi_j - X_j) \\ \min\limits_{j\geq 1}(\xi_j - Y_j) \end{array} \right),$$

where $(\xi_j)_{j\geq 1}$ are points of a Poisson point process of rate 1 on $(0, \infty)$, and are independent of $\{(X_j, Y_j)\}_{j\geq 1}$, which are *i.i.d.* having same distribution as of (X, Y).

- Thus $\sigma_n^2 \to 0$ if $X = Y$ a.s.

- So $X_\emptyset \in \mathcal{G}$ if $X = Y$ a.s.

Bivariate Uniqueness

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min\limits_{j \geq 1} (\xi_j - X_j) \\ \min\limits_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}$$

Definition 2 (Bivariate Uniqueness) *We say that* bivariate uniqueness *holds if the above bivariate fixed-point equation has unique solution as* $X = Y$ *a.s. and in this case with Logistic marginals.*

Theorem 3 *Bivariate uniqueness holds for the Logistic RDE on PWIT.*

Corollary 3.1 $X_\emptyset$ *is* $\mathcal{G}$*-measurable.*

Outline of the proof of Theorem 3

- The marginals of X and Y satisfy the Logistic RDE, so $X \stackrel{d}{=} Y$, and $X \sim$ Logistic distribution.

- Note that $(\xi_j; (X_j, Y_j))_{j \geq 1}$ is a Poisson point process on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $dt\, \nu(d(x,y))$, where $\nu = \text{Law}(X,Y)$. Let $G(x,y) := \mathbf{P}\,(X > x, Y > y)$, then

$$G(x,y) = \overline{H}(x)\,\overline{H}(y)\ \text{exp}\left(\int_0^\infty G(t-x, t-y)\,dt\right),$$

  where $\overline{H}(x) = e^{-x}/(1 + e^{-x})$ is the tail for the Logistic distribution function.

- To prove $X = Y$ a.s, it is enough to show that $X \wedge Y \stackrel{d}{=} X$.

- Let $g(x) := G(x,x) = \mathbf{P}\,(X \wedge Y > x)$, then

$$g(x) = \overline{H}^2(x)\ \text{exp}\left(\int_{-x}^\infty g(s)\,ds\right).$$

  It is enough to prove that $g = \overline{H}$ is the unique solution.

$$g(x) = \overline{H}^2(x) \; \text{exp} \left(\int_{-x}^{\infty} g(s) \, ds \right)$$

- Define $\mathfrak{F} := \left\{ f : \mathbb{R} \to [0,1] \mid \; \overline{H}^2(x) \leq f(x) \leq \overline{H}(x) \right\}$, and $T : \mathfrak{F} \to \mathfrak{F}$ as
$$T(f)(x) := \overline{H}^2(x) \; \text{exp} \left(\int_{-x}^{\infty} f(s) \, ds \right).$$

- Notice that, $g \in \mathfrak{F}$ and $g = T(g)$.

- It is enough to show that T has unique fixed point as $\overline{H}$ on $\mathfrak{F}$.

- Define a natural partial order, say, " $\preccurlyeq$ " on $\mathfrak{F}$ as $f \preccurlyeq h$ iff $f(x) \leq h(x), \; \forall \, x \in \mathbb{R}$.

- T is a monotone operator on $(\mathfrak{F}, \preccurlyeq)$, that is, $T(f) \preccurlyeq T(h) \iff f \preccurlyeq h$.

- Let $f_0 := \overline{H}^2$, define recursively $f_{n+1} := T(f_n)$.

- Observe that $f_n \preccurlyeq g \preccurlyeq \overline{H}$, so it is enough to prove that $f_n(x) \to \overline{H}(x)$ pointwise.

- Using induction one can show
$$f_n(x) = \overline{H}(x) \exp\left(-\beta_{n-1}\left(\overline{H}(x)\right)\right), \ n \geq 1;$$
where
$$\beta_n(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right) dw, \ n \geq 1,$$
with $\beta_0(s) = 1 - s$.

- Note that $\beta_n(1) = 0, \ \forall\, n$.

- It is enough to show that $\beta_n(x) \to 0$ pointwise.

$$\beta_n(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right)\, dw,\ n \geq 1$$

- Easy calculation shows that $\beta_n(s) \downarrow$ pointwise. Let $L(s) := \lim_{n\to\infty} \beta_n(s)$.

- $L(1) = 0$ and L satisfy the integral equation

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right)\, dw.$$

- Enough to show that $L \equiv 0$.

- Consider $\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1$, then it is easy to see that $\eta(0) = \eta(1) = 0$, and further

$$\eta'(w) = e^{-L(w)}\left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0.$$

  Thus $\eta \equiv 0 \iff L \equiv 0$.

Some final Remarks :

- Intuitively the natural approach to solve the bivariate uniqueness problem would be to use some kind of contraction argument. Unfortunately that seems to be rather hard in this case.

- For any *fixed-point equation* it is natural to ask what is its domain of attraction. In this case although both the Logistic RDE and its bivariate version have unique solutions but, it is not clear to us what are their domains of attraction.

Acknowledgment :

This work is a part of my doctoral dissertation, written under the guidance of Professor David J. Aldous, whom I would like to thank for suggesting the problem and for many illuminating discussions.