Recursive Distributional Equations and Associated Recursive Tree Processes

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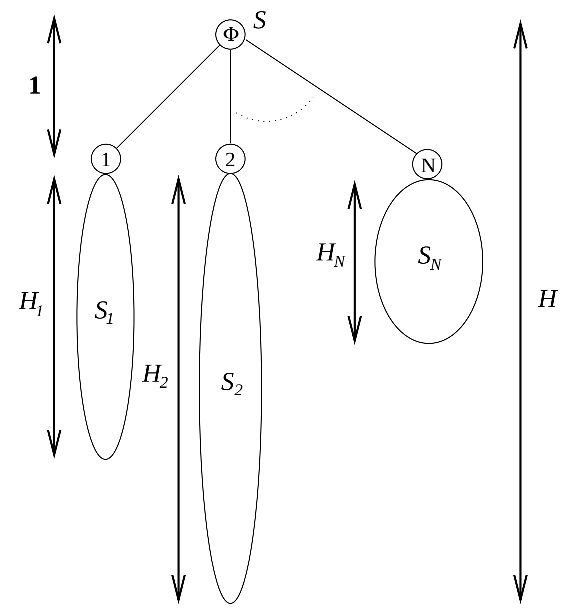
Two Easy Examples

Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$, and assume that $\mathbf{P}(N = 1) < 1$.



- **Size of the Population :** Let S be the total size of the population then

$$S \stackrel{d}{=} \sum_{j=1}^{N} S_j \quad \text{on} \quad \mathbb{Z}_+.$$

- **Height of the Tree :** Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s., and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on} \quad \mathbb{N}.$$

Some areas where RDEs appear :

- Probabilistic analysis of algorithms with suitable recursive structures (e.g. Quicksort Algorithm).
- Study of *branching random walks* (e.g. the rightmost position of a BRW).
- Statistical physics models on trees (e.g. *frozen percolation process* on infinite binary tree).
- Statistical physics and algorithmic questions in the mean-field model of distance (e.g. mean-field *traveling salesman problem*).
- Probabilistic study of combinatorial optimization and *local weak convergence* method (e.g. mean-field *random assignment problem*).

General Setup

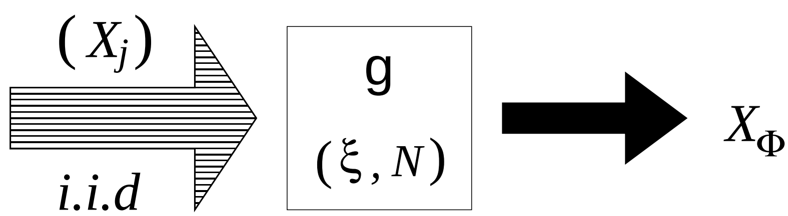
- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.
- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots ; \infty\}$.
- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).
- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

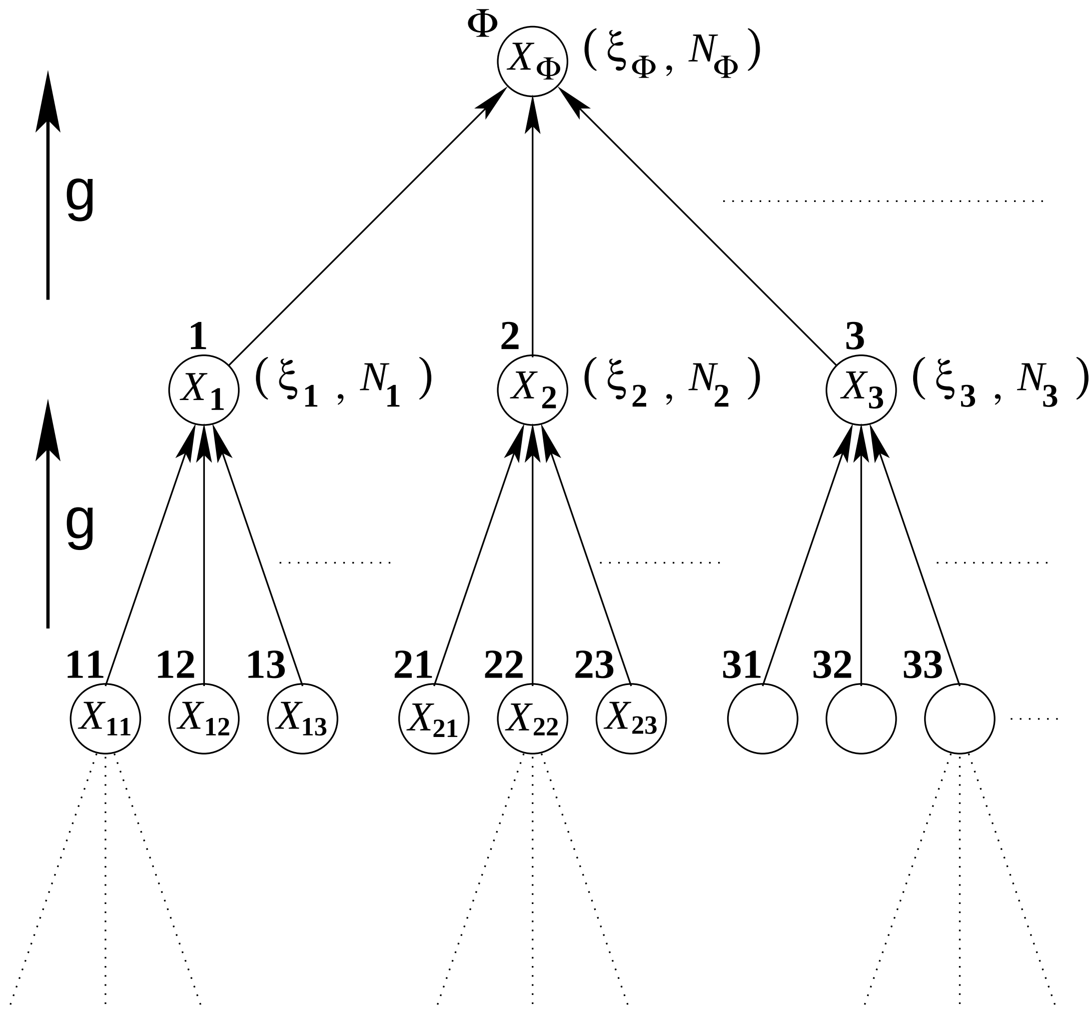
Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

$$X \stackrel{d}{=} g\left(\xi; X_j, 1 \leq j \leq^* N\right),$$

where X_j*'s has the same distribution as of* X*.*




Φ
X_Φ
(ξ_Φ, N_Φ)
g
g
1
2
3
X_1
(ξ_1, N_1)
X_2
(ξ_2, N_2)
X_3
(ξ_3, N_3)
11
12
13
21
22
23
31
32
33
X_{11}
X_{12}
X_{13}
X_{21}
X_{22}
X_{23}

Recursive Tree Framework (RTF) :

- **Skeleton :** $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ is the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,|\, \mathbf{i} \in \mathbb{N}^d,\, d \geq 1 \right\} \cup \{\emptyset\}$, and edge set $\mathcal{E} := \{ e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N} \}$, and root $\emptyset$.

- **Innovations :** Collection of **i.i.d** pairs $\{(\xi_\mathbf{i}, N_\mathbf{i}) \,|\, \mathbf{i} \in \mathcal{V}\}$.

- **Function :** The function $g(\cdot)$.

Recursive Tree Process (RTP) :

Consider a **RTF** and let μ be a solution of the associated **RDE** . A collection of S-valued random variables $(X_\mathbf{i})_{\mathbf{i} \in \mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $X_\mathbf{i} = g\left(\xi_\mathbf{i}; X_{\mathbf{i}j}, 1 \leq j \leq^* N_\mathbf{i}\right) \;\; \forall\, \mathbf{i} \in \mathcal{V}$.

- $X_\mathbf{i} \sim \mu \;\; \forall\, \mathbf{i} \in \mathcal{V}$.

- $X_\mathbf{i}$ is independent of $\{(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \,|\, |\mathbf{i}'| < |\mathbf{i}| \}$, for all $\mathbf{i} \in \mathcal{V} \setminus \{\emptyset\}$, where $|\mathbf{i}| = d$ if $\mathbf{i} \in \mathbb{N}^d$.

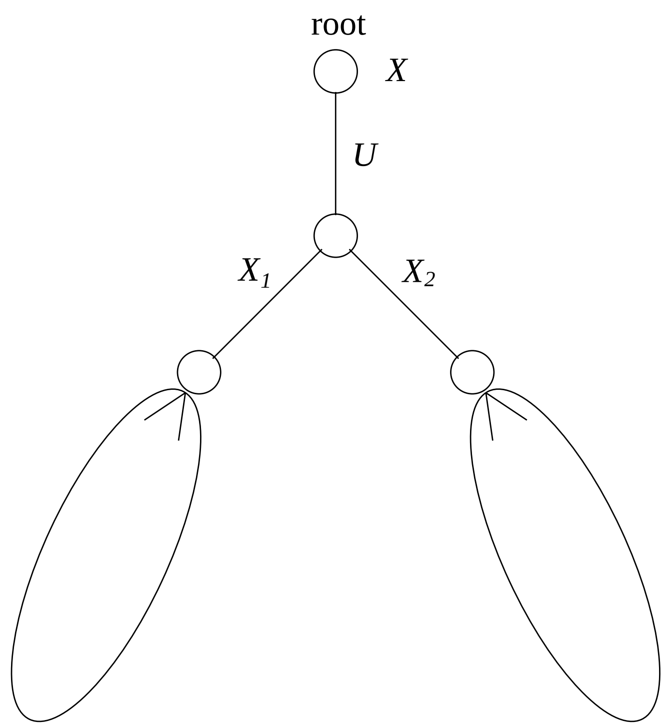
Remark : Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the associated RDE.

Example : Frozen Percolation on $\mathbb{T}_3$

Informal description :

- Each edge e of $\mathbb{T}_3$ has weight U_e which are **i.i.d** Unif$[0,1]$.
- Think of time is moving from 0 to 1.
- At a time t let $\mathcal{A}_t$ be the set of open edges of $\mathbb{T}_3$. For each edge e, at time $t = U_e$ set $\mathcal{A}_t = \mathcal{A}_{t-} \cup \{e\}$ if each end-vertex of e is in a finite cluster of $\mathcal{A}_{t-}$, otherwise set $\mathcal{A}_t = \mathcal{A}_{t-}$.

RDE for Frozen Percolation Process



Let $X :=$ time for the root to join ∞. Similarly define X_1 and X_2 which are **i.i.d** with same distribution as of X, then

$$X \stackrel{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases}$$

$$X \overset{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases}$$

Known [Aldous, 2000] :

- Above RDE has unique solution μ with support I, and it is given by

$$\mu(dx) = \tfrac{1}{2x^2}, \text{ on } (\tfrac{1}{2}, 1], \text{ and } \mu(\{\infty\}) = \tfrac{1}{2}.$$

- Using the invariant RTP associated with this RDE, one can rigorously construct an invariant *frozen percolation process* on $\mathbb{T}_3$, which satisfy the informal description.

- $\mathcal{A}_1$ is a random forest on $\mathbb{T}_3$ with both finite and infinite components.

- The process so constructed depends on the edge-weights as well as on the RTP (X_{i}).

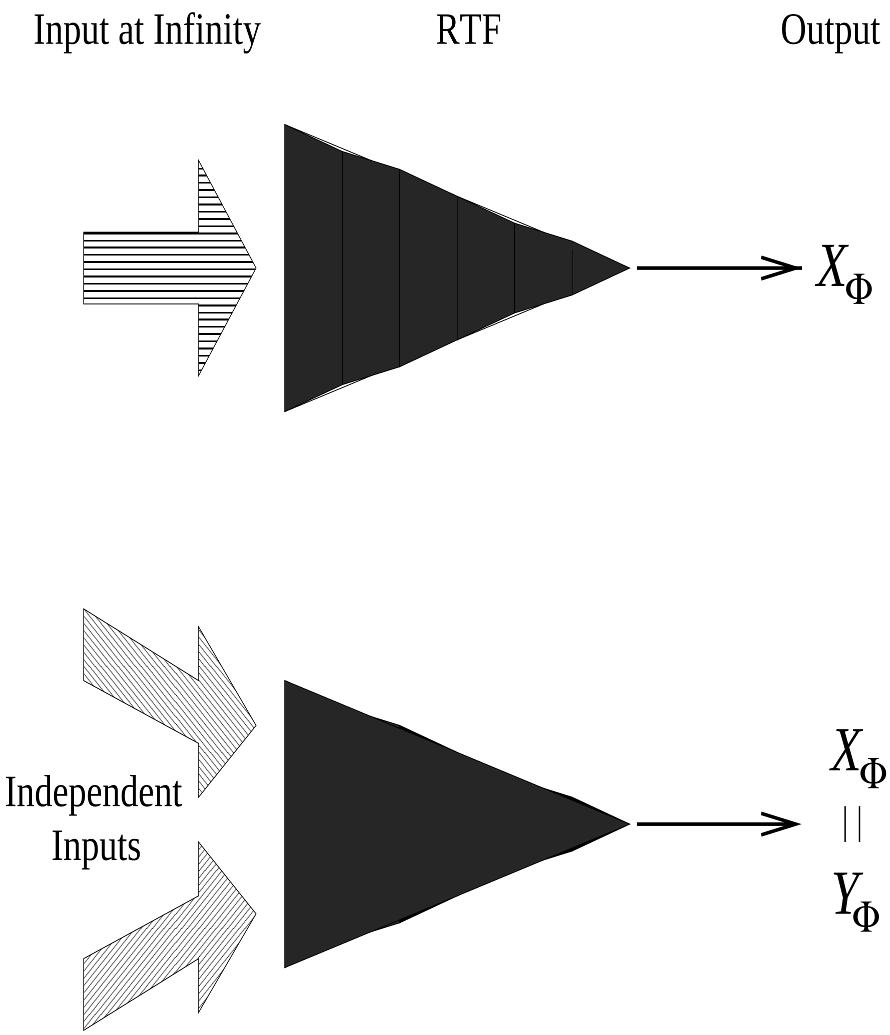
Natural Question : Is this process defined only through the edge-weights ?

Question : Let $\mathcal{G}$ be the σ-field generated by the *innovations* $\{(\xi_{\mathbf{i}}, N_{\mathbf{i}}) \,|\, \mathbf{i} \in \mathcal{V}\}$, then is $X_\emptyset$ $\mathcal{G}$-measurable ?

Definition 2 *We say an invariant RTP is* endogenous *if $X_\emptyset$ is $\mathcal{G}$-measurable.*

Motivation (from Frozen Percolation) : In *frozen percolation* process example if the associated RTP is *endogenous* then the informal description of the process through its edge-weights will be the rigorous description also.

Remark : Associated with each RTF there is a Galton-Watson branching process defined only through $\{N_{\mathbf{i}} \,|\, \mathbf{i} \in \mathcal{V}\}$, call it $\mathcal{T}$. It can be easily shown that if $|\mathcal{T}| < \infty$ a.s, then the RDE has unique solution and the RTP is endogenous.



Input at Infinity
RTF
Output
X_Φ
Independent
Inputs
X_Φ
||
Y_Φ

Bivariate Uniqueness

Consider the following **bivariate RDE**,

$$\left(\begin{array}{c} X \\ \\ Y \end{array}\right) \stackrel{d}{=} \left(\begin{array}{c} g\left(\xi; X_j, 1 \leq j \leq^* N\right) \\ \\ g\left(\xi; Y_j, 1 \leq j \leq^* N\right) \end{array}\right)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d and has the same law as of (X, Y), and are independent of the innovation (ξ, N).

Definition 3 *An invariant RTP with marginal μ has* **bivariate uniqueness** *property if the above* bivariate RDE *has unique solution as $X = Y$ a.s on the space of joint probabilities with both marginals μ.*

Equivalence Theorem

Theorem 1 *Suppose the S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the endogenous property holds then the bivariate uniqueness property holds.

(b) Conversely, under some technical conditions, if the bivariate uniqueness property holds and then the endogenous property holds.

(c) If $T^{(2)}$ be the operator associated with the bivariate RDE *then endogenous property holds if and only if*

$$T^{(2)^n}(\mu \otimes \mu) \xrightarrow{d} \mu^{\nearrow},$$

where $\mu \otimes \mu$ is the product measure, and $\mu^{\nearrow}$ is the measure concentrated on the diagonal with both marginal μ.

Remark : Similar kind of result can be found in the study of Gibbs measure and Markon random fields.

Logistic RDE and RTP

Consider the following RDE,

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right)$$

where $(\xi_j)_{j \geq 1}$ are points of a rate 1 Poisson point process on $(0, \infty)$.

- (Aldous, 2001) The above RDE occurs in the study of *mean-field random assignment problem*

- (Aldous, 2001) It has *unique* solution as *Logistic distribution*, given by

$$\mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \;\; x \in \mathbb{R}.$$

- So we can construct an invariant RTP associated with the *Logistic RDE* on $\mathbb{T}_\infty$, which we call *Logistic RTP*.

Theorem 2 *The invariant* Logistic RTP *has bivariate uniqueness property, that is, the following* bivariate RDE *has* unique *solution as* $X = Y$ *a.s with* Logistic *marginal*

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min\limits_{j \geq 1} (\xi_j - X_j) \\ \\ \min\limits_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}$$

where $(\xi_j)_{j\geq 1}$ *are points of a rate* 1 *Poisson point process on* $(0,\infty)$, *and* $(X_j, Y_j)_{j\geq 1}$ *are* i.i.d *with same joint distribution as of* (X,Y), *and are independent of* $(\xi_j)_{j\geq 1}$.

Corollary 2.1 *The invariant* Logistic RTP *is endogenous.*

Outline of the proof of Theorem 2

- The marginals of X and Y satisfy the Logistic RDE, so $X \stackrel{d}{=} Y$, and $X \sim$ Logistic distribution.

- Note that $(\xi_j; (X_j, Y_j))_{j \geq 1}$ is a Poisson point process on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $dt\, \nu(d(x,y))$, where $\nu = \text{Law}(X, Y)$. Let $G(x,y) := \mathbf{P}\left(X > x, Y > y\right)$, then

$$G(x,y) = \overline{H}(x)\, \overline{H}(y) \, \exp\left(\int_0^\infty G(t-x, t-y)\, dt \right),$$

  where $\overline{H}(x) = e^{-x}/(1 + e^{-x})$ is the tail for the Logistic distribution function.

- To prove $X = Y$ a.s, it is enough to show that $X \wedge Y \stackrel{d}{=} X$.

- Let $g(x) := G(x,x) = \mathbf{P}\left(X \wedge Y > x\right)$, then

$$g(x) = \overline{H}^2(x) \, \exp\left(\int_{-x}^\infty g(s)\, ds \right).$$

  It is enough to prove that $g = \overline{H}$ is the unique solution.

$$g(x) = \overline{H}^2(x) \text{ exp} \left(\int_{-x}^{\infty} g(s)\, ds \right)$$

- Define $\mathfrak{F} := \left\{ f : \mathbb{R} \to [0,1] \mid \overline{H}^2(x) \le f(x) \le \overline{H}(x) \right\}$, and $T : \mathfrak{F} \to \mathfrak{F}$ as

$$T(f)(x) := \overline{H}^2(x) \text{ exp} \left(\int_{-x}^{\infty} f(s)\, ds \right).$$

- Notice that, $g \in \mathfrak{F}$ and $g = T(g)$.

- It is enough to show that T has unique fixed point as $\overline{H}$ on $\mathfrak{F}$.

- Define a natural partial order, say, " $\preccurlyeq$ " on $\mathfrak{F}$ as $f \preccurlyeq h$ iff $f(x) \le h(x), \ \forall\, x \in \mathbb{R}$.

- T is a monotone operator on $(\mathfrak{F}, \preccurlyeq)$, that is, $T(f) \preccurlyeq T(h) \iff f \preccurlyeq h$.

- Let $f_0 := \overline{H}^2$, define recursively $f_{n+1} := T(f_n)$.

- Observe that $f_n \preccurlyeq g \preccurlyeq \overline{H}$, so it is enough to prove that $f_n(x) \to \overline{H}(x)$ pointwise.

- Using induction one can show

$$f_n(x) = \overline{H}(x) \exp\left(-\beta_{n-1}\left(\overline{H}(x)\right)\right), \; n \geq 1;$$

  where

$$\beta_n(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right) dw, \; n \geq 1,$$

  with $\beta_0(s) = 1 - s$.

- Note that $\beta_n(1) = 0, \; \forall\, n$.

- It is enough to show that $\beta_n(x) \to 0$ pointwise.

$$\beta_n(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right) dw,\ n \geq 1$$

- Easy calculation shows that $\beta_n(s) \downarrow$ pointwise. Let $L(s) := \lim_{n\to\infty} \beta_n(s)$.

- $L(1) = 0$ and L satisfy the integral equation

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw.$$

- Enough to show that $L \equiv 0$.

- Consider $\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1$, then it is easy to see that $\eta(0) = \eta(1) = 0$, and further

$$\eta'(w) = e^{-L(w)}\left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0.$$

Thus $\eta \equiv 0 \iff L \equiv 0$.

Open Questions :

- Is the RTP associated with the *frozen percolation* process endogenous ?

  - Simulation suggests that it is **not**. In other words the process can not be defined only through the edge-weights.

- What is the domain of attraction for the *Logistic RDE* ?

  - Do **not** have any “natural” contraction.
  - The associated operator is *discontinuous* at every point with respect to the weak-convergence topology.