Measurability of the Frozen Percolation Process on Infinite Regular Trees

Antar Bandyopadhyay

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Barcelona, Spain

Institute for Mathematics and Its Applications
University of Minnesota, USA

(http://www.ima.umn.edu/~antar)

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Frozen Percolation on a Regular Binary Tree

The Setup :

- Let $\mathbb{T}_3 = (\mathbb{V}, \mathbb{E})$ be the infinite regular binary tree.
- Each edge $e \in \mathbb{E}$ is equipped with independent edge weight $U_e \sim$ Uniform$[0, 1]$.
- Think of time moving from 0 to 1.

Frozen Percolation Process (informal description):

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e if each of its end vertex is in a finite component; otherwise do not open e.
- Let $(\mathcal{A}_t)_{t \geq 0}$ be set process of open edges starting from $\mathcal{A}_0 = \emptyset$.

The Standard Percolation Process :

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e.

- If $(\mathcal{B}_t)_{t \geq 0}$ be the set process of open edges the it can be described as

$$\mathcal{B}_t = \{ e \in \mathbb{E} \,|\, U_e \leq t \}$$

Remarks : Unlike the regular percolation process it is not clear whether the *frozen percolation process* exists and if so whether it admits a simpler description using only the edge weights.

Two Easy Observations : If frozen percolation process exists then following must hold

- $\mathcal{A}_t \subseteq \mathcal{B}_t$ for all $t \in [0, 1]$.

- $\mathcal{A}_t = \mathcal{B}_t$ if $t \leq \frac{1}{2}$ (since the critical probability for infinite binary tree is $\frac{1}{2}$).

Aldous' Construction of the Process

- Aldous (2000) showed that there exists a version of the process on $\mathbb{T}_3$ which is automorphism invariant but the construction not only used the edge weights (U_e) but also (possibly) *external* random variables.

- One of the key ingredient of the construction is the following distributional identity

$$X \stackrel{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases},$$

where (X_1, X_2) are i.i.d with same law as X and are independent of $U \sim \text{Uniform}[0,1]$.

We will call such identities a *recursive distributional equation* (RDE).

Frozen Percolation RDE

$$X \stackrel{d}{=} \Phi\left(X_1 \wedge X_2; U\right)$$

where (X_1, X_2) are independent copies of X and are independent of $U \sim$ Uniform$[0,1]$ and the function Φ is given by

$$\Phi(x;u) := \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases} .$$

Remark : [Aldous (2000)] The RDE has a unique non-atomic solution with full support, given by

$$\mu(dy) = \frac{dy}{2y^2},\ \frac{1}{2} < y < 1,\ \ \mu(\{\infty\}) = \frac{1}{2}.$$

Frozen Percolation RTP

$$X \stackrel{d}{=} \Phi\,(X_1 \wedge X_2; U)$$

Using the natural recursive structure and Kolmogorov's consistency theorem we can now construct a tree indexed process $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ such that for all $\mathbf{i} \in \mathcal{V}$

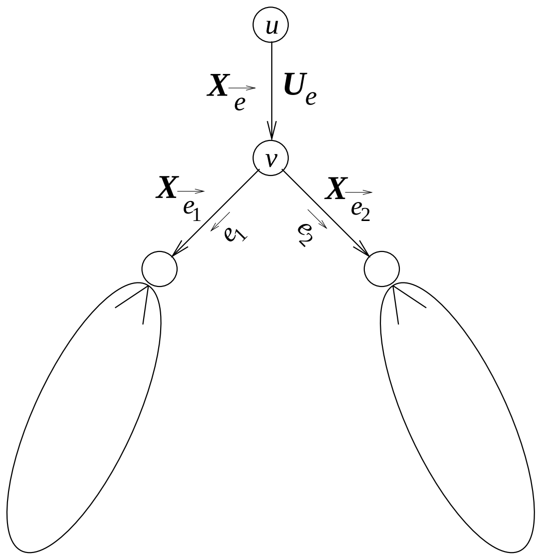
- $X_{\mathbf{i}} \sim \mu$,
- $X_{\mathbf{i}} = \Phi\,(X_{\mathbf{i}1} \wedge X_{\mathbf{i}2}; U_{\mathbf{i}})$, and
- $X_{\mathbf{i}}$ is independent of $\{U_{\mathbf{i}'} \,|\, \text{length}\,(\mathbf{i}') < \text{length}\,(\mathbf{i})\}$;

where $\mathcal{V} := \cup_{m \geq 1} \{1, 2\}^m \cup \{\emptyset\}$, which is endowed with the natural rooted binary tree structure with $\emptyset$ as the root, and $(U_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ are i.i.d Uniform$[0, 1]$ random variables.

We will call $(X_{\mathbf{i}})$ the invariant *recursive tree process* (RTP) with marginal μ associated with the frozen percolation RDE.

Back to the Construction ...

- If the process exists on $\mathbb{T}_3$ then for a directed edge $e := \overrightarrow{(u,v)}$, we define $X_{\overrightarrow{e}} \in [1/2, 1] \cup \{\infty\}$ as the "time for e to join to ∞ in the direction $\overrightarrow{e}$"; then these collection forms the RTP associated with the frozen percolation RDE.



$$X_{\overrightarrow{e}} = \Phi\left(X_{\overrightarrow{e}_1} \wedge X_{\overrightarrow{e}_2}; U_e\right)$$

- However the process is constructed by defining these times *externally* using the RTP.

Question : Is it possible to show that this construction is measurable only with respect to the edge weights ?

Endogeny

Definition 1 *We will say the RTP with marginal μ is* endogenous *if it is measurable with respect to the i.i.d uniform variables.*

Bivariate Uniqueness

$$\left(\begin{array}{c} X \\ Y \end{array}\right) \stackrel{d}{=} \left(\begin{array}{c} \Phi\left(X_1 \wedge X_2; U\right) \\ \Phi\left(Y_1 \wedge Y_2; U\right) \end{array}\right)$$

Definition 2 *We say that* bivariate uniqueness *holds if the above bivariate fixed-point equation has unique solution as $X = Y$ a.s. and in this case with marginal μ.*

Equivalence Theorem

Theorem 1 (Aldous and B. (2004)) *The RTP with marginal μ is* endogenous *if and only if the* bivariate uniqueness *property holds.*

Theorem 2 *The invariant RTP with marginal* μ *has bivariate uniqueness property, that is, the following* bivariate RDE *has* unique *solution as* $X = Y$ *a.s with marginal* μ

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2; U\right) \\ \Phi\left(Y_1 \wedge Y_2; U\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j=1,2}$ *are independent copies of* (X, Y)*, and are independent of* $U \sim$ *Uniform*$[0, 1]$.

Remarks :

- Thus by the *equivalence theorem* the RTP with marginal μ is *endogenous*.
- This then proves that the Aldous' construction of the frozen percolation process is measurable with respect to the uniform edge weights, and hence do not exhibit any *spatial chaos* property.
- Same conclusion holds for any infinite r-regular tree with $r \geq 3$.

Outline of the proof of Theorem 2

- Notice that X and Y have the same distribution μ. So if $F(x,y) = \mathbf{P}(X \le x, Y \le y)$ and $G(x,y) = \mathbf{P}(X > x, Y > y)$ then for every $x, y \in [\frac{1}{2}, 1]$

$$G(x,y) = F(x,y) + \frac{1}{2x} + \frac{1}{2y} - 1.$$

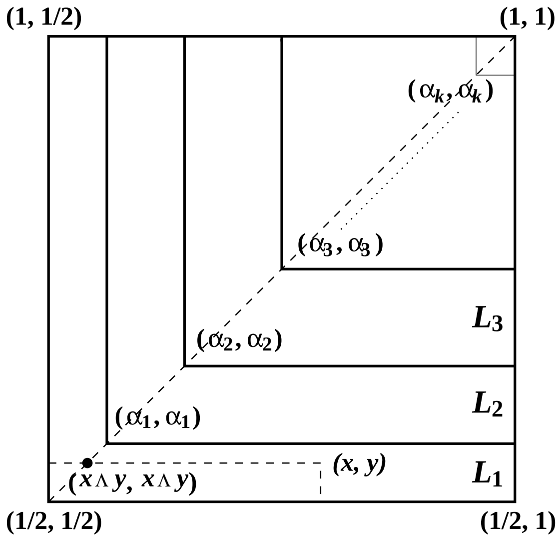
- From the bivariate RDE we get

$$F(x,y) =$$

$$\int_0^{x \wedge y} \left(G^2(x,y) - G^2(x,u) - G^2(u,y) + G^2(u,u) \right) du.$$

- We know that $X = Y$ a.s. is a solution so $G_0(x,y) = \frac{1}{2(x \vee y)}$ is a solution of the integral equation. It is enough to prove that $G = G_0$ is the *only* solution.

- Let $H(x,y) = 1 - \frac{G(x,y)}{G_0(x,y)}$, so we need to show $H \equiv 0$ on $D := [\frac{1}{2}, 1]^2$.



- Substituting back into the equation and after some algebra we get

$$H(x,y) = \frac{1}{G_0(x,y)} \int_0^{x \wedge y} \Lambda(x,y,u)\, du,$$

where Λ is a function (has long expression !) which satisfies the estimate

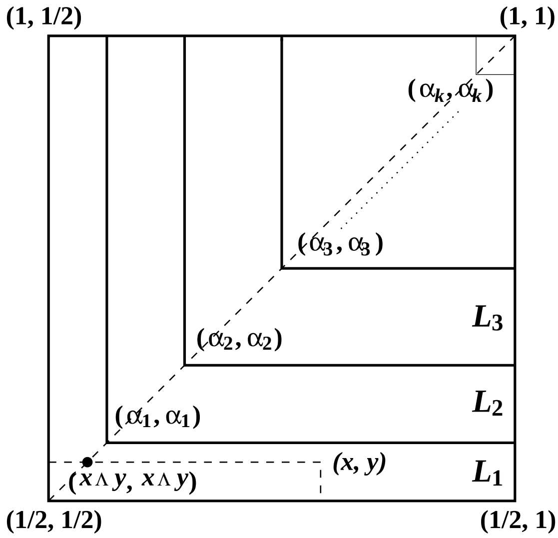
$$|\Lambda(x,y,u)| \leq 4G_0^2(u,u)\left(2|H(x,y)| + |H^2(x,y)|\right),$$

whenever $u \leq x \wedge y$.

- Find $\frac{1}{2} = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$ such that

$$\int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u,u)\, du < \frac{1}{48}.$$

- We partition D into L-shape parts (as in the figure) where $L_i := \{(x,y) \mid \alpha_{i-1} \leq x \wedge y < \alpha_i\}$.



- Define $\| H \|_i := \sup_{x,y \in L_i} |H(x,y)|$.

- Start with $i = 1$, let $(x,y) \in L_1$. Note $G_0(x,y) \geq \frac{1}{2}$. Thus from the estimate of Λ we get

$$\begin{aligned} |H(x,y)| &\leq 24 \| H \|_i \int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u,u)\, du \\ &\leq \frac{1}{2} \| H \|_i \end{aligned}$$

- Thus $H \equiv 0$ on L_1. Now proceed inductively for $i = 2, 3, \ldots k$ to conclude $H \equiv 0$ on D.

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References :

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