Recursive Distributional Equations and Associated Recursive Tree Processes

Antar Bandyopadhyay

(Joint work with Professor David J. Aldous)

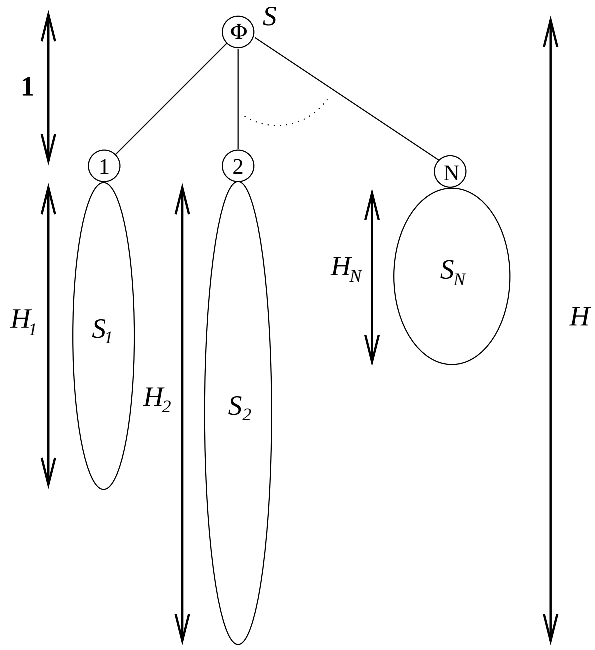
Institute for Mathematics and Its Applications
University of Minnesota

(http://www.ima.umn.edu/~antar)

February 26, 2004

An Easy Examples

Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.



Height of the Tree : Let $H :=$ 1+ height of the G-W tree, then $H < \infty$ a.s., and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on } \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

We will call such an equation a *Recursive Distributional Equation* (RDE).

Typical features of RDE

$$X \stackrel{d}{=} 1 + \max(X_1, X_2, \ldots, X_N) \quad \text{on } \mathbb{N}.$$

- **Unknown Quantity :** Distribution of X.
- **Known Quantities :**
  - $N \leq \infty$ which may or may not be random.
  - Possibly some more randomness whose distribution is known (not present in this example but will be so in others).
  - How we combine the known and unknown randomness (the "$1+\max$" operation in the example).
- **What is the RDE doing ?** It is to find a distribution μ such that when we take i.i.d. samples $(X_j)_{j \geq 1}$ from it and only use N many of them (where N is independent of the samples) and do the manipulation then we end up with another sample $X \sim \mu$.

Remark : In the case $N = 1$ a.s. it reduces to the question of finding a stationary distribution of a discrete time Markov chain.

Two main uses of RDEs

- **Direct use :** The RDE is used directly to define a distribution. Examples include,
  - The height (and also the size) of a (sub)-critical Galton-Watson tree (our first example).
  - The Quicksort distribution.
  - Discounted tree sums / inhomogeneous percolation on trees.
  - ... *and many others*.

- **Indirect use:** The RDE is used to define some auxiliary variables which help in defining/characterizing some other quantity of interest. Among others the following two type of applications are of special interest
  - 540° *argument* ! (will give an example).
  - Determining critical points and scaling laws (will not give an example).

General Setup

- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.

- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots; \infty\}$.

- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).

- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

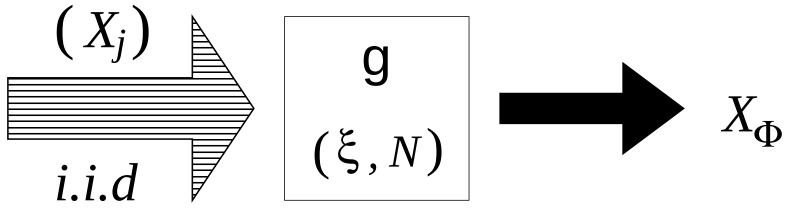
$$X \stackrel{d}{=} g\left(\xi; X_j, 1 \le j \le^* N\right), \quad \textit{on} \;\; S$$

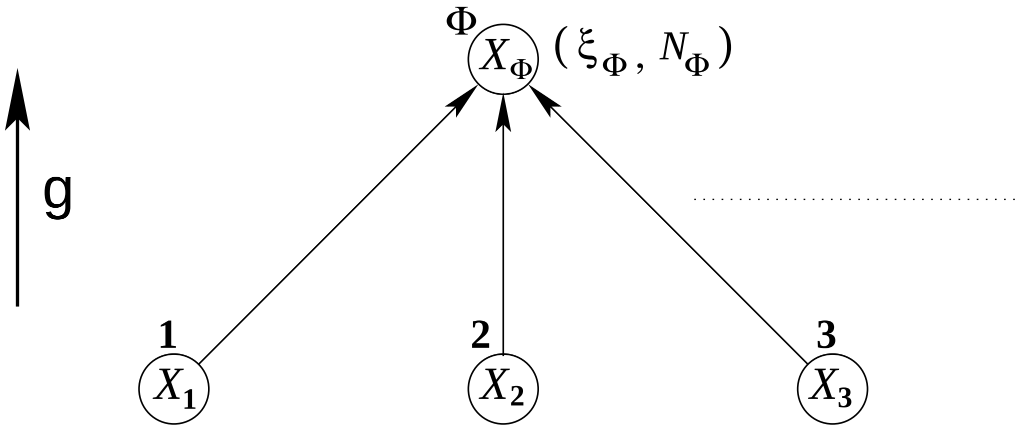
where $(X_j)_{j \ge 1}$ *are independent copies of* X *and are independent of* (ξ, N).

Remark : A more conventional (analysis) way of writing the equation would be

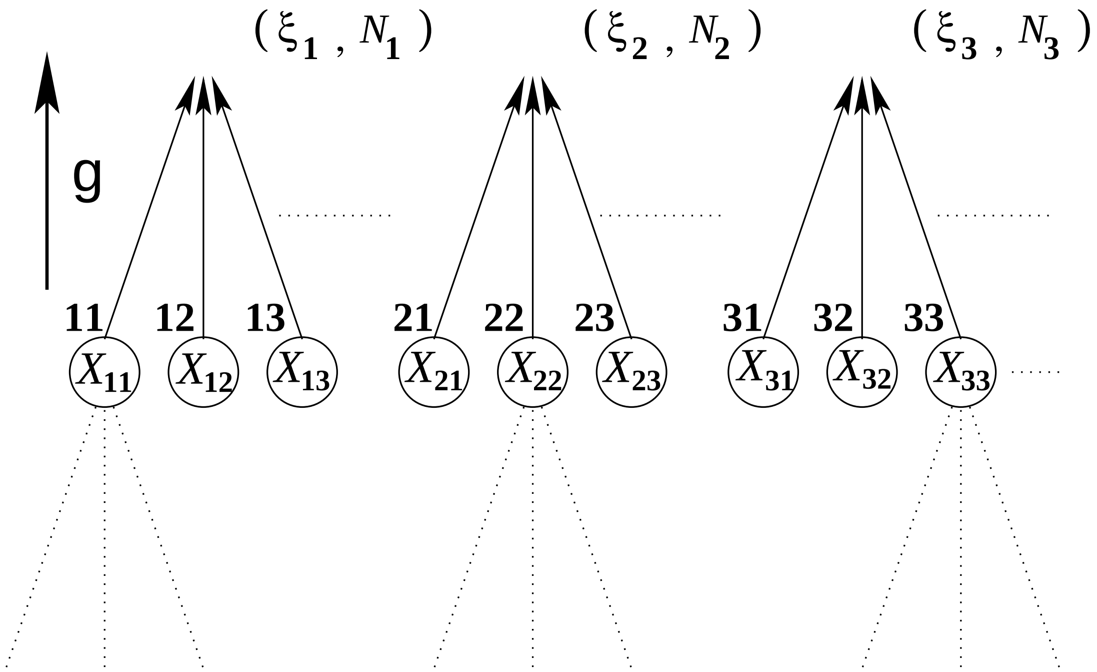
$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.



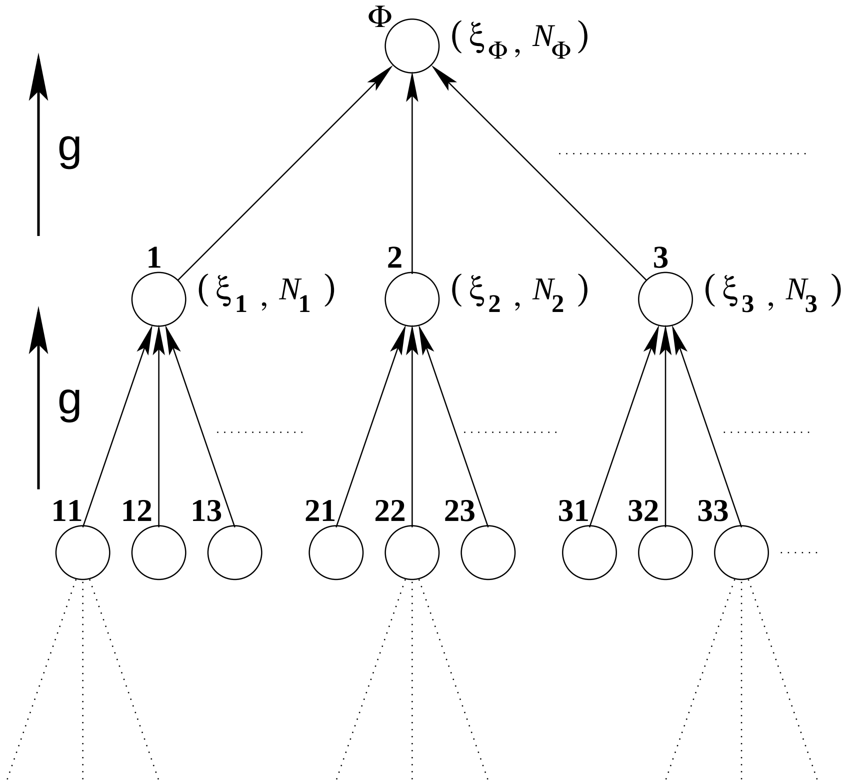


g
Φ
X_Φ
(ξ_Φ, N_Φ)
1
X_1
2
X_2
3
X_3



(ξ1 , N1)
(ξ2 , N2)
(ξ3 , N3)
g
11
12
13
21
22
23
31
32
33
X11
X12
X13
X21
X22
X23
X31
X32
X33

Recursive Tree Framework (RTF)

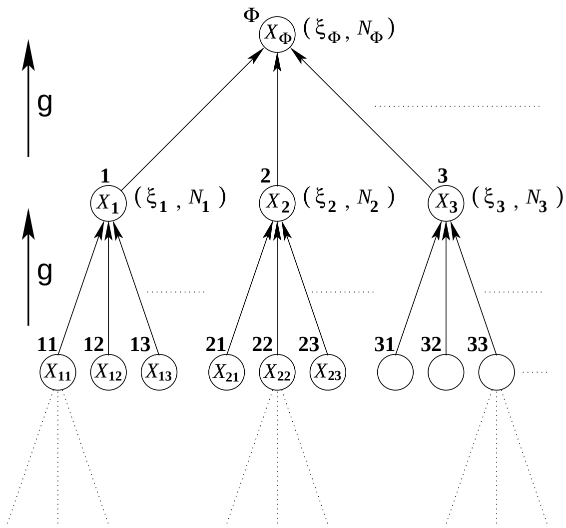


- **Skeleton :** $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ is the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,|\, \mathbf{i} \in \mathbb{N}^d, \, d \geq 1 \right\} \cup \{\emptyset\}$, and edge set $\mathcal{E} := \{ e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N} \}$, and root $\emptyset$.

- **Innovations :** Collection of **i.i.d** pairs $\{ (\xi_\mathbf{i}, N_\mathbf{i}) \,|\, \mathbf{i} \in \mathcal{V} \}$.

- **Function :** The function $g(\cdot)$.

Recursive Tree Process (RTP)



Consider a **RTF** and let μ be a solution of the associated **RDE** . A collection of S-valued random variables $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $X_{\mathbf{i}} \sim \mu \ \forall \ \mathbf{i} \in \mathcal{V}$.
- $X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; X_{\mathbf{i}j}, 1 \leq j \leq^* N_{\mathbf{i}}\right) \ \ \forall \ \mathbf{i} \in \mathcal{V}$.
- $X_{\mathbf{i}}$ is independent of $\{(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \mid |\mathbf{i}'| < |\mathbf{i}| \}$, for all $\mathbf{i} \in \mathcal{V} \backslash \{\emptyset\}$, where $|\mathbf{i}| = d$ if $\mathbf{i} \in \mathbb{N}^d$.

Remark : Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the associated RDE.

Example of a 540° Argument
Frozen Percolation on Regular Binary Tree

The Setup :

- Let $\mathbb{T}_3 = (\mathbb{V}, \mathbb{E})$ be the infinite regular binary tree.
- Each edge $e \in \mathbb{E}$ is equipped with independent edge weight $U_e \sim \text{Uniform}[0, 1]$.
- Think of time moving from 0 to 1.

Frozen Percolation Process (informal description):

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e if each of its end vertex is in a finite component; otherwise do not open e.
- Let $(\mathcal{A}_t)_{t \geq 0}$ be set process of open edges starting from $\mathcal{A}_0 = \emptyset$.

The Regular Percolation Process :

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e.

- If $(\mathcal{B}_t)_{t \geq 0}$ be the set process of open edges the it can be described as

$$\mathcal{B}_t = \{e \in \mathbb{E} \,|\, U_e \leq t\}$$

Remarks : Unlike the regular percolation process it is not clear whether the *frozen percolation process* exists and if so whether it admits a simpler description using only the edge weights.

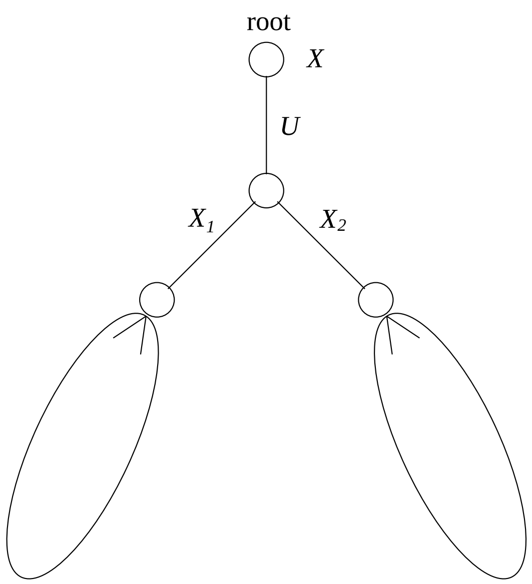
Two Easy Observations : If frozen percolation process exists then following must hold

- $\mathcal{A}_t \subseteq \mathcal{B}_t$ for all $t \in [0, 1]$.

- $\mathcal{A}_t = \mathcal{B}_t$ if $t \leq \frac{1}{2}$ (since the critical probability for infinite binary tree is $\frac{1}{2}$).

540° **Argument** [Aldous, 2000]

- **Stage 1 :** Suppose that the process exists on $\mathbb{T}_3$. Let $\widetilde{\mathbb{T}_3}$ be the *planted* binary tree which is a modification of $\mathbb{T}_3$ where we distinguish a vertex of degree 1 as the *root* and all other vertices have degree 3.

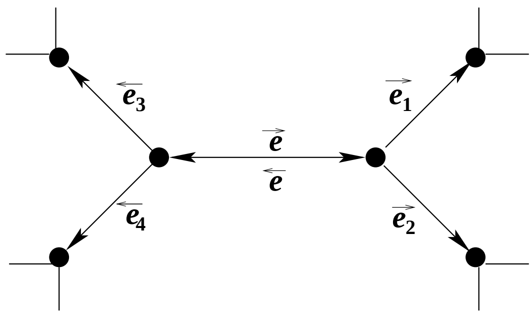


  - ▶ $X :=$ Time it takes for the root to join ∞ (will write $X = \infty$ if it never joins).
  - ▶ $X_j :=$ Time it takes for the root to join to ∞ in the j^{th} sub-tree for $j = 1, 2$.
  - ▶ X_1 and X_2 are independent copies of X.
  - ▶ It is easy to see that

$$X \stackrel{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases}$$

- **Stage 2 :**

  - ▶ The RDE has only one solution with full support given by
$$\mu(dy) = \frac{dy}{2y^2},\ \frac{1}{2} < y < 1,\ \mu(\{\infty\}) = \frac{1}{2}.$$
So using the general theory we can construct the invariant RTP with marginal μ.



  - ▶ Each edge $e \in \mathbb{E}$ defines two directed edges, and each directed edge $\overrightarrow{e}$ defines one *planted* tree, let $X_{\overrightarrow{e}}$ be the corresponding X variable.

  - ▶ Each directed edge $\overrightarrow{e}$ has two children say $\overrightarrow{e}_1$ and $\overrightarrow{e}_2$ then $\left\{X_{\overrightarrow{e}_1}, X_{\overrightarrow{e}_2}\right\}$ and $X_{\overrightarrow{e}}$ satisfies the equation with the edge weight U_e.

  - ▶ Each edge $e \in \mathbb{E}$ has a set of four *children* which are the four directed edges away from e. We denote it by $\partial\{e\}$.

  - ▶ Define $\mathcal{A}_1 := \left\{e \in \mathbb{E} \,|\, U_e < \min\left(X_f : f \in \partial\{e\}\right)\right\}$ and $\mathcal{A}_t := \{e \in \mathcal{A}_1 \,|\, U_e \le t\}$ for $0 \le t < 1$.

- **Stage 3 :** Using this *external* random variables $\left(X_{\overrightarrow{e}}\right)$ repeat the original computation to prove the existence of a frozen percolation process on $\mathbb{T}_3$. In fact it is easy to see that this construction gives an automorphism invariant version of the process.

Remark : The construction of the process not only uses the edge weights (U_e) but also (possibly) *external* random variables, namely $\left(X_{\overrightarrow{e}}\right)$.

Endogeny

Natural Question : Does $X_\emptyset$ only depend on the innovation process (the *data*) $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$?

Remark : For the frozen percolation process on $\mathbb{T}_3$ the above question translates to whether or not the variables $\left(X_{\overrightarrow{e}}\right)$ are measurable only with respect to the edge weights (U_e). If so then the process $(\mathcal{A}_t)$ will not have any *external* randomness in it. This will then imply that the informal description defines a process for $\mathbb{T}_3$.

Definition 2 *Let $\mathcal{G}$ be the σ-field generated by the* innovation process $\{ (\xi_{\mathbf{i}}, N_{\mathbf{i}}) \,|\, \mathbf{i} \in \mathcal{V} \}$*. We will say an invariant RTP is* endogenous *if $X_\emptyset$ is $\mathcal{G}$-measurable.*

Motivations

- Presence / absence of *external* randomness.
- Influence of the boundary at infinity !
- Relation with *long-range independence* ?

One easy fact to built our confidence

Remark : Associated with a RTF there is a Galton-Watson branching process tree rooted at $\emptyset$ defined only through $\{N_{\mathbf{i}} \,|\, \mathbf{i} \in \mathcal{V}\}$, call it $\mathcal{T}$. Essentially any associated invariant RTP lives on $\mathcal{T}$.

Proposition 1 *If $\mathcal{T}$ is almost surely finite (equivalently $\mathbf{E}[N] \leq 1$) then the associated RDE has unique solution and the RTP is endogenous.*

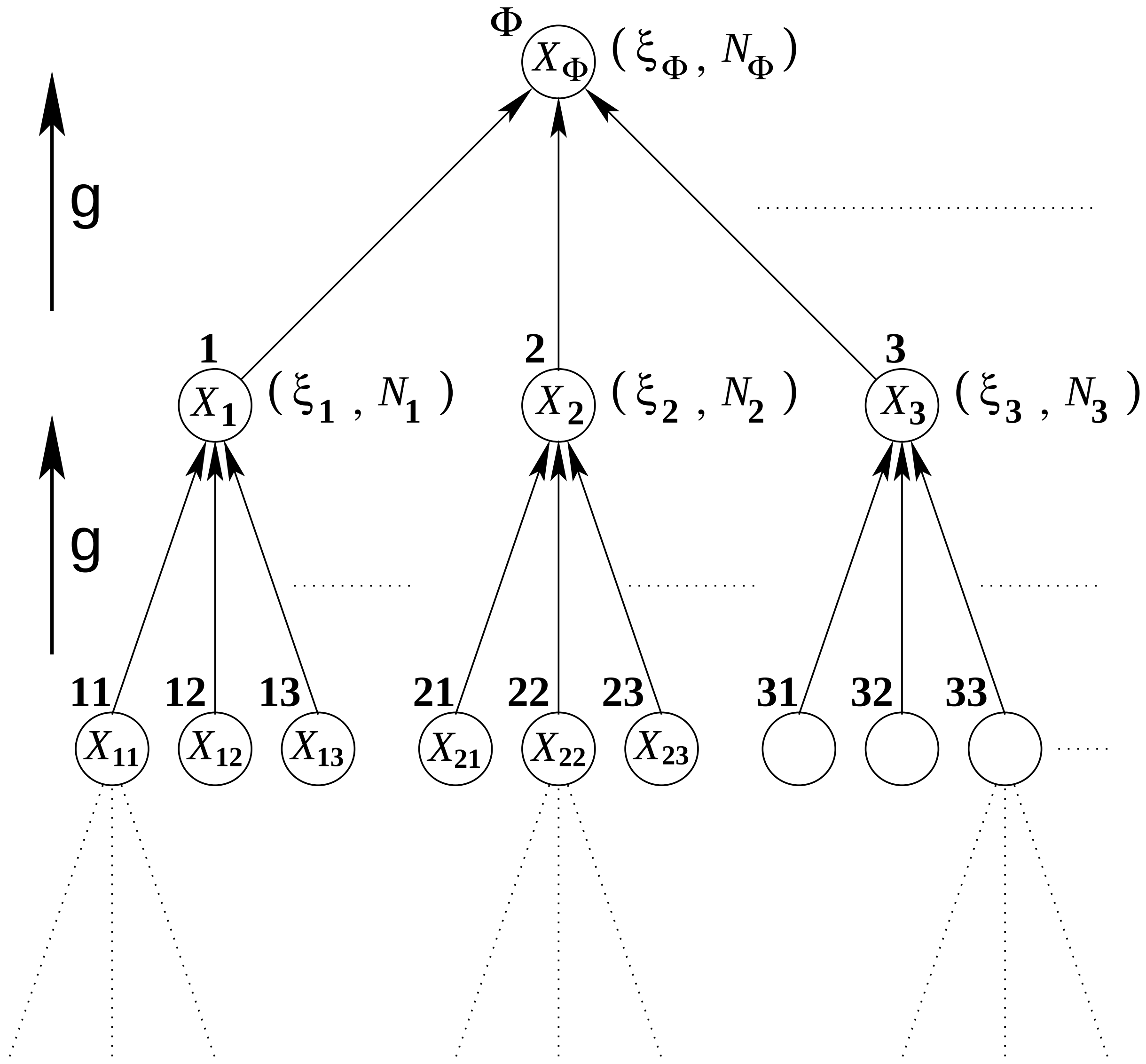
Remarks :

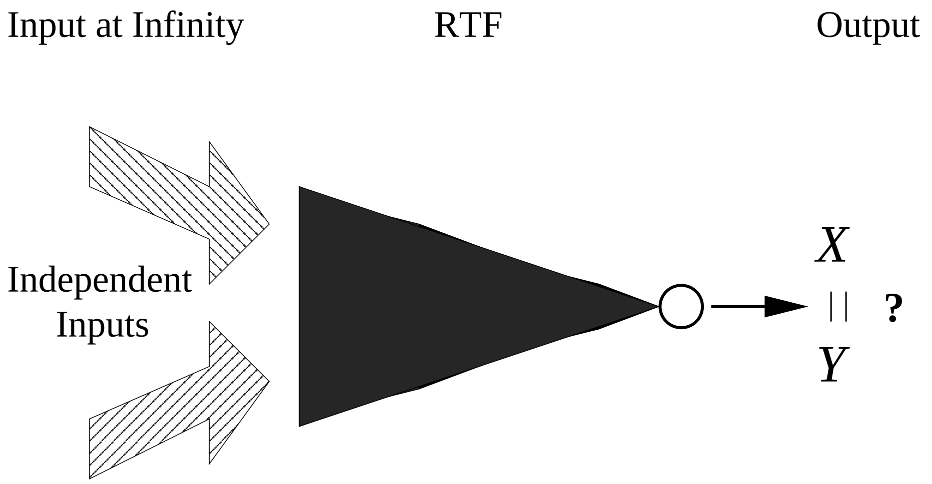
- The RDE in our first example has unique solution and it is endogenous.

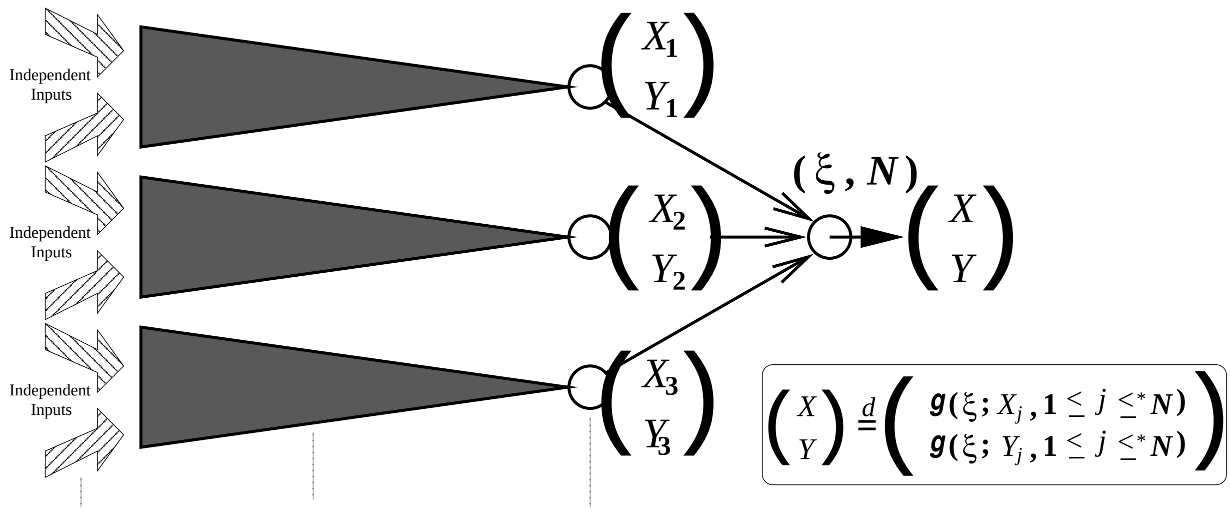
- Perhaps the simplest example of an RDE where no solution is endogenous is the following

$$X \stackrel{d}{=} \frac{X_1 + X_2}{\sqrt{2}}.$$

The solution set is the Normal$(0, \sigma^2)$ family. But the associated RTF has *no randomness* involved and hence none of the RTP is endogenous.


Φ
X_Φ
(ξ_Φ, N_Φ)
g
g
1
2
3
X_1
(ξ_1, N_1)
X_2
(ξ_2, N_2)
X_3
(ξ_3, N_3)
11
12
13
21
22
23
31
32
33
X_{11}
X_{12}
X_{13}
X_{21}
X_{22}
X_{23}


Input at Infinity
RTF
Output
Independent Inputs
X
||
?
Y


Independent Inputs
Independent Inputs
Independent Inputs
X1 Y1
X2 Y2
X3 Y3
(ξ, N)
X Y
(X, Y) =d (g(ξ; Xj, 1 ≤ j ≤* N), g(ξ; Yj, 1 ≤ j ≤* N))

Bivariate Uniqueness

Consider the following **bivariate RDE**,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g\left(\xi; X_j, 1 \leq j \leq^* N\right) \\ g\left(\xi; Y_j, 1 \leq j \leq^* N\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d and has the same law as of (X, Y), and are independent of the innovation (ξ, N).

Definition 3 *An invariant RTP with marginal μ has* **bivariate uniqueness** *property if the above* bivariate RDE *has unique solution as $X = Y$ a.s on the space of joint probabilities with both marginals μ.*

An Equivalence Theorem

Theorem 1 *Suppose the* S *is a* Polish *space. Consider an invariant RTP with marginal distribution* μ.

(a) If the endogenous property holds then the bivariate uniqueness property holds.

(b) Conversely, (under some technical conditions) if the bivariate uniqueness property holds and then the endogenous property holds.

(c) If $T^{(2)}$ *be the operator associated with the* bivariate RDE *then endogenous property holds if and only if*

$$T^{(2)^n}(\mu \otimes \mu) \xrightarrow{d} \mu^{\nearrow},$$

where $\mu \otimes \mu$ *is the product measure, and* $\mu^{\nearrow}$ *is the measure concentrated on the diagonal with both marginal* μ.

Remark : Similar kind of result appears in the study of Gibbs measures and Markov random fields.

Back to Frozen Percolation RDE

- Recall the RDE associated with the frozen percolation process,

$$X \stackrel{d}{=} \Phi\left(X_1 \wedge X_2; U\right)$$

where X_1, X_2 are independent copies of X and are independent of $U \sim$ Uniform$[0,1]$ and the function Φ is given by

$$\Phi(x;u) := \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases}.$$

- Also recall that it has *unique* solution with full support given by

$$\mu(dy) = \frac{dy}{2y^2},\ \frac{1}{2} < y < 1,\ \ \mu(\{\infty\}) = \frac{1}{2}.$$

Theorem 2 *The invariant RTP with marginal μ has bivariate uniqueness property, that is, the following* bivariate RDE *has* unique *solution as $X = Y$ a.s with marginal μ*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2; U\right) \\ \Phi\left(Y_1 \wedge Y_2; U\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j=1,2}$ are independent copies of (X, Y), and are independent of $U \sim$ Uniform$[0, 1]$.

Corollary 2.1 *The invariant RTP with marginal μ is* endogenous. *Thus the* frozen percolation process *on $\mathbb{T}_3$ as constructed is measurable with respect to the edge weights.*

Outline of the proof of Theorem 2

- Notice that X and Y have the same distribution μ. So if $F(x,y) = \mathbf{P}(X \le x, Y \le y)$ and $G(x,y) = \mathbf{P}(X > x, Y > y)$ then for every $x, y \in [\frac{1}{2}, 1]$

$$G(x,y) = F(x,y) + \frac{1}{2x} + \frac{1}{2y} - 1.$$

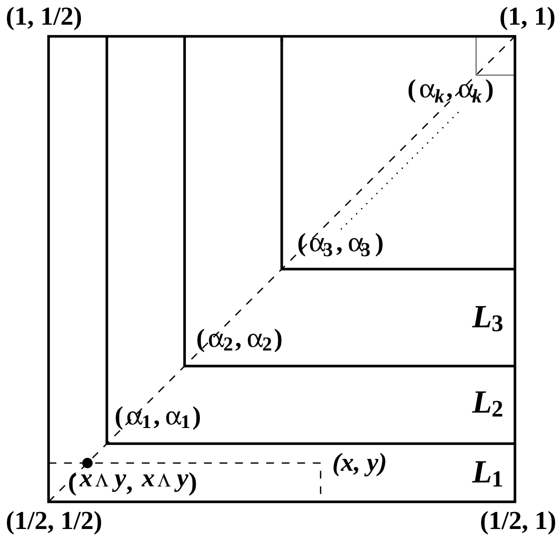
- From the bivariate RDE we get

$$F(x,y) =$$

$$\int_0^{x \wedge y} \left(G^2(x,y) - G^2(x,u) - G^2(u,y) + G^2(u,u) \right) du.$$

- We know that $X = Y$ a.s. is a solution so $G_0(x,y) = \frac{1}{2(x \vee y)}$ is a solution of the integral equation. It is enough to prove that $G = G_0$ is the *only* solution.

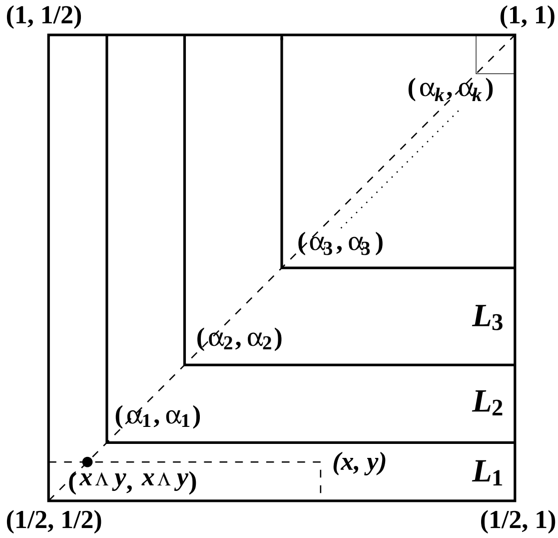
- Let $H(x,y) = 1 - \frac{G(x,y)}{G_0(x,y)}$, so we need to show $H \equiv 0$ on $D := [\frac{1}{2}, 1]^2$.



- Substituting back into the equation and after some algebra we get
$$H(x,y) = \frac{1}{G_0(x,y)} \int_0^{x\wedge y} \Lambda(x,y,u)\,du,$$
where Λ is a function (has long expression !) which satisfies the estimate
$$|\Lambda(x,y,u)| \le 4G_0^2(u,u)\left(2|H(x,y)| + |H^2(x,y)|\right),$$
whenever $u \le x \wedge y$.

- Find $\frac{1}{2} = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$ such that
$$\int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u,u)\,du < \frac{1}{48}.$$

- We partition D into L-shape parts (as in the figure) where $L_i := \{(x,y) \,|\, \alpha_{i-1} \le x \wedge y < \alpha_i\}$.



- Define $\| H \|_i := \sup_{x,y \in L_i} |H(x,y)|$.

- Start with $i = 1$, let $(x,y) \in L_1$. Note $G_0(x,y) \geq \frac{1}{2}$. Thus from the estimate of Λ we get

$$
\begin{aligned}
|H(x,y)| &\leq 24 \| H \|_i \int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u,u)\,du \\
&\leq \frac{1}{2} \| H \|_i
\end{aligned}
$$

- Thus $H \equiv 0$ on L_1. Now proceed inductively for $i = 2, 3, \ldots k$ to conclude $H \equiv 0$ on D.