More notes for Math 8302, Manifolds and Topology, Spring 2005
Smooth Manifolds

Definition. Let M be a topological space and let $d \geq 1$ be an integer. A **(topological) d-chart** on M is a continuous, open, injective function $\mathbb{R}^d \to M$.

Note that such a function is a homeomorphism onto its image.

Definition. Let M be a topological space, let $d \geq 1$ be an integer and let $\phi, \psi : \mathbb{R}^d \to M$ be d-charts. The **overlap map** of ϕ and ψ is the map $\psi^{-1} \circ \phi : \phi^{-1}(\psi(\mathbb{R}^d)) \to \mathbb{R}^d$.

For all integers $d \geq 0$, for all integers $k \geq 1$, for all $i \in \{1, \ldots, d\}^k$, we define $\partial_i := (\partial/\partial x_{i_1}) \cdots (\partial/\partial x_{i_k})$. We define $\{1, \ldots, d\}^0 := \{\emptyset\}$ and we define $\partial_\emptyset$ to be the identity operator; that is, for any function $f : \mathbb{R}^d \to \mathbb{R}^d$ we define $\partial_\emptyset f = f$.

With these conventions:

Definition. Let $d, e \geq 0$ be integers. Let U be an open subset of $\mathbb{R}^d$. Let $V \subset \mathbb{R}^d$. For any integer $k \geq 1$, we say that a function $\chi : U \to V$ is C^k if, for all $i \in \{1, \ldots, d\}^k$, we have that $\partial_i \chi : U \to \mathbb{R}^e$ exists and is continuous. We say that $\chi : U \to V$ is C^∞ or **smooth** if: for all integers $k \geq 0$, χ is C^k.

Definition. Let M be a topological space. A **d-atlas** on M is a set $\mathcal{A}$ of d-charts on M such that

(1) $\displaystyle\bigcup_{\phi \in \mathcal{A}} \phi(\mathbb{R}^d) = M$; and

(2) for all $\phi, \psi \in \mathcal{A}$, the overlap map $\psi^{-1} \circ \phi : \phi^{-1}(\psi(\mathbb{R}^d)) \to \mathbb{R}^d$ is C^∞.

Definition. Let $d \geq 0$ be an integer. Let M be a topological space and let $\mathcal{A}$ be a d-atlas on M. We say that $\mathcal{A}$ is **maximal** if there exists no d-atlas $\mathcal{A}'$ on M such that $\mathcal{A} \subsetneq \mathcal{A}'$.

EXERCISE 21C: Let $d \geq 0$ be an integer. Let M be a topological space. Let $\mathcal{A}_0$ be any d-atlas on M. Show that there is a unique maximal d-altas $\mathcal{A}$ on M such that $\mathcal{A}_0 \subseteq \mathcal{A}$.

Definition. Let $d \geq 0$ be an integer. A **(smooth) d-manifold** is a topological space M together with a maximal d-atlas on M.

A **manifold** is just a d-manifold, for some integer $d \geq 0$. By invariance of domain, the dimension of a manifold is well defined. That is, if M is both a d-manifold and a d'-manifold, then $d = d'$. It is denoted $\dim(M)$.

Example. Any (finite dimensional) vector space is a manifold. If V is a d-dimensional vector space, then the set of vector space isomorphisms $\mathbb{R}^d \to V$ gives a d-atlas, and it extends uniquely to a maximal d-atlas on V.

Example. Any (finite dimensional) sphere is a manifold. The stereographic projection maps $\mathbb{R}^d \to S^d$ give an atlas that extends to a maximal d-atlas on S^d.

Example. Any product of manifolds is again a manifold. Let $d, e \geq 0$ be integers. Let M be a d-manifold and let N be an e-manifold. Let $\mathcal{A}$ be the maximal d-atlas on M and let $\mathcal{B}$ be the maximal e-atlas on N. For any $\phi \in \mathcal{A}$, $\psi \in \mathcal{B}$, define $\phi \times \psi : \mathbb{R}^{d+e} \to M \times N$ by

$(\phi \times \psi)(x, y) = (\phi(x), \psi(y))$. Then $\{\phi \times \psi \,|\, \phi \in \mathcal{A}, \psi \in \mathcal{B}\}$ is a $(d+e)$-atlas on M which extens to a maximal $(d+e)$-atlas on M.

Example. Any open subset of a manifold is again a manifold. Let $d \geq 0$ be an integer and let M be a d-manifold, with maximal d-atlas $\mathcal{A}$. Let U be an open subset of M. Then we leave it as an unassigned exercise to show that $\{\phi \in \mathcal{A} \,|\, \phi(\mathbb{R}^d) \subseteq U\}$ is a maximal d-atlas on U.

We have defined above a (topological) chart for on a topological space. If M is a manifold, then a **(smooth) chart** on M is an element of the maximal atlas. For both, we will typically say "chart", but the context should make the meaning (topological or smooth) clear. Typically in the sequel, I expect charts will be smooth charts.

Definition. Let $d, e \geq 0$ be integers. Let M be a d-manifold and let N be an e-manifold. Let $f : M \to N$ be a function. Then we say that f is C^∞ or **smooth** if, first, f is continuous and, second, for any charts ϕ on M and ψ on N, the map

$$\psi^{-1} \circ f \circ \phi \quad : \quad \phi^{-1}(f^{-1}(\psi(\mathbb{R}^e))) \quad \to \quad \mathbb{R}^e$$

is C^∞.

Smooth maps are the arrows in the category of manifolds.

A **diffeomorphism** is an isomorphism in the category of manifolds; it is a smooth bijection with smooth inverse.

Definition. Let $d, e, k \geq 0$ be integers. Let $U, U' \subseteq \mathbb{R}^d$ be open. Let $V, V' \subseteq \mathbb{R}^e$. Let $f : U \to V$ and $f' : U' \to V'$ be smooth. Let $x \in U \cap U'$. We say that f and f' **agree at** x **to order** k if: for all integers $j \in [0, k]$, for all $i \in \{1, \ldots, d\}^k$, we have $(\partial_i f)(x) = (\partial_i f')(x)$.

Definition. Let $d, e, k \geq 0$ be integers. Let M be a d-manifold and let N be an e-manifold. Let $f, f' : M \to N$ be smooth. Let $m \in M$. We say that f and f' **agree at** x **to order** k if: for all charts $\phi : \mathbb{R}^d \to M$, for all charts $\psi : \mathbb{R}^e \to N$, if $\phi(0) = m$, then $\phi^{-1} \circ f \circ \phi$ and $\phi^{-1} \circ f' \circ \phi$ agree to order k at 0.

Definition. Let $d, e, k \geq 0$ be integers. Let M be a d-manifold and let N be an e-manifold. Let $f : M \to N$ be smooth. Let $m \in M$. Then the k**-jet of** f **at** M **into** N is the union over all neighborhoods M_0 of m in M of: the collection of all smooth $f' : M_0 \to N$ such that $f|M_0$ and f' agree to order k at m. It is denoted by $J^k_{mN} f$; when no confusion will arise, we will denote this alternatively by the simpler notation $[f]$.

Definition. Let $d, e, k \geq 0$ be integers. Let M be a d-manifold and let N be an e-manifold. Let $m \in M$ and $n \in N$. We then define $J^k_{mn}(M, N)$ to be

$$\{J^k_{mN} f \quad | \quad U \text{ is open in } M, \quad f : U \to N \text{ is smooth}, \quad f(m) = n\}.$$

Definition. Let $k \geq 0$ be an integer. Let N be an e-manifold. Then, for all $n \in N$, the **tangent space** to N at n is defined to be $T_n N := J^1_{0n}(\mathbb{R}, N)$. The **tangent bundle** of N

is defined to be $TN := \bigcup_{n \in N} T_n N$. If U is an open neighborhood of 0 in $\mathbb{R}$, if $c : U \to N$ is smooth and if $c(0) = n$, then $J^1_{0N}(c) \in T_n N$ is usually denoted $(d/dt)_{t=0}(c(t))$.

Definition. Let $e, h \geq 0$ be integers. Let N be an e-manifold and let P be a h-manifold. Let $\phi : N \to P$ be smooth. Then, for all $n \in N$, we define $(d\phi)_n : T_n N \to T_{\phi(n)} P$ by $(d\phi)_n([c]) = [\phi \circ c]$. We define $d\phi : TN \to TP$ by: for all $n \in N$, $(d\phi)|(T_n N) = (d\phi)_n$.

We sometimes denote $d\phi$ by $T\phi$; it is called the **differential of** ϕ. Then T is a functor from {manifolds} to {sets}. For all $n \in N$, $(d\phi)_n$ is called the **differential of** ϕ **at** n.

Definition. Let $d \geq 0$ be an integer. Let M be a d-manifold and let $\phi : \mathbb{R}^d \to M$ be a chart. We define $D\phi : \mathbb{R}^{2d} \to TM$ by: for all $p, v \in \mathbb{R}^d$,

$$(D\phi)(p, v) = (d/dt)_{t=0}(\phi(p + tv)).$$

For all $p \in \mathbb{R}^d$, we define $D_p \phi : \mathbb{R}^d \to T_{\phi(p)} M$ by $(D_p \phi)(v) = (D\phi)(p, v)$.

EXERCISE 22A: Let $d \geq 0$ be an integer. Let M be a d-manifold and let $\phi : \mathbb{R}^d \to M$ be a chart. Let $p \in \mathbb{R}^d$ and let $m := \phi(p)$. Show that $D_p : \mathbb{R}^d \to T_m M$ is a bijection.

EXERCISE 22B: Let S and I be sets. For all $i \in I$, let X_i be a topological space and let $\phi_i : X_i \to S$ be an injective function. Assume that $S = \bigcup_{i \in I} \phi_i(X_i)$. Assume, for all $i, j \in I$, that $X_i^j := \phi_i^{-1}(\phi_j(X_j))$ is open in X_i and that $\phi_j^{-1} \circ \phi_i : X_i^j \to X_j$ is continuous. Show that there is a unique topology $\mathcal{T}$ on S such that, for all $i \in I$, $\phi_i : X_i \to (S, \mathcal{T})$ is continuous and open.

We leave the following fact as an unassigned exercise.

Fact. Let I_0 be a countable set. Let X be a topological space. For all $i \in I_0$, let X_i be a second countable topological space and let $\phi_i : X_i \to X$ be an open, continuous function. Assume that $X = \bigcup_{i \in I_0} \phi_i(X_i)$. Then X is second countable.

Putting Exercise 22B together with the preceding fact, one gets:

Proposition. Let S be a set and let $d \geq 0$ be an integer. Let Φ be a set of injective maps $\mathbb{R}^d \to S$. Assume that there is a countable subset Φ_0 of Φ such that $X = \bigcup_{i \in I_0} \phi_i(X_i)$. Assume, for all $\phi, \psi \in \Phi$, that $X_i^j := \phi^{-1}(\psi(\mathbb{R}^d))$ is open in $\mathbb{R}^d$ and that $\psi^{-1} \circ \phi : X_i^j \to \mathbb{R}^d$ is smooth. Then there is a unique topology $\mathcal{T}$ on S such that Φ is an atlas on $(S, \mathcal{T})$.

In other words, if you have a set S, and you have an "set-theoretic atlas" on S (meaning only that overlaps are smooth with open domains, and that countably many of them cover) then issues of topology will take care of themselves, and S becomes a manifold.

Using the preceding proposition we have:

Fact. Let $d \geq 0$ be an integer. Let M be a d-manifold. Let $\mathcal{A}$ be the maximal atlas on M, *i.e.*, the set of charts on M. Then there is a unique topology on TM with respect to which $\{D\phi \mid \phi \in \mathcal{A}\}$ is an atlas on TM.

Extending the atlas described above uniquely to a maximal atlas makes TM into a $(2d)$-manifold. Thus T is now a functor $\{\text{manifolds}\} \to \{\text{manifolds}\}$.

For each such chart ϕ, we get a vector space structure on T_mM simply by pushing the vector space structure on $\mathbb{R}^d$ forward along the bijection $D_{\phi^{-1}(m)}\phi : \mathbb{R}^d \to T_mM$.

EXERCISE 22C: Let $d \geq 0$ be an integer. Let M be a d-manifold, let $m \in M$, let $p, q \in \mathbb{R}^d$ and let $\phi, \psi : \mathbb{R}^d \to M$ be charts. Assume that $\phi(p) = m = \psi(q)$. Show that $(D_q\psi)^{-1} \circ (D_p\phi) : \mathbb{R}^d \to \mathbb{R}^d$ is a vector space isomorphism.

Let $\mathcal{V}$ denote the standard vector space structure on $\mathbb{R}^d$. (That is, $\mathcal{V}$ consists of the addition map $(x, y) \mapsto x + y : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, together with the scalar multiplication map $(t, x) \mapsto tx : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.) Then $(D_p\phi)_*(\mathcal{V})$ and $(D_q\psi)_*(\mathcal{V})$ are vector space structures on T_mM. However, Exercise 22C asserts that $((D_q\psi)^{-1} \circ (D_p\phi))_*(\mathcal{V}) = \mathcal{V}$, so, applying $(D_q\psi)_*$ to both sides, we get $(D_p\phi)_*(\mathcal{V}) = (D_q\psi)_*(\mathcal{V})$. That is, we get a well-defined vector space structure on T_mM, independent of the choice of chart.

We also have a map $\pi : TM \to M$ defined by $\pi([c]) = c(0)$. This map is the **tangent bundle map**. Note, for all $m \in M$, that $\pi^{-1}(m) = T_mM$. If one pictures tangent vectors as little arrows pointing tangent to the manifold, then this map associates to each such arrow its starting point, sometimes called its "footpoint". Formally, the **footpoint** of $v \in TM$ is defined to be $\pi(v)$.

We are still making TM into a "vector bundle", as yet undefined, but meaning roughly: a collection of vector spaces filling up a manifold and parameterized by another manifold and varying somehow "coherently" as one moves from point to point. So far we have the collection $\{T_mM \mid m \in M\}$ which fills up the manifold TM and which is parameterized by points m varying across the manifold M.

Next we work on the meaning of "coherently" above. Before we can accomplish that, though, we need to develop the definition of a submanifold.

WARNING: Different authors have different definitions of submanifold. We are using the right definitions here, and all others are wrong. Please let everyone know.

A small unassigned exercise: A subset S of a topological space X is closed iff, for all $x \in X$, there is a neighborhood U in X of x such that $S \cap U$ is closed in U, where U has the inherited topology from X.

Definition. A subset of a S of a topological space X is said to be **locally closed** if, for all $s \in S$, there is a neighborhood U in X of s such that $S \cap U$ is closed in U, where U has the inherited topology from X.

More unassigned exercises: A subset is locally closed iff it can be written as the intersection of an open set with a closed set. Also, a subset is locally closed iff it is an open subset of its closure, where the closure is given the inherited topology. The locally closed sets are obtained by taking the collection of all open sets and closing it up under complement and finite intersection.

For example, $(0,1) \times \{0\}$ is a locally closed subset of $\mathbb{R}^2$, although it is not closed.

Definition. Let $d \geq 0$ be an integer and let $c \in [0, d]$ be an integer. Let M be a d-manifold and let $S \subseteq M$. A chart $\phi : \mathbb{R}^d \to M$ will be said to be **c-adapted** to S if $\phi^{-1}(S) = \mathbb{R}^c \times \{0\}^{d-c}$. We say that S is a **locally closed c-submanifold** of M if: for all $s \in S$, there exists a chart $\phi : \mathbb{R}^d \to M$ such that $s \in \phi(\mathbb{R}^d)$ and such that ϕ is c-adapted to S. We say that S is a **closed c-submanifold** of M if it is both a closed subset of M and is a locally closed c-submanifold of M.

Note: S is a closed c-submanifold of M iff: for all $m \in M$, there is a chart $\phi : \mathbb{R}^d \to M$ such that $m \in \phi(\mathbb{R}^d)$ and such that either $\phi^{-1}(S) = \emptyset$ or $\phi^{-1}(S) = \mathbb{R}^c \times \{0\}^{d-c}$.

Let S be a locally closed c-submanifold of the d-manifold M. Give S the topology inherited from M. Any c-adapted chart gives rise to a map $\mathbb{R}^c \to \mathbb{R}^c \times \{0\}^{d-c} \to S$. The maps so obtained from the c-adapted charts form an maximal atlas on S called the **submanifold atlas** on S.

A **locally closed submanifold** of a manifold M is just a locally closed c-submanifold of M, for some integer $c \in [0, \dim(M)]$. A **closed submanifold** of a manifold M is just a closed c-submanifold of M, for some integer $c \in [0, \dim(M)]$.

Any locally closed submanifold of a manifold M is locally closed as a subset of the topological space M. Any closed submanifold of a manifold M is closed as a subset of the topological space M.

Definition. Let M and N be manifolds. A map $f : M \to N$ is an **immersion** if f is smooth and, for all $m \in M$, $(df)_m : T_m M \to T_{f(m)} N$ is injective.

We leave it as an unassigned exercise to show that, for any manifold M, for any locally closed submanifold S of M, there is a unique manifold structure on S with respect to which the inclusion map $S \to M$ becomes an immersion. The topology underlying this manifold structure is just the inherited topology from M, and the maximal atlas is the submanifold atlas.

From now on, any locally closed submanifold is defined to have the unique manifold structure making inclusion an immersion.

NOTE: A figure "8" in the plane $\mathbb{R}^2$ is not a locally closed submanifold, even though it is the image of two essentially different injective immersions $\mathbb{R} \to \mathbb{R}^2$. (Both send $-\infty$ to the crossing point, but one starts out northeast, whereas the other starts out northwest.) So, in our terminology, a locally closed manifold is *not* the same as the image of an immersion, and for good reason: The figure "8" has two different reasonable manifold structures on it, and we prefer our submanifolds to have a nice well-defined manifold structure that is somehow "inherited" from the ambient manifold.

EXERCISE 22D: Let M be a manifold and let $m \in M$. Show that $T_m M$ is a closed submanifold of TM.

We now have a situation where the manifold TM is filled up by closed submanifolds $T_m M$ parameterized by points $m \in M$ and each of which has a vector space structure.

We are still moving toward the definition of vector bundle, of which TM is the archetype. We will organize the material so that all sorts of other "bundles", like principal bundles and fiber bundles, are defined at the same time.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor. Let S be a manifold. An **$\mathcal{F}$-structure** on S is an object $C \in \mathcal{C}$ such that $\mathcal{F}(C) = S$.

So an $\mathcal{F}$-structure is just a "lift" of S into $\mathcal{C}$ via $\mathcal{F}$.

For example, let $\mathcal{C}_0 := \{\text{finited dimensional vector spaces}\}$ and let $\mathcal{F}_0 : \mathcal{C}_0 \to \{\text{manifolds}\}$ be the standard functor. Let S_0 be the unit disk centered at the origin in $\mathbb{R}^2$; then S_0 is a manifold. Let $\phi : \mathbb{R}^2 \to S_0$ be some diffeomorphism. Let $\mathcal{V}$ denote the standard vector space structure on $\mathbb{R}^2$. Then $C_0 := (S_0, \phi_*(\mathcal{V}))$ is a vector space and $\mathcal{F}(C_0) = S_0$. So C_0 is an $\mathcal{F}_0$-structure on S_0. For this functor, $\mathcal{F}_0$, one thinks of an $\mathcal{F}_0$-structure as being the same as a vector spaces structure. That is, to "lift" a manifold into $\{\text{vector spaces}\}$ is just to give it a vector space structure.

Recall (from Exercise 22C) that in TM, each submanifold T_mM has been given a vector space structure, *i.e.*, an $\mathcal{F}_0$-structure.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor. A **pre-$\mathcal{F}$ bundle** on M is:

(1) a manifold X;
(2) a smooth map $\pi : X \to M$; and
(3) a map $\zeta : M \to \mathcal{C}$

such that, for all $m \in M$,

(A) $\pi^{-1}(m)$ is a submanifold of M; and
(B) $\mathcal{F}(\zeta(m)) = \pi^{-1}(m)$.

According to (B), (3) simply assigns an $\mathcal{F}$-structure to each fiber of π.

Let $\mathcal{F}_0 : \{\text{finite dimensional vector spaces}\} \to \{\text{manifolds}\}$ be the standard functor, described above. Let M be a manifold. Then a pre-$\mathcal{F}_0$ bundle on M will be called a **pre-vector bundle** on M.

Thus, for any manifold M, TM (as a manifold, together with the smooth tangent bundle map $TM \to M$, together with the vector spaces structures on the tangent spaces) is a pre-vector bundle on M. This pre-vector bundle is also denoted TM.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor. We will say that $\mathcal{F}$ has **unique diffeomorphism lifting** if, for all $C \in \mathcal{C}$, for all manifolds S, for any diffeomorphism $\phi : \mathcal{F}C \to S$, there is a unique arrow $\psi : C \to D$ in $\mathcal{C}$ such that $\mathcal{F}\psi = \phi$.

Note that, in the above notation, $\mathcal{F}D = S$, so D is an $\mathcal{F}$-structure on S.

So, if $\mathcal{F}$ has unique diffeomorphism lifting, if S is a manifold, and if we have a diffeomorphism between S and some manifold C_0 with a $\mathcal{F}$-structure C, then S obtains an $\mathcal{F}$-structure, denoted D above. We will write $\phi_*(C)$ to denote D.

In particular, if $\mathcal{F}_0 : \{\text{finite dimensional vector spaces}\} \to \{\text{manifolds}\}$ is the standard functor, then $\mathcal{F}_0$ has unique diffeomorphism lifting, so, for example, if S is a manifold and if we have a diffeomorphism specified bewteen S and $\mathbb{R}^d$, then we can transfer the standard vector space structure on $\mathbb{R}^d$ to S.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor with unique diffeomorphism lifting. Let M be a manifold and let C be an object in $\mathcal{C}$. Let $C_0 := \mathcal{F}C$, so C is an $\mathcal{F}$-structure on C_0. Let $\pi : M \times C_0 \to M$ be projection onto the first coordinate. For all $m \in M$, let $\phi_m : C_0 \to \pi^{-1}(m)$ be the diffeomorphism $\phi_m(c) = (m, c)$. Define $\zeta : M \to \mathcal{C}$ by $\zeta(m) = (\phi_m)_*(C)$. Then $(M \times C_0, \pi, \zeta)$ is called the **trivial $\mathcal{F}$ bundle** on M with fiber C. It be denoted $M \times C$.

Let $\mathcal{F}_0 : \{\text{finite dimensional vector spaces}\} \to \{\text{manifolds}\}$ be the standard functor, let V be a finite dimensional vector space and let M be a manifold. Then the trivial $\mathcal{F}_0$ bundle $M \times V$ on M with fiber V is called the **trivial vector bundle** on M with fiber V. In particular, if M is a manifold and V is a finite dimensional vector space, then $M \times \mathbb{R}^k$ denotes the trivial vector bundle with fiber $\mathbb{R}^k$.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor. Let M be a manifold and let M_0 be a nonempty open subset of M. Let $X = (X, \pi, \zeta)$ be a pre-$\mathcal{F}$ bundle on M. Let $X_0 := \pi^{-1}(M_0)$. Let $\pi_0 := \pi | X_0 : X_0 \to M_0$. Let $\zeta_0 := \zeta | M_0$. Then the pre-$\mathcal{F}$ bundle on M_0 given by (X_0, π_0, ζ_0) will be denoted $X | M_0$.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor with unique diffeomorphism lifting. Let M be a manifold. Let $X = (X, \pi, \zeta)$ be a pre-$\mathcal{F}$ bundle on M. We say that X is **trivial** if there exists an object $C \in \mathcal{C}$ such that X is isomorphic (in the category of pre-$\mathcal{F}$ bundles on M) to $M \times C$. We say that X is **locally trivial** if, for all $m \in M$, there is an open neighborhood M_0 of m in M such that $X | M_0$ is trivial.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor with unique diffeomorphism lifting. Let M be a manifold. An $\mathcal{F}$ **bundle** on M is a locally trivial pre-$\mathcal{F}$ bundle on M.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C} \to \{\text{manifolds}\}$ be a functor with unique diffeomorphism lifting. If $X = (X, \pi, \zeta)$ is a pre-$\mathcal{F}$ bundle on M then a **section** of X is a smooth function $\sigma : M \to X$ such that $\pi \circ \sigma : M \to M$ is the identity map.

Let $\mathcal{F}_0 : \{\text{finite dimensional vector spaces}\} \to \{\text{manifolds}\}$ be the standard functor. Let M be a manifold. Then a **vector bundle** on M is an $\mathcal{F}_0$ bundle on M. Let $V = (V, \pi, \zeta)$ be a vector bundle on M. A section σ of V is said to be **nowhere vanishing** if, for all $m \in M$, we have that $\sigma(m)$ is not the zero vector in the vector space $\zeta(m)$. (Recall that $\zeta(m)$ is just a vector space structure on $\pi^{-1}(m)$, *i.e.*, $\zeta(m)$ is a vector space whose underlying manifold is $\mathcal{F}_0(\zeta(m)) = \pi^{-1}(m)$.)

EXERCISE 23A: Let $d \geq 0$ be an integer. Let M be a d-manifold. Show that the pre-vector bundle TM on M is locally trivial. That is, show, for all $m \in M$, that there is a neighborhood M_0 of m in M such that $TM | M_0$ is isomorphic to $M_0 \times \mathbb{R}^d$ in the category of pre-vector bundles on M_0.

Then, for any manifold M, TM is a vector bundle on M, called the **tangent vector bundle** of M. A **vector bundle** is a manifold M together with a vector bundle on M. Then T is a functor from the category of manifolds to the category of vector bundles.

It's an unassigned exercise to show that a vector bundle over a connected manifold has the property that every fiber has the same dimension as every other. When the dimension of the fibers of a vector bundle does not vary, this constant is called the **rank** of the vector bundle. In particular, if M is a d-manifold, then TM is a vector bundle of rank d.

Note that TM, while locally trivial need not be trivial: Let $M := S^2$. Then $TM = TS^2$ has no nowhere vanishing sections, because you can't comb the hairs on a hedgehog. On the other hand, it is an (easy) unassigned exercise to show that any trivial vector bundle of positive rank admits nowhere vanishing sections. Note that TM has rank 2.

Fact and Definition. Let V be a finite dimensional vector space. Let $v_0 \in V$. Then the map $v \mapsto (d/dt)_{t=0}(v_0 + tv) : V \to T_{v_0}V$ is an isomorphism of vector spaces. It is called the **standard identification** of $T_{v_0}V$ with V.

EXERCISE 23B: Let V and W be finite dimensional vector spaces. Let $L : V \to W$ be linear. Let $v_0 \in V$ and let $w_0 := L(v_0)$. Let $p : V \to T_{v_0}V$ and $q : W \to T_{w_0}W$ be the standard identifications. Show that $(dL)_{v_0} \circ p = q \circ L$.

That is, up to standard identifications, the differential of L at v_0 is just L itself. One sometimes says that any linear function is its own differential, but you should realize that this is only after appropriate identifications, and only when one takes the differential at a particular point, and not the full differential $dL = TL : TV \to TW$.

Let $\mathcal{F}_0$: {finite dimensional vector spaces} $\to$ {manifolds} be the standard functor. For any vector space V, we will often denote $\mathcal{F}_0 V$ by V. That is, "any vector space is a manifold", automatically.

Definition. Let V and W be finite dimensional vector spaces. Let $h : V \to W$ be smooth and let $v \in V$. Let $p : V \to T_vV$ and $q : W \to T_{h(v)}W$ be the standard identifications. Then $h'(v) : V \to W$ is the composite $h'(v) := q^{-1} \circ (dh)_v \circ p$.

That is, $h'(v)$ is the same as the differential of h at v, up to standard identifications.

Definition. A **pointed smooth map** consists of

(1) an arrow $f : M \to N$ in the category {manifolds}; and
(2) a point $m \in M$.

EXERCISE 23C: Let W be a vector space, let $w \in W$ and let $h : W \to W$ be smooth. Assume that $h'(w) : W \to W$ is invertible. Show that there is a smooth map $f : W \to W$ such that $f(0) = 0$, such that $f'(0) : W \to W$ is the identity map and such that (h, w) is isomorphic to $(f, 0)$ in the category {pointed smooth maps}.

In the next result, $\cdot$ denotes the ordinary dot product in Euclidean space, and a dot over a letter denotes ordinary calculus differentiation.

Fact. Let $n \geq 0$ be an integer. Let $\gamma : R \to \mathbb{R}^n$ be differentiable and assume that $\gamma(0) = \gamma(1)$. Let $w \in \mathbb{R}^n$. Then, for some $t \in (0,1)$, we have $(\dot{\gamma}(t)) \cdot w = 0$.

The geometric meaning of this is that, given a smooth loop in Euclidean space and given a hyperplane, there is some velocity vector of the loop that is parallel to the hyperplane.

Proof: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = (\gamma(t)) \cdot w$. then $f(0) = f(1)$. By Rolle's theorem, there is some $t \in (0,1)$ such that $\dot{f}(t) = 0$. Then $(\dot{\gamma}(t)) \cdot w = \dot{f}(t) = 0$. **QED**

In the next definition, $|\cdot|$ denotes the usual Euclidean length, given by $|v| = \sqrt{v \cdot v}$.

Definition. Let $n \geq 0$ be an integer and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear. We define the **norm** or **operator norm** of L to be $\|L\| := \sup \left\{ |L(v)| \,\middle|\, v \in S^{n-1} \right\}$.

Note that, for all L and v we have $|Lv| \leq \|L\| \cdot |v|$. Letting $\cdot$ denote both dot product and ordinary multiplcation of real numbers, note that, for all L, v and w, we have $|Lv \cdot w| \leq \|L\| \cdot |v| \cdot |w|$.

Remark. Let $n \geq 0$ be an integer and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Assume that $\|L - I\| < 1$. Let $w \in \mathbb{R}^n \backslash \{0\}$. Then $(Lw) \cdot w \neq 0$.

The intuition behind this is that if L is sufficiently close to the identity, then L cannot "turn a vector sideways", *i.e.*, move a vector into its orthogonal complement.

In particular, this result implies that the kernel of L is $\{0\}$, so L is a vector space isomorphism.

Proof: We have $|w|^2 - [(Lw) \cdot w] = [(Iw) \cdot w] - (Lw) \cdot w] = [(I - L)w] \cdot w$. Then $|w|^2 - [(Lw) \cdot w] \leq |[(I - L)w] \cdot w| \leq \|I - L\| \cdot |w| \cdot |w| < |w|^2$. Then $[(Lw) \cdot w] \neq 0$. **QED**

Note that the proof actually shows that $(Lv) \cdot v > 0$ which says that (the symmetric part of) L is positive definite.

Recall that a map $f : M \to N$ is an **immersion** if, for all $m \in M$, we have that $(df)_m : T_m M \to T_{f(m)} N$ is injective. Note that this can only happen if $\dim(M) \leq \dim(N)$.

Definition. Let M and N be manifolds and let $f : M \to N$ be a smooth map. We say that f is **submersive** at $m \in M$ if $(df)_m : T_m M \to T_{f(m)} N$ is surjective. We say that f is **submersive** if, for all $m \in M$, f is submersive at m. We say that f is **bimersive** at $m \in M$ if $(df)_m : T_m M \to T_{f(m)} N$ is a vector space isomorphism. We say that f is **bimersive** if, for all $m \in M$, f is bimersive at m.

Note that f can be submersive at a point only if $\dim(M) \geq \dim(N)$. Note that f can be bimersive at a point only if $\dim(M) = \dim(N)$.

Lemma. Let M and N be manifolds and let $h : M \to N$ be smooth. Let $x \in M$ and assume that h is bimersive at x. Then there is a neighborhood U of x in M such that $h|U : U \to N$ is injective.

That is, bimersivity at a point implies injectivity near the point.

Proof: Let $d := \dim(M) = \dim(N)$. Let $I : \mathbb{R}^d \to \mathbb{R}^d$ be the identity map.

Replacing N by a neighborhood of $h(x)$ and M by a neighborhood of x, we may assume that M and N are both diffeomorpic to $\mathbb{R}^d$. We may therefore assume $M = \mathbb{R}^d = N$. Then, by Exercise 23C, we may assume that $x = 0$, that $h(0) = 0$ and that $h'(0) = I$.

We have $\|(h'(0)) - I\| = 0$. Let U be a convex neighborhood of 0 in $\mathbb{R}^d$ such that, for all $u \in U$, we have $\|(h'(u)) - I\| < 1$. Let $u, v \in U$ and assume both that $u \neq v$ and that

$h(u) = h(v)$. We wish to obtain a contradiction.

Let $\alpha : \mathbb{R} \to \mathbb{R}^d$ be defined by $\alpha(t) = (1-t)u + tv$. By convexity of U, for all $t \in [0,1]$, we have $\alpha(t) \in U$. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be defined by $\gamma(t) = h(\alpha(t))$. Then $\gamma(0) = \gamma(1)$. Also, by the chain rule, for all $t \in \mathbb{R}$, we have $\dot\gamma(t) = [h'(\alpha(t))][\dot\alpha(t)]$.

Let $w := v - u$. Then, for all $t \in \mathbb{R}$, we have $\dot\alpha(t) = w$. By the preceding Fact, choose $t_0 \in (0,1)$ such that $(\dot\gamma(t_0)) \cdot w = 0$. Let $L := h'(\alpha(t_0)) : \mathbb{R}^d \to \mathbb{R}^d$. Then $\dot\gamma(t_0) = Lw$. Thus $(Lw) \cdot w = 0$.

Since $\alpha(t_0) \in U$, we conclude that $\|L - I\| < 1$. Then, by the preceding Remark, $(Lw) \cdot w \neq 0$, a contradiction. **QED**

Recall that *Invariance of Domain* asserts that any injective continuous map between equidimensional topological manifolds is open. The preceding lemma shows that any bimersion between (smooth) manifolds is locally injective and is therefore (by Invariance of Domain) locally open. (A map is "locally injective" if, for any point of its domain, there is a neighborhood such that the restriction to that neighborhood is injective. A map is "locally open" if, for any point of its domain, there is a neighborhood such that the restriction to that neighborhood is open.) It is an unassigned exercise in point-set topology to show that there is no difference between a map that is locally open versus a map that is simply open. Thus we have:

Corollary. Any bimersion is open.

Remark. If M_0 is a locally closed submanifold of a manifold M, and if $\dim(M_0) = \dim(M)$, then M_0 is open in M.

Proof: The inclusion map $M_0 \to M$ is a bimersion, so its image is open. That is, M_0 is open. **QED**

Proposition. Let $n \geq 0$ be an integer. Let $W \neq \emptyset$ be an open subset of $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to W$ be a bijective bimersion. Then $h^{-1} : W \to \mathbb{R}^n$ is differentiable.

Note: In the statement of the preceding proposition, we are using the ordinary calculus definition of "differentiable", which is indicated in the proof below.

Proof: Let $z_0 \in \mathbb{R}^n$ and let $f := h^{-1}$. We wish to show that f is differentiable at $h(z_0)$.

Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. By Exercise 23C, we may assume that $z_0 = 0$, that $h(0) = 0$ and that $h'(0) = I$. We wish to show that f is differentiable at 0.

Let $\mathcal{U}$ denote the set of open neighborhoods of 0 in $\mathbb{R}^n$. For any $U \in \mathcal{U}$, let o_U denote the set of all functions $\alpha : U \to \mathbb{R}^n$ such that $\alpha(0) = 0$ and such that $|\alpha(x)|/|x| \to 0$ as $x \to 0$. Let $o := \bigcup_{U \in \mathcal{U}} o_U$. For any $U \in \mathcal{U}$, let O_U denote the set of all functions $\alpha : U \to \mathbb{R}^n$ such that $\alpha(0) = 0$ and such that $\sup\{|\alpha(x)|/|x| \,\big|\, x \in U\backslash\{0\}\} < \infty$. Let $O := \bigcup_{U \in \mathcal{U}} O_U$.

For any function ρ, let $\mathrm{dom}(\rho)$ denote the domain of ρ. By definition of "differentiable at 0", we wish to show that there is a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ and an $o_1 \in o$ such that, for all $x \in \mathrm{dom}(o_1)$, we have: $f(x) = [f(0)] + [Lx] + [o_1(x)]$. We have $f(0) = 0$. Since $h'(0) = I$, we expect the derivative of $f = h^{-1}$ at 0 to be I. We therefore wish to show, for some $o_1 \in o$, for all $x \in \mathrm{dom}(o_1)$, that:

$$f(x) = x + [o_1(x)].$$

By the first order form of Taylor's Theorem in Advanced Calculus, choose $o_2 \in o$ such that, for all $x \in \mathrm{dom}(o_2)$, we have: $h(x) = [h(0)] + [(h'(0))x] + [o_2(x)]$. Since $h(0) = 0$ and $h'(0) = I$, this reads: For all $x \in \mathrm{dom}(o_2)$,

$$h(x) = x + [o_2(x)].$$

As $o_2 \in o$, replacing o_2 by a restriction to a smaller neighborhood of 0, we may assume, for all $x \in \mathrm{dom}(o_2)$, that $|o_2(x)| \leq |x|/100$.

By the preceding Corollary, bimersions are open, so $h : \mathbb{R}^n \to W$ is open. Let $U := f^{-1}(\mathrm{dom}(o_2)) = h(\mathrm{dom}(o_2))$. Then U is an open neighborhood of 0 in $\mathbb{R}^n$. For all $x \in U$, we have $f(x) \in \mathrm{dom}(o_2)$, so both $h(f(x)) = [f(x)] + [o_2(f(x))]$ and $|o_2(f(x))| \leq |f(x)|/100$. As $h(f(x)) = h(h^{-1}(x)) = x$, this reads: For all $x \in U$,

$$\text{both} \qquad x = [f(x)] + [o_2(f(x))] \qquad \text{and} \qquad |o_2(f(x))| \leq (1/100)|f(x)|,$$

which implies that

$$|x| \geq |f(x)| - (1/100)|f(x)| = (99/100)|f(x)|,$$

which implies that

$$|f(x)| \leq (100/99)|x|.$$

Let $O_1 := f|U$. Then $O_1 \in O$ and for all $x \in U$, we have

$$f(x) = O_1(x).$$

Then, for all $x \in \mathrm{dom}(O_1) = U$, we have

$$x = [f(x)] + [o_2(f(x))] = [f(x)] + [o_2(O_1(x))].$$

Define $o_1 : U \to \mathbb{R}^d$ by $o_1(x) = -o_2(O_1(x))$. Then $o_1 \in o$, and, for all $x \in \mathrm{dom}(o_1) = U$, we have $f(x) = x - [o_2(O_1(x))] = x + o_1(x)$, as desired. **QED**

A summary of this proof would read: We wish to show that $f := h^{-1}$ is differentiable at 0. We may assume, for x near 0, that $h(x) = x + o(x)$. Then, for x near 0,

$$x = h(f(x)) = f(x) + o(f(x)).$$

Then, for x near 0, $|x| \geq |f(x)| - (1/100)|f(x)|$, so $f(x)$ is $O(x)$. Then

$$x = f(x) + o(f(x)) = f(x) + o(O(x)) = f(x) + o(x).$$

Then $f(x) = x - o(x) = x + o(x)$, so f is differentiable at 0.

Definition. Let $m, n \geq 0$ be integers. Let V be a nonempty open subset of $\mathbb{R}^m$ and let W be a nonempty open subset of $\mathbb{R}^n$. Let $f : V \to W$ be differentiable. (That is, assume, for all $v \in V$, that there exists a linear $L_v : \mathbb{R}^m \to \mathbb{R}^n$ such that, for all $x \in V$ close to zero, we have $f(v + x) = [f(v)] + [L_v x] + [o(x)]$. Then we define $Df : V \times \mathbb{R}^m \to W \times \mathbb{R}^n$ by $(Df)(v, x) = (f(v), L_v x)$.

We say that f is D^1 if f is differentiable. We say that f is D^2 if f is differentiable and Df is also differentiable. We say that f is D^3 if f is differentiable and Df is also differentiable and $D^2 f := D(Df)$ is also differentiable. And so on.

It is an unassigned exercise to show all three of the following:

(1) For all integers $k \geq 1$, f is C^k iff both f is D^k and $D^k f$ is continuous.
(2) $D(g \circ f) = (Dg) \circ (Df)$.
(3) For all integers $k \geq 1$, f is D^k implies that f is C^{k-1}.

Item (2) above is called the **Chain Rule of Advanced Calculus**.

Lemma. Let $n \geq 0$ be an integer. Let V and W be nonempty open subsets of $\mathbb{R}^n$. Let $h : V \to W$ be a bijective bimersion. Then $Dh : V \times \mathbb{R}^n \to W \times \mathbb{R}^n$ is a bijective bimersion.

Proof: By definition, if h is a bijective bimersion, then Dh is bijective. We must show that $Dh : V \times \mathbb{R}^n \to W \times \mathbb{R}^n$ is a bimersion.

Let $a := (v, x) \in V \times \mathbb{R}^n$. Let $b := (Dh)(a)$. We wish to show that

$$(d(Dh))_a : T_a(V \times \mathbb{R}^n) \to T_b(W \times \mathbb{R}^n)$$

is a vector space isomorphism. By equality of dimension, it suffices to show that the kernel of $(d(Dh))_a$ is zero.

Let $(\dot{v}, \dot{x}) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\sigma : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ be defined by $\sigma(t) = (v + t\dot{v}, x + t\dot{x})$. Let $\kappa := (d/dt)_{t=0}(\sigma(t)) \in T_a(V \times \mathbb{R}^n)$. Assume that $(T(Dh))\kappa = 0 \in T_b(W \times \mathbb{R}^n)$. We wish to show that $\kappa = 0 \in T_a(V \times \mathbb{R}^n)$. That is, we wish to show that $\dot{v} = 0 = \dot{x}$.

Let $p : V \times \mathbb{R}^n \to V$ and $\pi : W \times \mathbb{R}^n \to W$ be projections. Since $h \circ p = \pi \circ (Dh)$, it follows that $(Th) \circ (Tp) = (T\pi) \circ (T(Dh))$. Applying this to κ, because $(T(Dh))\kappa = 0$, we get $((Th) \circ (Tp))\kappa = 0 \in T_{h(v)}W$. Since h is bimersive, this implies that $(Tp)\kappa = 0 \in T_v V$.

We have $p(\sigma(t)) = v + t\dot{v}$. Then $0 = (Tp)\kappa = (d/dt)_{t=0}(p(\sigma(t))) = (d/dt)_{t=0}(v + t\dot{v})$. Then $\dot{v} = 0$. It remains to show that $\dot{x} = 0$.

For all $t \in \mathbb{R}$, we have $\sigma(t) = (v, x + t\dot{x})$, so $(Dh)(\sigma(t)) = (h(v), (h'(v))(x + t\dot{x}))$. Let $q : W \times \mathbb{R}^n \to \mathbb{R}^n$ be projection. Let $r := q \circ (Dh) : V \times \mathbb{R}^n \to \mathbb{R}^n$. for all $t \in \mathbb{R}$, we have $r(\sigma(t)) = (h'(v))(x + t\dot{x}) = (h'(v))x + t[(h'(v))\dot{x}]$. Also, we have

$$(d/dt)_{t=0}(r(\sigma(t)) = (Tr)\kappa = [(Tq) \circ (T(Dh))]\kappa.$$

So, as $(T(Dh))\kappa = 0$, we get $(d/dt)_{t=0}(r(\sigma(t)) = 0 \in T_x\mathbb{R}^n$. Then

$$0 = (d/dt)_{t=0}(r(\sigma(t)) = (d/dt)_{t=0}\big((h'(v))x + t[(h'(v))\dot{x}]\big).$$

Then $(h'(v))\dot{x} = 0$. Since h is bimersive, this implies that $\dot{x} = 0$. **QED**

We can now state the multivariable calculus version of the Inverse Function Theorem:

Theorem. Let W be a nonempty open subset of $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to W$ be a bijective bimersion. Then $h^{-1} : W \to \mathbb{R}^n$ is smooth.

Proof: By the preceding lemma, Dh is a bijective bimersion. Then, by the preceding proposition, $(Dh)^{-1}$ is differentiable. Because D distributes over composition, it follows that $D(h^{-1}) = (Dh)^{-1}$. Therefore $D(h^{-1})$ is differentiable. That is, h^{-1} is D^2.

By the preceding lemma, D^2h is a bijective bimersion. Then, by the preceding proposition, $(D^2h)^{-1}$ is differentiable. Because D distributes over composition, it follows that $D^2(h^{-1}) = (D^2h)^{-1}$. Therefore $D^2(h^{-1})$ is differentiable. That is, h^{-1} is D^3.

Continuing in this way, we show, for all integers $k \geq 0$, that h^{-1} is D^{k+1}, and is therefore C^k. Then h^{-1} is C^∞. **QED**

Definition. Let M, N, M_0 and N_0 be manifolds. Let $m \in M$. Let $f : M \to N$ and $f_0 : M_0 \to N_0$ be smooth maps. We say that f_0 is a **localization** of f near m if:

(1) M_0 is a neighbhorhood of m in M;
(2) N_0 is a neighbhorhood of $f(m)$ in N;
(3) $f(M_0) \subseteq N_0$; and
(4) $f_0 = f|M_0 : M_0 \to N_0$.

Inverse Function Theorem (first avatar). Let M and N be manifolds and let $f : M \to N$ be smooth. Let $m_0 \in M$ and let $n_0 := f(m_0)$. Assume that $(df)_{m_0} : T_{m_0}M \to T_{n_0}N$ is a vector space isomorphism. Then there is a localization $f_0 : M_0 \to N_0$ of f near m_0 such that $f|M_0 : M_0 \to N_0$ is a diffeomorphism.

Proof: We have $\dim(M) = \dim(T_{m_0}M) = \dim(T_{n_0}N) = \dim(N)$. Let $d := \dim(M) = \dim(N)$. There is a neighborhood of n_0 in N that is diffeomorphic to $\mathbb{R}^d$, so we may assume that $N \subseteq \mathbb{R}^d$. There is a neighborhood of m_0 in M that is diffeomorphic to $\mathbb{R}^d$, so we may assume that $M \subseteq \mathbb{R}^d$. Let $I := \mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$. By Exercise 23C, we may assume that $m_0 = 0 = n_0$ and that $f'(0) = I$. Let $\|\cdot\|$ denote the operator norm on the vector space of linear functions $\mathbb{R}^d \to \mathbb{R}^d$. We have $\|(f'(0)) - I\| = 0$. Replacing M by a sufficiently small neighborhood of 0 in $\mathbb{R}^d$, we may assume, for all $m \in M$, that $\|(f'(m)) - I\| < 1$. Then, by the remark following Exercise 23C, we see, for all $m \in M$, that $f'(m)$ is a vector space isomorphism. That is, $f : M \to N$ is a bimersion.

By the lemma following Exercise 23C, by replacing M by a smaller neighborhood of 0 in $\mathbb{R}^d$, if necessary, we may assume that $f : M \to N$ is injective. Since M is a manifold, there is a neighborhood M_0 of 0 in M and a diffeomorphism $\phi : \mathbb{R}^d \to M_0$ such that $\phi(0) = 0$. Replacing f by $f \circ \phi$, we may assume that $M = \mathbb{R}^d$.

By Invariance of Domain, an injective continuous map between manifolds of the same dimension is open, so $f : \mathbb{R}^d \to N$ is open. In particular $W := f(\mathbb{R}^d)$ is open in N. Then, by the preceding theorem, $f^{-1} : W \to \mathbb{R}^d$ is smooth. Let $M_0 := \mathbb{R}^d$ and let $N_0 := W$. Then $f : M_0 \to N_0$ is smooth with smooth inverse, *i.e.*, is a diffeomorphism. **QED**

Inverse Function Theorem (second avatar). Let M and N be manifolds and let $f : M \to N$ be smooth. Let $m_0 \in M$ and let $n_0 := f(m_0)$. Assume that $(df)_{m_0} : T_{m_0}M \to T_{n_0}N$ is a vector space isomorphism. Let $d := \dim(M)$. Then there is a localization $f_0 : M_0 \to N_0$ of f near m_0 such that, in the arrow category of {manifolds}, we have that $f_0 : M_0 \to N_0$

is isomorphic to the identity map $\mathbb{R}^d \to \mathbb{R}^d$.

Proof: By the first avatar of the Inverse Function Theorem, choose a localization $f_1 : M_1 \to N_1$ of f near m_0 such that $f_1 : M_1 \to N_1$ is a diffeomorphism. Let M_0 be a neighborhood of m_0 in M_1 such that M_0 is diffeomorphic to $\mathbb{R}^d$. Let $N_0 := f(M_0)$. Then $f_0 := f_1|M_0 : M_0 \to N_0$ is isomorphic to a diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ and any diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ is isomorphic to the identity map $\mathbb{R}^d \to \mathbb{R}^d$. **QED**

Definition. Let M and N be manifolds and let $f : M \to N$ be smooth. Define $r_f : M \to \mathbb{Z}$ by $r_f(m) = \dim((df)_m(T_m M))$.

That is, for all $m \in M$, $r_f(m)$ is defined to be the dimension of the image of the differential $(df)_m : T_m M \to T_{f(m)} N$; this number is called the **rank** of f at m. Note that $r_f(m) \leq \dim(M)$ and $r_f(m) \leq \dim(N)$.

For example, if $M = \mathbb{R} = N$ and if $f : M \to M$ is defined by $f(x) = x^2$, then

$$r_f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$$

EXERCISE 24A: Let M and N be manifolds and let $f : M \to N$ be smooth. For all $k \in \mathbb{Z}$, show that $\{m \in M \,|\, r_f(m) \leq k\}$ is a closed subset of M.

That is, r_f has closed sublevel sets. In particular, r_f is semicontinuous. (I think one says "upper" semicontinuous here...)

Now, more generally, let M be a topological space and let $p : M \to \mathbb{Z}$ be a function. For any $m \in M$, we say that p **is constant near** m if: there exists an open neighborhood M_0 of m in M such that $p|M_0$ is constant. We define C_p to be the set of all $m \in M$ such that p is constant near m. It is clear that C_p is open in M, but the following property is also useful:

EXERCISE 24B: Assume that $p(M)$ is finite and that p has closed sublevel sets. Show that C_p is dense in M.

Now go back to the situation where M and N are manifolds and where $f : M \to N$ is smooth. Let $c := \min\{\dim(M), \dim(N)\}$. Then $r_f(M) \subseteq \{1, \ldots, c\}$ so $r_f(M)$ is finite. Then by Exercise 24A and Exercise 24B, we conclude that C_{r_f} is a dense open subset of M. Because of this, the following result is quite interesting:

Implicit Function Theorem (first avatar). Let M and N be manifolds, let $f : M \to N$ be smooth and let $m \in C_{r_f}$. Let $d := \dim(M)$, let $e := \dim(N)$ and let $r := r_f(m)$. Then there is a localization of f near m_0 which, in the arrow category of {manifolds}, is isomorphic to

$$(x_1, \ldots, x_d) \quad \mapsto \quad (x_1, \ldots, x_r, 0, \ldots, 0) \qquad : \qquad \mathbb{R}^d \quad \to \quad \mathbb{R}^e.$$

We will prove this later, after setting up some preliminary results.

Definition. Let $r, k, l \geq 0$ be integers. Let $U \subseteq \mathbb{R}^{r+k}$ be a nonempty open subset. Let $V \subseteq \mathbb{R}^{r+l}$ be a nonempty open subset. Let $f : U \to V$ be smooth. Let $p : \mathbb{R}^{r+k} \to \mathbb{R}^r$ be projection onto the first r coordinates. Let $q : \mathbb{R}^{r+l} \to \mathbb{R}^r$ be projection onto the first r coordinates. We say that f **fibers directly** over r if $q \circ f = p$.

Definition. Let M and N be manifolds and let $f : M \to N$ be smooth. Let $r \geq 0$ be an integer. We say that f **fibers over** r if there exist integers $k, l \geq 0$ there exist nonempty open subsets $U \subseteq \mathbb{R}^{r+k}$ and $V \subseteq \mathbb{R}^{r+l}$ and there exists $f_0 : U \to V$ fibering directly over r such that, in the arrow category of {manifolds}, $f : M \to N$ is isomorphic to $f_0 : U \to V$.

Definition. Let M and N be manifolds and let $f : M \to N$ be smooth. Let $r \geq 0$ be an integer and let $m \in M$. We say that f **fibers over** r **near** m if there is a localization of f near m that fibers over r.

Lemma. Let M and N be manifolds and let $f : M \to N$ be smooth. Let $m_0 \in M$ and let $r := r_f(m_0)$. Then f fibers over r near m_0.

Proof: Let $k := (\dim(M)) - r$ and $l := (\dim(N)) - r$. We may assume that $N = \mathbb{R}^{r+l}$. We may assume that $M = \mathbb{R}^{r+k}$. We may assume that $m_0 = 0$ and that $f(m_0) = 0$.

Then r is equal to the dimension of the image of $L := f'(0) : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$. The matrix of L is $(r+l) \times (r+k)$. Let $W := (f'(0))(\mathbb{R}^{r+k})$. Let V be a vector space complement in $\mathbb{R}^{r+k}$ to the kernel of $L : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$. Then $\dim(V) = r = \dim(W)$, $L(V) = W$ and $L|V : V \to W$ is a vector space isomorphism.

By precomposing and postcomposing $f : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ with vector space isomorphisms that move V to $\mathbb{R}^r \times \{0\}^k$ and W to $\mathbb{R}^r \times \{0\}^l$, we may assume that $V = \mathbb{R}^r \times \{0\}^k$ and that $W = \mathbb{R}^r \times \{0\}^l$.

Then the upper left $r \times r$ block B of the matrix of L is the matrix of $L|V : V \to W$ and is therefore invertible. So $\det(B) \neq 0$.

Let $p : \mathbb{R}^{r+k} \to \mathbb{R}^r$ be projection onto the first r coordinates. Let $q : \mathbb{R}^{r+l} \to \mathbb{R}^r$ be projection onto the first r coordinates. Let $g := q \circ f : \mathbb{R}^{r+k} \to \mathbb{R}^r$. Then the matrix of $g'(0)$ is the upper $r \times (r+k)$ block of $L = f'(0)$. Define $\phi : \mathbb{R}^{r+k} \times \mathbb{R}^{r+k}$ by $\phi(x, y) = (g(x, y), y)$. Note that $p \circ \phi = g = q \circ f$.

The upper $r \times (r+k)$ block of $\phi'(0)$ is the matrix of $g'(0)$, which is the upper $r \times (r+k)$ block of L. Then the upper left $r \times r$ block of $\phi'(0)$ is the same as that of L; it is equal to B. Moreover, the lower $k \times (r+k)$ block of $\phi'(0)$ is the matrix of $(x, y) \mapsto y : \mathbb{R}^{r+k} \to \mathbb{R}^k$. So the lower left $k \times r$ block of $\phi'(0)$ is equal to zero and the lower right $k \times k$ block of $\phi'(0)$ is equal to the $k \times k$ identity matrix I. Then $\det(\phi'(0)) = [\det(B)][\det(I)] = \det(B) \neq 0$, so $\phi'(0) : \mathbb{R}^{r+k} \to \mathbb{R}^{r+k}$ is invertible.

Then, by the first avatar of the Inverse Function Theorem, choose a localization $\phi_0 : V \to U$ of $\phi : \mathbb{R}^{r+k} \to \mathbb{R}^{r+k}$ such that $\phi_0 : V \to U$ is a diffeomorphism. Then, in the arrow category of {manifolds}, $f|V : V \to \mathbb{R}^{r+l}$ is isomorphic to $(f|V) \circ \phi_0^{-1} : U \to \mathbb{R}^{r+l}$.

Since $p \circ \phi = g = q \circ f$, it follows that $p \circ \phi_0 = q \circ (f|V)$, or $p = q \circ [(f|V) \circ \phi_0^{-1}]$. That is, $(f|V) \circ \phi_0^{-1}$ fibers directly over r. Since $f|V : V \to \mathbb{R}^{r+l}$ is a localization of $f : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ and since $f|V : V \to \mathbb{R}^{r+l}$ is isomorphic to $(f|V) \circ \phi_0^{-1} : U \to \mathbb{R}^{r+l}$, we see that a localization of f fibers over r. **QED**

Definition. Let $k, l, r \geq 0$ be integers. We define $\sigma_r^{kl} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ by: for all $x \in \mathbb{R}^r$, for all $y \in \mathbb{R}^k$, $(\sigma_r^{kl})(x, y) = (x, 0)$.

Definition. Let $k, l, r \geq 0$ be integers. Let $\tau : \mathbb{R}^r \to \mathbb{R}^l$ be a smooth map. We define $\sigma_\tau^{kl} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ by: for all $x \in \mathbb{R}^r$, for all $y \in \mathbb{R}^k$, $(\sigma_\tau^{kl})(x, y) = (x, \tau(x))$.

EXERCISE 25A: Let $k, l, r \geq 0$ be integers. Let $\tau : \mathbb{R}^r \to \mathbb{R}^l$ be a smooth map. Show that, in the arrow category of {manifolds}, $\sigma_\tau^{kl} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ is isomorphic to $\sigma_r^{kl} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$.

EXERCISE 25B: Let M be a connected manifold, let N be a manifold and let $g : M \to N$ be smooth. Assume, for all $m \in M$, that $(dg)_m : T_m M \to T_{g(m)} N$ is the zero map. Show that g is constant, *i.e.*, that there exists $n_0 \in N$ such that, for all $m \in M$, $g(m) = n_0$.

EXERCISE 25C: Let $k, l, r \geq 0$ be integers. Let $f : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ be a smooth map which fibers directly over r. Assume, for all $p \in \mathbb{R}^{r+k}$, that $r_f(p) = r$. Show that there exists a smooth map $\tau : \mathbb{R}^r \to \mathbb{R}^l$ such that $f = \sigma_\tau^{kl}$. (*Hint:* Use Exercise 25B.)

Definition. Let $d \geq 0$ be an integer. A subset $U \subseteq \mathbb{R}^d$ is an **open box** if there exist open intervals $I_1, \ldots, I_d \subseteq \mathbb{R}$ such that $U = I_1 \times \cdots \times I_d$.

EXERCISE 25D: Let $k, l, r \geq 0$ be integers. Let $M \subseteq \mathbb{R}^{r+k}$ and $N \subseteq \mathbb{R}^{r+l}$ be open boxes. Let $f : M \to N$ be a smooth map which fibers directly over r. Show that there is an open subset $N_0 \subseteq N$ and a smooth map $f_0 : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ which fibers directly over r and which is isomorphic, in the arrow category of {manifolds}, to $f : M \to N_0$.

The first avatar of the Implicit Function Theorem (see above) is implied by

Implicit Function Theorem (second avatar). Let M and N be manifolds, let $f : M \to N$ be smooth Let $m_0 \in M$. Let $r \geq 0$ be an integer. Assume, for some neighborhood W of m_0 in M, that: for all $w \in W$, $r_f(w) = r$. Let $k := (\dim M) - r$. Let $l := (\dim N) - r$. Then there is a localization of f near m_0 which is isomorphic to $\sigma_r^{kl} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$.

Proof: Passing to a localization, we may assume, for all $m \in M$, that $r_f(m) = r$. By the preceding lemma, f fibers over r near m_0. Passing to a localization, we may assume that f fibers over r. Passing to an isomorphic map, we may assume that M is an open subset of $\mathbb{R}^{r+k}$, that N is an open subset of $\mathbb{R}^{r+l}$ and that $f : \mathbb{R}^{r+k} \to \mathbb{R}^{r+l}$ fibers directly over r. Passing to a localization, we may assume that N is an open box in $\mathbb{R}^{r+l}$ and that M is an open box in $\mathbb{R}^{r+k}$. By Exercise 25D, we may assume that $M = \mathbb{R}^{r+l}$ and that $N = \mathbb{R}^{r+k}$. By Exercise 25C, choose a smooth map $\tau : \mathbb{R}^r \to \mathbb{R}^l$ such that $f = \sigma_\tau^{kl}$. Then, by Exercise 25A, we are done. **QED**

Definition. Let M be a manifold and let $\pi : TM \to M$ be the tangent bundle map. A **(tangent) vector field** on M is a section of TM, *i.e.*, a smooth map $V : M \to TM$ such that $\pi \circ V : M \to M$ is the identity. We denote the set of all vector fields on M by $\mathrm{VF}(M)$. For all $V \in \mathrm{VF}(M)$, for all $m \in M$, we denote $V(m)$ alternatively by V_m.

Definition. Let M be a manifold and let $\pi : E \to M$ be a vector bundle on M. For all $m \in M$, let E_m be the vector space $\pi^{-1}(m)$, and let 0_m be the zero element of that vector space. The **zero section** of E is the map $0 : M \to E$ defined by: for all $m \in M$, $0(m) = 0_m$.

Definition. Let M be a manifold. The zero section 0 of TM is called the **zero vector field**. For any $V \in VF(M)$, for any $m \in M$, we say that V **vanishes at** m if $V_m = 0_m$. For any $V \in VF(M)$, we say that V is **nowhere vanishing** if, for all $m \in M$, we have $V_m \neq 0_m$.

We have argued that we cannot comb the hairs on a hedgehog, which implies that S^2 has no nonvanishing vector fields, which, in turn, implies that TS^2 is not a trivial vector bundle on S^2.

Definition. Let M be a manifold, let $V \in \mathrm{VF}(M)$ and let $m \in M$. Let $I \subseteq \mathbb{R}$ be an open interval and assume that $0 \in I$. Let $\gamma : I \to M$ be smooth. Then we say that γ is an **integral curve** for V at m if:

(1) $\gamma(0) = m$; and
(2) for all $t \in I$, we have $(d/dt)_{t=0}(\gamma(t)) = V_{\gamma(t)}$.

Definition. Let $n \geq 0$ be an integer, let $V \in \mathrm{VF}(\mathbb{R}^n)$. For all $p \in \mathbb{R}^n$, let $\phi_p : T_p\mathbb{R}^n \to \mathbb{R}^n$ be the standard identification. Then we define $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}^n$ by $\widetilde{V}(p) = \phi_p(V_p)$.

We leave it as an unassigned exercise to show, for any integer $n \geq 0$, for any $V \in \mathrm{VF}(\mathbb{R}^n)$, that $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

Theorem. Let M be a manifold, let $V \in \mathrm{VF}(M)$ and let $m \in M$. Then there exists an integral curve for V at m.

Proof: Let $n := \dim(M)$.

We may assume that $M = \mathbb{R}^n$ and that $m = 0$. We wish to show that there exist $\epsilon > 0$ and a smooth $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and such that, for all $t \in (-\epsilon, \epsilon)$, we have: $\dot{\gamma}(t) = \widetilde{V}(\gamma(t))$.

Let C be the closed ball of radius 1 about the origin in $\mathbb{R}^n$. Recall, for all $p \in \mathbb{R}^n$, that $\|\widetilde{V}'(p)\|$ denotes the operator norm of the linear transformation $\widetilde{V}'(p) : \mathbb{R}^n \to \mathbb{R}^n$. Let

$$K := \sup\left\{|\widetilde{V}(p)| \,\middle|\, p \in C\right\} \qquad \text{and} \qquad K_1 := \sup\left\{\|\widetilde{V}'(p)\| \,\middle|\, p \in C\right\}.$$

For all $p, q \in C$, the multivariable mean value theorem yields: $|\widetilde{V}(p) - \widetilde{V}(q)| \leq K_1|p - q|$. Choose $\epsilon > 0$ such that $K\epsilon \leq 1$ and such that $K_1\epsilon \leq 1/2$.

Let $X := C([-\epsilon, \epsilon], C)$ be the set of continuous functions $[-\epsilon, \epsilon] \to C$. For all $\alpha, \beta \in X$, define $d(\alpha, \beta) := \sup\left\{|(\alpha(p)) - (\beta(p))| \,\middle|\, p \in C\right\}$. For all $\alpha \in X$, define $\widetilde{\alpha} : [-\epsilon, \epsilon] \to C$ by $\widetilde{\alpha}(t) = \int_0^t \widetilde{V}(\alpha(s))\, ds$.

For all $\alpha \in X$, for all $t \in [-\epsilon, \epsilon]$, we have

$$|\alpha(t)| \leq \left| \int_0^t |\widetilde{V}(\alpha(s))|\, ds \right| \leq \left| \int_0^t K\, ds \right| = K|t| \leq K\epsilon \leq 1,$$

so $\alpha(t) \in C$. Thus, for all $\alpha \in X$, we have $\widetilde{\alpha} \in X$.

For all $\alpha, \beta \in X$, for all $t \in [-\epsilon, \epsilon]$, we have

$$
\begin{aligned}
|(\widetilde{\alpha}(t)) - (\widetilde{\beta}(t))| &\leq \left| \int_0^t |\widetilde{V}(\alpha(s)) - \widetilde{V}(\beta(s))| \, ds \right| \\
&\leq \left| \int_0^t K_1 |(\alpha(s)) - \beta(s)| \, ds \right| \\
&\leq \left| \int_0^t K_1 (d(\alpha, \beta)) \, ds \right| \\
&= K_1(d(\alpha, \beta))|t| \leq K_1(d(\alpha, \beta))\epsilon \leq (1/2)(d(\alpha, \beta)).
\end{aligned}
$$

Thus, for all $\alpha, \beta \in X$, we have $d(\widetilde{\alpha}, \widetilde{\beta}) \leq (1/2)(d(\alpha, \beta))$.

EXERCISE 25E: Show, for some $\zeta \in X$, that $\widetilde{\zeta} = \zeta$.

Choose ζ as in Exercise 25E, and let $\gamma := \zeta|(-\epsilon, \epsilon)$. Then, for all $t \in (-\epsilon, \epsilon)$, we have:

$$\gamma(t) = \int_0^t \widetilde{V}(\gamma(s)) \, ds.$$

By the Fundamental Theorem of Calculus, it follows that γ is C^1, and, for all $t \in (-\epsilon, \epsilon)$, that $\dot{\gamma}(t) = \widetilde{V}(\gamma(t))$. It remains to show that γ is C^∞.

We have $\dot{\gamma} = \widetilde{V} \circ \gamma$, so, as $\widetilde{V}$ is C^∞ and γ is C^1, we see that $\dot{\gamma}$ is C^1, so γ is C^2.

We have $\dot{\gamma} = \widetilde{V} \circ \gamma$, so, as $\widetilde{V}$ is C^∞ and γ is C^2, we see that $\dot{\gamma}$ is C^2, so γ is C^3.

Continuing in this way, for all integers $k \geq 1$, γ is C^k. Then γ is C^∞. **QED**

EXERCISE 25F: Let M be a manifold, let $V \in \mathrm{VF}(M)$ and let $m \in M$. Let $\gamma : I \to M$ be an integral curve for V at m. Let $\delta : J \to M$ be an integral curve for V at m. Show that $\gamma|(I \cap J) = \delta|(I \cap J)$.

Definition. Let M be a manifold, let $V \in \mathrm{VF}(M)$ and let $m \in M$. An integral curve $\gamma : I \to M$ for V at m is said to be **maximal** if, for any integral curve $\gamma_0 : I_0 \to M$ for V at m, we have both $I_0 \subseteq I$ and $\gamma_0 = \gamma|I_0$.

We leave it as an unassigned exercise to show: For any manifold M, for any $V \in \mathrm{VF}(M)$, for any $m \in M$, there is a unique maximal integral curve for V at m.

Definition. Let M be a manifold and let $V \in \mathrm{VF}(M)$. We say that V is **complete** if: for all $m \in M$, there exists an integral curve γ for V at m such that the domain of γ is $\mathbb{R}$.

A vector field which is not complete is said to be **incomplete**.

Let $M := (-1, 1)$ be the open interval from -1 to 1. For all $m \in M$, let $V_m := (d/dt)_{t=0}(m + t)$. Then $V \in \mathrm{VF}(M)$ and we leave it as an unassigned exercise to show that V is incomplete.

Let $f : M \to \mathbb{R}$ be a diffeomorphism. Let $W := f_*(V)$ be the vector field on $\mathbb{R}$ corresponding to V on M. Since V is incomplete, W is as well. Thus $\mathbb{R}$ admits an incomplete vector field.

For any $S \subseteq \mathbb{R}$, for any $t \in \mathbb{R}$, we define $S - t := \{s - t \,|\, s \in S\}$.

Remark. Let M be any manifold, let $V \in \mathrm{VF}(M)$ and let $\epsilon > 0$. Suppose, for all $m \in M$, that there is an integral curve $(-\epsilon, \epsilon) \to M$ for V at m. Then V is complete.

Proof: Fix $m_0 \in M$. We wish to show that there is an integral curve $\mathbb{R} \to M$ for V at m_0. Let $\gamma_0 : I \to M$ be the maximal integral curve for V at m_0. We wish to show that $I = \mathbb{R}$. That is, we wish to show that the interval I is neither bounded above, nor bounded below. We will show that I is not bounded above; the proof that I is not bounded below is similar. Suppose I is bounded above. We aim for a contradiction.

Choose $t_1 \in I$ such that $t_1 + (1/2)\epsilon \notin I$. Let $m_1 := \gamma_0(t_1)$. Let $I_1 := I - t_1$. Define $\gamma_1 : I_1 \to M$ by $\gamma_1(t) = \gamma_0(t + t_1)$. Then γ_1 is a maximal integral curve for V at m_1. By assumption, there is an integral curve $(-\epsilon, \epsilon) \to M$ for V at m_1, so, by maximality, $(-\epsilon, \epsilon) \subseteq I_1 = I - t_1$. Then $(t_1 - \epsilon, t_1 + \epsilon) \subseteq I$. However, $t_1 + (1/2)\epsilon \notin I$, so we have a contradiction. **QED**

Corollary. Any vector field on a compact manifold is complete.

Proof: Let M be a compact manifold and let $V \in \mathrm{VF}(M)$. We wish to show that V is complete.

For all $m \in M$, choose $\epsilon_m > 0$ such that there is an integral curve $(-2\epsilon_m, 2\epsilon_m) \to M$ for V at m. We then leave it as an unassigned exercise to show that there is a neighborhood U_m of m in M such that: for all $u \in U_m$, there is an integral curve $(-\epsilon_m, \epsilon_m) \to M$ for V at u. (*Hint:* To do this exercise, it helps to have the existence of local flows, see below.)

By compactness, choose a finite subset $F \subseteq M$ such that $\bigcup_{f \in F} U_f = M$. Let $\epsilon := \min\{\epsilon_f \,|\, f \in F\}$. Then, for all $m \in M$, there is an integral curve $(-\epsilon, \epsilon) \to M$ for V at m. Then V is complete, by the preceding remark. **QED**

Definition. Let V be a vector field on a manifold M and let $\mathcal{N}$ denote the set of all open subsets U of $M \times \mathbb{R}$ such that $M \times \{0\} \subseteq U$. For all $U \in \mathcal{N}$, for all $m \in M$, let $U_m := \{t \in \mathbb{R} \,|\, (m, t) \in U\}$. A **local flow** for V is

(1) a $U \in \mathcal{N}$; and

(2) a smooth $\phi : U \to M$

such that, for all $m \in M$, we have:

(A) U_m is an open interval in $\mathbb{R}$; and

(B) $t \mapsto \phi(m, t) : U_m \to M$ is an integral curve for V at m.

Theorem. For any manifold M, for any $V \in \mathrm{VF}(M)$, there is a local flow for V.

We leave the proof as an unassigned exercise for the interested reader. It is quite similar to the proof of the existence of integral curves, except that one must create an integral curve for V at *every* point of M, not just at one. Moreover, one must do all of them simultaneously in a smoothly varying way.

For any function f, let $\mathrm{dom}(f)$ denote the domain of f.

Remark. Let M be a manifold and let $V \in \mathrm{VF}(M)$. Let ϕ and ψ be local flows for V. Let

$W := (\mathrm{dom}(\phi)) \cap (\mathrm{dom}(\psi))$. Then $\phi|W = \psi|W$.

Definition. Let M be a manifold and let $V \in \mathrm{VF}(M)$. Let $\mathcal{F}$ be the set of flows for V and let $\phi \in \mathcal{F}$. We say ϕ is **maximal** if, for all $\psi \in \mathcal{F}$, we have $\psi = \phi|(\mathrm{dom}(\psi))$.

EXERCISE 25G: For all $V \in \mathrm{VF}(M)$, show that there is a unique maximal flow for V.

We now begin a new topic: Tensors and tensor bundles. From this point on, $\otimes$ means $\otimes_{\mathbb{R}}$. By "vector space", we will always mean real vector space, unless otherwise specified.

Recall that, if V and W are vector spaces, then $V \otimes W$ is defined in such a way that, for any vector space X, the set of bilinear maps $V \times W \to X$ is naturally in one-to-one correspondence with the set of linear maps $V \otimes W \to X$.

Similarly, if V, W and X are vector spaces, then $V \otimes W \otimes X$ is defined in such a way that, for any vector space Y, the set of bilinear maps $V \times W \times X \to Y$ is naturally in one-to-one correspondence with the set of linear maps $V \otimes W \otimes X \to Y$.

For any vector space V, for any integer $n \geq 1$, we define $\otimes^n V := V \otimes V \otimes \cdots \otimes V$ (n factors). For any vector space V, we define $\otimes^0 V := \mathbb{R}$.

Definition. Let V be a vector space and let $n \geq 2$ be an integer. Let Σ be the set of all permutations on n symbols, *i.e.*, the set of all bijective maps $\{1, \ldots, n\} \to \{1, \ldots, n\}$. Let $W := \otimes^n V$. Let

$$Q \quad := \quad \{ \quad (v_1 \otimes \cdots \otimes v_n) - (v_{\sigma(1)} \otimes v_{\sigma(n)}) \quad | \quad v_1, \ldots, v_n \in V, \quad \sigma \in \Sigma \quad \}.$$

Let X be the linear span of Q in W. Then we define $S^n V := W/X$.

For all vector spaces V, we also define $S^0 V := \mathbb{R}$ and $S^1 V := V$.

We leave it as an unassigned exercise to show, for any vector spaces V and W, for any integer $n \geq 1$, that the set of symmetric multilinear maps $V \times \cdots \times V \to W$ (where the Cartesian product has n factors) is in one-to-one correspondence with the set of linear maps $S^n V \to W$.

Definition. Let V be a vector space and let $n \geq 2$ be an integer. Let Σ be the set of all permutations on n symbols, *i.e.*, the set of all bijective maps $\{1, \ldots, n\} \to \{1, \ldots, n\}$. Let $\mathrm{sgn} : \Sigma \to \{-1, 1\}$ be the sign homomorphism. Let $W := \otimes^n V$. Let

$$Q \quad := \quad \{ \quad (v_1 \otimes \cdots \otimes v_n) - (\mathrm{sgn}(\sigma))(v_{\sigma(1)} \otimes v_{\sigma(n)}) \quad | \quad v_1, \ldots, v_n \in V, \quad \sigma \in \Sigma \quad \}.$$

Let X be the linear span of Q in W. Then we define $\wedge^n V := W/X$.

For all vector spaces V, we also define $\wedge^0 V := \mathbb{R}$ and $\wedge^1 V := V$.

We leave it as an unassigned exercise to show, for any vector spaces V and W, for any integer $n \geq 1$, that the set of antisymmetric multilinear maps $V \times \cdots \times V \to W$ (where the Cartesian product has n factors) is in one-to-one correspondence with the set of linear maps $\wedge^n V \to W$.

Let $\mathcal{VS}$ denote the category of finite dimensional vector spaces. Recall that, if $n \geq 1$ is an integer and if $V_1, \ldots, V_n \in \mathcal{VS}$, then $\dim(V_1 \otimes \cdots \otimes V_n) = (\dim(V_1)) \cdots (\dim(V_n))$.

Let $V \in \mathcal{VS}$ and let $d := \dim(V)$. We leave it as an unassigned exercise to show that $\dim(S^n V)$ is equal to the cardinality of the set of: monomials in d variables with total degree n. We also leave it as an unassigned exercise to show that this is equal to the binomial coefficient "$d + n - 1$ choose d".

Let $V \in \mathcal{VS}$ and let $d := \dim(V)$. We leave it as an unassigned exercise to show that $\dim(\wedge^n V)$ is equal to the cardinality of the set of: square-free monomials in d variables with total degree n. We also leave it as an unassigned exercise to show that this is equal to the binomial coefficient "d choose n".

For all integers $n \geq 1$, let $\mathcal{VS}^n$ denote the category of n-tuples of finite dimensional vector spaces.

Let $\mathcal{VB}$ denote the category of vector bundles. For any vector bundle E over a manifold M, for all $m \in M$, let $E_m \in \mathcal{VS}$ denote the fiber over m of E.

An **n-tuple of vector bundles** consists of a manifold M and an n-tuple of vector bundles on M. Let $\mathcal{VB}^n$ denote the category of n-tuples of vector bundles. For any vector bundle $E = (E^1, \ldots, E^n)$ over a manifold M, for all $m \in M$, let

$$E_m \qquad := \qquad (E^1_m, \ldots, E^n_m) \qquad \in \qquad \mathcal{VS}^n.$$

Let $\mathcal{M}$ denote the cateogry of manifolds. For any $V \in \mathcal{VS}$, for any $M \in \mathcal{M}$, recall that $M \times V$ is a trivial vector bundle over M. For any $V = (V_1, \ldots, V_n) \in \mathcal{VS}^n$, for any $M \in \mathcal{M}$, let $M \times V := (M \times V_1, \ldots, M \times V_n) \in \mathcal{VB}^n$.

A **smooth family of vector spaces** consists of a manifold M and a function from M to the class of vector spaces. Let $\mathcal{SFVS}$ denote the category of smooth family of vector spaces. Let $\mathcal{A} : \mathcal{VB} \to \mathcal{SFVS}$ be the forgetful functor, which associates, to any vector bundle E on a manifold M, the map $m \mapsto E_m$.

Now fix an integer $n \geq 1$ and let $\mathcal{F} : \mathcal{VS}^n \to \mathcal{VS}$ be a functor.

Let $\mathcal{P} : \mathcal{VB}^n \to \mathcal{M}$ be the functor which associates to any n-tuple of vector bundles, the underlying base manifold. Let $\mathcal{Q} : \mathcal{VB} \to \mathcal{M}$ be the functor which associates to any vector bundle, the underlying base manifold.

Let $\mathcal{C}$ be the full subcategory of $\mathcal{VB}^n$ whose objects are

$$\{ \qquad M \times V \qquad | \qquad M \in \mathcal{M}, \qquad V \in \mathcal{VS}^n \qquad \}.$$

(By "full subcategory", we mean that the arrows of $\mathcal{C}$ are those arrows in $\mathcal{VB}^n$ whose domain and target are objects in $\mathcal{C}$.) Define $\mathcal{F}_0 : \mathcal{C} \to \mathcal{VB}$ by $\mathcal{F}_0(M \times V) = M \times (\mathcal{F}(V))$. Define $\mathcal{B} : \mathcal{VB}^n \to \mathcal{SFVS}$ be defined by: for any n-tuple E of vector bundles over a manifold M, $\mathcal{B}(E)$ is the map $m \mapsto \mathcal{F}(E_m)$.

Proposition. There is a unique functor $\mathcal{F}' : \mathcal{VB}^n \to \mathcal{VB}$ such that $\mathcal{F}'|\mathcal{C} = \mathcal{F}_0$, such that $\mathcal{A} \circ \mathcal{F}' = \mathcal{B}$ and such that $\mathcal{Q} \circ \mathcal{F}' = \mathcal{P}$.

We omit the proof. Four important special cases:

First, if $\mathcal{F} = \otimes : \mathcal{VS}^n \to \mathcal{VS}$, then $\mathcal{F}'$ is also denoted $\otimes : \mathcal{VB}^n \to \mathcal{VB}$. That is, if $E^1, \ldots, E^n$ are vector bundles over a manifold M, then $\mathcal{F}'(E^1, \ldots, E^n)$ is a vector bundle over M which is typically denoted $E^1 \otimes \cdots \otimes E^n$. It has the property that, for all $m \in M$, we have $(E^1 \otimes \cdots \otimes E^n)_m$ is naturally isomorphic to $E^1_m \otimes \cdots \otimes E^n_m$ in $\mathcal{VB}$.

Second, if $\mathcal{F} = S^n : \mathcal{VS} \to \mathcal{VS}$, then $\mathcal{F}'$ is also denoted $S^n : \mathcal{VB} \to \mathcal{VB}$. That is, if E is a vector bundle over a manifold M, then $\mathcal{F}'(E)$ is a vector bundle over M which is typically denoted S^nE. It has the property that, for all $m \in M$, we have $(S^nE)_m$ is naturally isomorphic to $S^n(E_m)$ in $\mathcal{VB}$.

Third, if $\mathcal{F} = \wedge^n : \mathcal{VS} \to \mathcal{VS}$, then $\mathcal{F}'$ is also denoted $\wedge^n : \mathcal{VB} \to \mathcal{VB}$. That is, if E is a vector bundle over a manifold M, then $\mathcal{F}'(E)$ is a vector bundle over M which is typically denoted $\wedge^nE$. It has the property that, for all $m \in M$, we have $(\wedge^nE)_m$ is naturally isomorphic to $\wedge^n(E_m)$ in $\mathcal{VB}$.

Fourth, let $\mathcal{F} = V \mapsto V^* : \mathcal{VS} \to \mathcal{VS}$ be the (contravariant) functor which associates, to any finite dimensional vector space, its dual. Then $\mathcal{F}'$ is denoted $E \mapsto E^* : \mathcal{VB} \to \mathcal{VB}$. That is, if E is a vector bundle over a manifold M, then $\mathcal{F}'(E)$ is a vector bundle over M which is typically denoted E^*. It has the property that, for all $m \in M$, we have $(E^*)_m$ is naturally isomorphic to $(E_m)^*$ in $\mathcal{VB}$. We may therefore write E_m^* without fear of ambiguity.

Let $\widetilde{\mathcal{VS}}$ be the category whose objects are vector spaces, and whose arrows are isomorphisms between vector spaces. For all integers $n \geq 1$, let $\widetilde{\mathcal{VS}}^n$ be the category whose objects are n-tuples of vector spaces and whose arrows are n-tuples of isomorphisms between vector spaces. Let $\widetilde{\mathcal{VB}}$ be the category whose objects are vector bundles and whose arrows are isomorphisms between vector bundles. For all integers $n \geq 1$, let $\widetilde{\mathcal{VB}}^n$ be the category whose objects are n-tuples of vector bundles and whose arrows are isomorphisms between n-tuples vector bundles.

Let $n \geq 1$ be an integer. Given a functor $\mathcal{F} : \widetilde{\mathcal{VS}}^n \to \widetilde{\mathcal{VS}}$, we can mimic the preceding construction and construct a functor $\mathcal{F}' : \widetilde{\mathcal{VB}}^n \to \widetilde{\mathcal{VB}}$.

Note that $\mathcal{A} : \mathcal{VS} \to \mathcal{VS}$, defined by $\mathcal{A}(V) = V^*$ is contravariant, but we may define a covariant functor $\widetilde{\mathcal{A}} : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$ by, for all vector spaces V, $\widetilde{\mathcal{A}}(V) = V^*$ and by, for all vector space isomorphisms $f : V \to W$, $\widetilde{\mathcal{A}}(f) = (\mathcal{A}(f))^{-1}$.

For any vector space V, let $\mathrm{End}(V)$ be the vector space of all linear maps $V \to V$.

Fifth, let $\mathcal{F} : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$ be the covariant functor defined by $\mathcal{F}(V) = V^* \otimes V$. It is an unassigned exercise that $\mathcal{F}$ is equivalent to the functor $V \mapsto \mathrm{End}(V) : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. The resulting fucntor $\mathcal{F}' : \widetilde{\mathcal{VB}} \to \widetilde{\mathcal{VB}}$ may be thought of as associating to any vector bundle $E \to M$, a vector bundle over M whose fiber over the point $m \in M$ is the vector space of linear maps $T_mM \to T_mM$.

Sixth, fix integers $p, q \geq 0$ and define $\mathcal{F} : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$ by $\mathcal{F}(V) = (\otimes^p V^*) \otimes (\otimes^q V)$. Then, for any manifold M, a section of $\mathcal{F}'(TM)$ is called a **(p, q)-tensor field on M**. When $q = 0$, we have $\mathcal{F}(V) = \otimes^p V^*$, and so a $(p, 0)$-tensor field on M may be thought of as associating, in a smoothly varying way, to each point m of M, a p-multilinear map $T_mM \times \cdots \times T_mM \to \mathbb{R}$, where there are p factors in the Cartesian product. By polarization, this is equivalent to a homogeneous polynomial $T_mM \to \mathbb{R}$ of degree p. When $q = 1$, we have $\mathcal{F}(V) = (\otimes^p V^*) \otimes V$, and so a $(p, 1)$-tensor field on M may be thought of as associating, in a smoothly varying way, to each point m of M, a p-multilinear map $T_mM \times \cdots \times T_mM \to T_mM$, where there are p factors in the Cartesian product. By polarization, this is equivalent to a homogeneous polynomial $T_mM \to T_mM$ of degree p.

For any vector space V, for any integer $n \geq 0$, the map

$$l_1 \wedge \cdots \wedge l_n \qquad \mapsto \qquad (v_1 \wedge \cdots \wedge v_n) \mapsto \det[l_i(v_j)]$$

is an isomorphism $\wedge^n(V^*) \to (\wedge^n V)^*$. Thus the functors

$$V \mapsto \wedge^n(V^*) : \mathcal{VS} \to \mathcal{VS} \qquad \text{and} \qquad V \mapsto (\wedge^n V)^* : \mathcal{VS} \to \mathcal{VS}$$

are equivalent, it follows that the functors

$$E \mapsto \wedge^n(E^*) : \mathcal{VB} \to \mathcal{VB} \qquad \text{and} \qquad E \mapsto (\wedge^n E)^* : \mathcal{VB} \to \mathcal{VB}$$

are equivalent. We therefore write $\wedge^n V^*$ and $\wedge^n E^*$ without concern about ambiguity.

Definition. Let M be a manifold and let $n \geq 0$ be an integer. A **(differential) n-form** on M is a section of $\wedge^n(TM)^*$.

We now try to give some motivation for that definition.

For any $V \in \mathcal{VS}$, for any integer $n \geq 1$, let's say that an **ordered n-parallelpiped** in V is simply an element of $V \times \cdots \times V$, where there are n-factors in this Cartesian product. Let $P^n(V) := V \times \cdots \times V$ denote the set of ordered n-parallelpipeds in V.

Definition. A **signed n-parallelpiped measure** is a function $\mu : P^n(V) \to \mathbb{R}$ which is multilinear and which satisfies:

(∗) for all $v_1, \ldots, v_n \in V$, if, for some $i, j \in \{1, \ldots, d\}$, we have both $i \neq j$ and $v_i = v_j$, then $\mu(v_1, \ldots, v_n) = 0$.

The condition (∗) simply says that a degenerate parallelpiped has size zero. This is a reasonable geometric condition.

The multilinearity says that, if $i \in \{1, \ldots, n\}$, if v, w and x are three n-parallelpipeds which agree in all coordinates but the ith, if $c, d \in \mathbb{R}$ if $x_i = cv_i + dw_i$, then $\mu(x) = c(\mu(v)) + d(\mu(w))$. We leave it as an unassigned geometric exercise to show that this property holds, say, for the usual signed area of ordered 2-parallelpipeds in $\mathbb{R}^2$.

Note: A 2-parallelpiped is often called a **parallelogram**.

EXERCISE 25H: Let V be a vector space. Let $n \geq 1$ be an integer. Let $\mu : P^n(V) \to \mathbb{R}$ be multilinear. Show that μ is a signed n-parallelpiped measure iff μ is antisymmetric.

By Exercise 25H, the set of signed n-parallelpiped measures on V is naturally in one-to-one correspondence with the set of multilinear antisymmetric maps $V \times \cdots \times V \to \mathbb{R}$. This, in turn, is naturally in one-to-one correspondence with the set of linear maps $\wedge^n V \to \mathbb{R}$. This, by definition, is equal to $(\wedge^n V)^*$.

Therefore a differential n-form on a manifold M may be thought of as a smoothly varying system of signed n-parallelpiped measures, one on each tangent space of M.

For any vector space V, for any open subset U in V, for any $u \in U$, the **standard identification** of V with $T_u U$ is the vector space isomorphism given by

$$v \quad \mapsto \quad (d/dt)_{t=0}(u + tv) \quad : \quad V \quad \to \quad T_u U.$$

Definition. Let M and N be manifolds, let $S \subseteq M$ and let $f : S \to N$ be a function. We say that $f : S \to N$ has a **smooth extension in M to N** if there is an open subset U of M and a smooth function $F : U \to N$ such that both $S \subseteq U$ and $f = F|S$. The collection of such functions will be denoted $C^\infty_M(P, N)$.

Let M be a manifold, let $n \geq 1$ be an integer and let ω be a differential n-form on M. Let λ denote Lebesgue measure on $\mathbb{R}^n$. Let S be a compact subset of $\mathbb{R}^n$ and let $\sigma : S \to M$ have a smooth extension in $\mathbb{R}^n$ to M. Assume that $S \subseteq \mathrm{Cl}_{\mathbb{R}^n}(\mathrm{Int}_{\mathbb{R}^n}(S))$. For all $s \in S$, let $e_s \in P^n(T_s S)$ denote the basis of $T_s S$ which corresponds to the standard basis of $\mathbb{R}^n$, under the standard identification $T_s S \cong \mathbb{R}^n$. For all $s \in S$, let $p_s \in P^n(T_{\sigma(s)} S)$ be the image of e_s under $(d\sigma)_s : T_s S \to T_{\sigma(s)} M$. Define $g : S \to \mathbb{R}$ by $g(s) := \omega_s(p_s)$, where ω_s is the signed n-parallelpiped measure on $T_{\sigma(s)} M$ determined by ω. Then $g : S \to \mathbb{R}$ has a smooth extension in $\mathbb{R}^n$ to $\mathbb{R}$, and is, in particular, continuous. We then define $\int_\sigma \omega := \int_S g \, d\lambda$.

That is, given σ, a smooth parametric S inside M, and given a differential n-form ω on M, we can associate a number, $\int_\sigma \omega$, to be thought of as integrating the n-form, ω, along the parametric S, σ.

Definition. For any vector bundle E, the vector space of sections of E is denoted $\Gamma(E)$.

Definition. Let $\mathcal{F}$ be either a functor $\mathcal{VS} \to \mathcal{VS}$, or a functor $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. Then, for any vector space V, any element of $\mathcal{F}(V)$ is called an **$\mathcal{F}$-tensor** in V. For any manifold M, any element of $\Gamma(\mathcal{F}'(TM))$ is called an **$\mathcal{F}$-tensor field** on M.

If σ is an $\mathcal{F}$-tensor field on M and $m \in M$, then $\sigma(m) \in (\mathcal{F}'(TM))_m \cong \mathcal{F}(T_m M)$ is often denoted σ_m.

If $\mathcal{F} : \mathcal{VS} \to \mathcal{VS}$ is defined by $\mathcal{F}(V) = \mathbb{R}$, then an $\mathcal{F}$-tensor field on M is a section of $M \times \mathbb{R}$, which is equivalent to a smooth function $M \to \mathbb{R}$. If $\mathcal{F} : \mathcal{VS} \to \mathcal{VS}$ is defined by $\mathcal{F}(V) = V$, then an $\mathcal{F}$-tensor field is a vector field. If $\mathcal{F} : \mathcal{VS} \to \mathcal{VS}$ is defined by $\mathcal{F}(V) = \wedge^n V^*$, then an $\mathcal{F}$-tensor field is a differential n-form. If $\mathcal{F} : \mathcal{VS} \to \mathcal{VS}$ is defined by $\mathcal{F}(V) = S^n V$, then an $\mathcal{F}$-tensor field on M is a **symmetric n-tensor field** on M.

Let $p, q \geq 0$ be integers. If $\mathcal{F} : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$ is defined by $\mathcal{F}(V) = (\otimes^p V^*) \otimes (\otimes^q V)$, then an $\mathcal{F}$-tensor field is a (p, q)-tensor field.

Definition. Let $\mathcal{F}$ be a contravariant functor, either $\mathcal{VS} \to \mathcal{VS}$ or $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. Let V and W be vector spaces and let $L : V \to W$ be a linear map. Let σ be an $\mathcal{F}$-tensor in W, *i.e.*, let $\sigma \in \mathcal{F}W$. Then we define $L^*\sigma$ to be the $\mathcal{F}$-tensor in V defined by $L^*\sigma := (\mathcal{F}L)\sigma$.

Definition. Let $\mathcal{F} : \widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$ be a covariant functor. Let V and W be vector spaces, let $L : V \to W$ be a vector space isomorphism. Let σ be an $\mathcal{F}$-tensor in W, *i.e.*, let $\sigma \in \mathcal{F}W$. Then we define $L^*\sigma$ to be the $\mathcal{F}$-tensor in V defined by $L^*\sigma := (\mathcal{F}L)^{-1}\sigma$.

Definition. Let $\mathcal{F}$ be either a contravariant functor $\mathcal{VS} \to \mathcal{VS}$ or a covariant functor $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. Let M and N be manifolds and let $\phi : M \to N$ be a smooth map. Let σ be an $\mathcal{F}$-tensor field on N, *i.e.*, let $\sigma \in \Gamma(\mathcal{F}'(TN))$. Then $\phi^*\sigma$ is the $\mathcal{F}$-tensor field on M defined by $(\phi^*\sigma)_m = ((d\phi)_m)^*(\sigma_{\phi(m)})$.

Definition. Let $\mathcal{F}$ be a covariant functor $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. Let M and N be manifolds and let $\phi : M \to N$ be a diffeomorphism. Let σ be an $\mathcal{F}$-tensor field on N, *i.e.*, let $\sigma \in \Gamma(\mathcal{F}'(TN))$. Then $\phi^*\sigma$ is the $\mathcal{F}$-tensor field on M defined by $(\phi^*\sigma)_m = ((d\phi)_m)^*(\sigma_{\phi(m)})$.

Let $\mathcal{F}$ be either a contravariant functor $\mathcal{VS} \to \mathcal{VS}$ or a functor $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$. Fix a manifold M and $X \in \mathrm{VF}(M)$. Let $\phi : U \to M$ be the maximal local flow of M. For all $m \in M$, let $U_m := \{t \in \mathbb{R} \,|\, (m,t) \in U\}$, and define $\phi_m : U_m \to M$ by $\phi_m(t) = \phi(m,t)$. For all $t \in \mathbb{R}$, let $U^t := \{m \in M \,|\, (m,t) \in U\}$, and define $\phi^t : U^t \to M$ by $\phi^t(m) = \phi(m,t)$. Let $V^t := \phi^t(U^t)$.

EXERCISE 26A: For all $t \in \mathbb{R}$, show that V^t is open in M and that $\phi^t : U^t \to V^t$ is a diffeomorphism.

Definition. Let σ be an $\mathcal{F}$-tensor field on M. For all $m \in M$, define $\gamma_m : U_m \to (\mathcal{F}(TM))_m$ by $\gamma_m(t) = ((\phi_t)^*(\sigma|V^t))_m$. Then the **Lie derivative** of σ along X, denoted $\mathcal{L}_X\sigma$ is the $\mathcal{F}$-tensor on M defined by $(\mathcal{L}_X\sigma)_m = \dot{\gamma}_m(0)$.

For any manifold M, $C^\infty(M)$ denotes the vector space of smooth functions $M \to \mathbb{R}$.

Definition. Let $f \in C^\infty(M)$ For all $m \in M$, define $\gamma_m := f \circ \phi_m : U_m \to \mathbb{R}$. Then the **Lie derivative** of f along X, denoted $\mathcal{L}_X f$ is the $\mathcal{F}$-tensor on M defined by $(\mathcal{L}_X\sigma)_m = \dot{\gamma}_m(0)$.

We next seek to practice calculations of Lie derivatives.

Let $M := \mathbb{R}^2$. Let $e_1 := (1,0)$ and $e_2 := (0,1)$. Let $E_1 \in \mathrm{VF}(M)$ be defined by $(E_1)_p = (d/dt)_{t=0}(p + te_1)$. Let $E_2 \in \mathrm{VF}(M)$ be defined by $(E_2)_p = (d/dt)_{t=0}(p + te_2)$. Define $x, y : M \to \mathbb{R}$ by $x(s,t) = s$ and $y(s,t) = t$. Let $X = x^2E_1 + xyE_2 \in \mathrm{VF}(M)$. Let $f_0 := x^5 + y^6 : M \to \mathbb{R}$. Let $\widetilde{f}_0 \in \Gamma(M \times \mathbb{R})$ be defined by $\widetilde{f}_0(p) = (p, f_0(p))$. Let us compute $\mathcal{L}_X f_0$ and $\mathcal{L}_X \widetilde{f}_0$.

We first note that E_1 is often denoted $\partial/\partial x$, while E_2 is often denoted $\partial/\partial y$. One often writes $X\widetilde{f}_0$ for $\mathcal{L}_X\widetilde{f}_0$. One often writes Xf_0 for $\mathcal{L}_X f_0$. Then we have:

$$\mathcal{L}_X f_0 = X f_0 = (x^2(\partial/\partial x) + xy(\partial/\partial y))(x^5 + y^6).$$

Our habits from calculus suggest that the answer is $x^2(5x^4) + xy(6y^5)$, or $5x^6 + 6xy^6$. Define $f_1 := 5x^6 + 6xy^6$ and let $\widetilde{f}_1 \in \Gamma(M \times \mathbb{R})$ be defined by $\widetilde{f}_1(p) = (p, f_1(p))$. We leave it as an unassigned exercise to show that $\mathcal{L}_X f_0 = f_1$ and that $\mathcal{L}_X\widetilde{f}_0 = \widetilde{f}_1$. Equivalently, one writes $Xf_0 = f_1$ and $X\widetilde{f}_0 = \widetilde{f}_1$.

Note: The shorthand of Xf for $\mathcal{L}_X f$ is only used when one is computing the Lie derivative of a section of $M \times \mathbb{R}$ along a vector field on a manifold M, or the Lie derivative of a smooth function along a vector field on M. If X and Y are vector fields on a manifold M, for example, XY is *not* a shorthand for $\mathcal{L}_X Y$. In this case, the accepted shorthand for $\mathcal{L}_X Y$ is $[X, Y]$, and the reason for this choice of shorthand notation will become clear later.

Definition. Let V be a finite dimensional vector space, let I be an open interval in $\mathbb{R}$ and let $t \mapsto v_t : I \to V$ be smooth. Assume that $0 \in I$. Let $\phi : T_{v_0}V \to V$ be the standard identification. Then we define $(D/dt)_{t=0}\, v_t := \phi((d/dt)_{t=0}\, v_t)$.

Let $\mathcal{F}$ be either a contravariant functor $\mathcal{VS} \to \mathcal{VS}$ or a covariant functor $\widetilde{\mathcal{VS}} \to \widetilde{\mathcal{VS}}$.

Definition. Let M and N be manifolds, and let ω be an $\mathcal{F}$-tensor field on N. Let U be an open subset of M and let V be an open subset of N. Let $\phi : U \to V$ be a diffeomorphism. Then we define $\phi^*\omega$ to be $\phi^*(\omega|V)$.

Definition. Let M be a manifold and let U be an open subset of $M \times \mathbb{R}$. For all $m \in M$, let $U_m := \{t \in \mathbb{R} \,|\, (m,t) \in U\}$. For all $t \in \mathbb{R}$, let $U^t := \{m \in M \,|\, (m,t) \in U\}$. Assume, for all $m \in M$, that U_m is a interval in $\mathbb{R}$ and that $0 \in U_m$. For all $t \in \mathbb{R}$, let ω^t be an $\mathcal{F}$-tensor field on U^t. Assume that $(m,t) \mapsto (\omega^t)_m : U \to \mathcal{F}'(TM)$ is smooth. Then $(D/dt)_{t=0}\,\omega^t$ is the $\mathcal{F}$-tensor field on M defined by $((D/dt)_{t=0}\,\omega^t)_m = (D/dt)_{t=0}\,((\omega^t)_m)$.

Now let X be a vector field on a manifold M with maximal flow $\phi : U \to M$. For all $t \in \mathbb{R}$, let $U^t := \{m \in M \,|\, (m,t) \in U\}$, and let $\phi^t : U^t \to M$ be defined by $\phi^t(m) = \phi(m,t)$. Then note that, with the definitions made above, we have $\mathcal{L}_X\omega = (D/dt)_{t=0}((\phi^t)^*\omega)$.

A small unassigned calculus exercise: Let V be a finite dimensional vector space and let $f : \mathbb{R}^2 \to V$ be smooth. Show that

$$(D/dt)_{t=0}(f(t,t)) \quad = \quad [(D/dt)_{t=0}(f(t,0))] \quad + \quad [(D/dt)_{t=0}(f(0,t))].$$

Now let $M := \mathbb{R}^2$ and let $X := x^2(\partial/\partial x) + xy(\partial/\partial y)$. Let $Y := x^3(\partial/\partial x)$. That is, $X = x^2E_1 + xyE_2 \in \mathrm{VF}(M)$ and $Y = x^3E_2 \in \mathrm{VF}(M)$.

We wish to compute $[X,Y]$, *i.e.*, to compute $\mathcal{L}_XY$.

Lemma. For all $f \in C^\infty(M)$, we have $\mathcal{L}_X(Yf) = (\mathcal{L}_XY)f + Y(\mathcal{L}_Xf)$.

Proof: For all $t \in \mathbb{R}$, we have $(\phi^t)^*(Yf) = ((\phi^t)^*Y)((\phi^t)^*f)$. Taking $(D/dt)_{t=0}$ of both sides, we get

$$\mathcal{L}_X(Yf) \quad = \quad (D/dt)_{t=0}[((\phi^t)^*Y)((\phi^0)^*f)] \quad + \quad (D/dt)_{t=0}[((\phi^0)^*Y)((\phi^t)^*f)].$$

Since $\phi^0 : M \to M$ is the identity, this reduces to

$$\begin{aligned} \mathcal{L}_X(Yf) \quad &= \quad (D/dt)_{t=0}[((\phi^t)^*Y)f] \quad + \quad (D/dt)_{t=0}[Y((\phi^t)^*f)] \\ &= \quad (\mathcal{L}_XY)f \quad + \quad Y(\mathcal{L}_Xf). \qquad \textbf{QED} \end{aligned}$$

Using alternate notation, the preceding lemma asserts that $XYf = [X,Y]f + YXf$, or $[X,Y]f = XYf - YXf$, which explains why $[X,Y]$ is used as an abbreviation for $\mathcal{L}_XY$.

Remark. Let N be a manifold and let $P, Q \in \mathrm{VF}(N)$. Then $P = Q$ iff, for all $f \in C^\infty(M)$, we have $Pf = Qf$.

We leave this last remark as an unassigned exercise.

We now compute $[X,Y]$, with X and Y on $M = \mathbb{R}^2$ as defined above: Let ∂_x be shorthand for $\partial/\partial x$. Let ∂_y be shorthand for $\partial/\partial y$. Let ∂_{xx} be shorthand for $\partial^2/\partial x^2$. Let ∂_{xy} be shorthand for $\partial^2/\partial x\,\partial y$. For any $f \in C^\infty(M)$, we have

$$XYf \quad = \quad (x^2\partial_x + xy\partial_y)(x^3\partial_x)f \quad = \quad (x^2\partial_x + xy\partial_y)(x^3\partial_xf),$$

so

$$XYf \quad = \quad x^2(3x^2)\,\partial_x f + x^5\,\partial_{xx} f + x^4 y\,\partial_{xy} f.$$

Similarly, for all $f \in C^\infty(M)$, we have

$$YXf \quad = \quad 2x^4\,\partial_x f + x^3 y\,\partial_y f + x^5\,\partial_{xx} f + x^4 y\,\partial_{xy} f.$$

So, by the preceding lemma, for all $f \in C^\infty(M)$, we have

$$[X,Y]f \quad = \quad XYf - YXf \quad = \quad x^4\,\partial_x f - x^3 y\,\partial_y f.$$

By the preceding remark, this shows that

$$[X,Y] \quad = \quad x^4\,\partial_x - x^3 y\,\partial_y.$$

EXERCISE 27A: In the notation established above, let $X := x\partial_y$ and $Y := \partial_x$ be two vector fields on $\mathbb{R}^2$. These two vector fields are complete. Let $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the flows of X and Y, respectively. As usual, for all $t \in \mathbb{R}$, define $\phi^t, \psi^t : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi^t(p) = \phi(p,t)$ and $\psi^t(p) = \psi(p,t)$.

(1) For all $t \in \mathbb{R}$ compute $\phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ and $\psi^t : \mathbb{R}^2 \to \mathbb{R}^2$.
(2) For all $t \in \mathbb{R}$ compute $(\phi^t)^* Y$ and $(\psi^t)^* X$.
(3) Let $q := (1,0) \in \mathbb{R}^2$. For all $t \in \{2, 1, 1/2\}$, show a picture of $((\phi^t)^* Y)_q$. For all $t \in \{2, 1, 1/2\}$, show a picture of $((\psi^t)^* X)_q$.
(4) Compute $[X,Y]$ and $[Y,X]$.
(5) Let $q := (1,0) \in \mathbb{R}^2$. Show a picture of $[X,Y]_q$. Show a picture of $[Y,X]_q$.

Note that, if M is a manifold and if $X \in \mathrm{VF}(M)$, then $f \mapsto Xf : C^\infty(M) \to C^\infty(M)$ is a **function-valued derivation** i.e., is an $\mathbb{R}$-linear function which satisfies: for all $f, g \in C^\infty(M)$, we have $X(fg) = (Xf)g + f(Xg)$.

Let M be a manifold and let $v \in TM$. Let $\epsilon > 0$ and let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth function such that $\dot\gamma(0) = v$. For any $f \in C^\infty(M)$ we define $vf \in \mathbb{R}$ by $vf = (D/dt)_{t=0}(f(\gamma(t)))$. We leave it as an exercise to show that this is well-defined, *i.e.*, that the value of vf does not depend on the choice of γ.

Note that, if M is a manifold, if $m \in M$ and if $v \in T_m M$, then $f \mapsto vf : C^\infty(M) \to \mathbb{R}$ is a **scalar-valued derivation** at m, i.e., is an $\mathbb{R}$-linear function which satisfies: for all $f, g \in C^\infty(M)$, we have $v(fg) = (vf)(g(m)) + (f(m))(vg)$.

Note that, if M is a manifold, if $X \in \mathrm{VF}(M)$ and if $f \in C^\infty(M)$, then, for all $m \in M$, we have $(Xf)(m) = X_m f$.

Lemma. Let M be a manifold and let $X, Y \in \mathrm{VF}(M)$. Let $\phi : U \to M$ and $\psi : V \to M$ be the maximal flows of X and Y, respectively. Let $m_0 \in M$. Choose $\epsilon > 0$ such that, for all $t \in [0, \epsilon)$, we have: $m_0 \in \mathrm{dom}(\psi^{-\sqrt{t}} \circ \phi^{-\sqrt{t}} \circ \psi^{\sqrt{t}} \circ \phi^{\sqrt{t}})$. Define $\alpha : [0, \epsilon) \to M$ by $\alpha(t) = (\psi^{-\sqrt{t}} \circ \phi^{-\sqrt{t}} \circ \psi^{\sqrt{t}} \circ \phi^{\sqrt{t}})(m_0)$. Let $f \in C^\infty(M)$. Then $(D/dt)_{t=0+}(f(\alpha(t))) = [X,Y]_{m_0} f$.

Proof: Let I be an open interval in $\mathbb{R}$ such that $0 \in I$ and such that, for all $s, t, u, v \in I$, we have $m_0 \in \mathrm{dom}(\psi^{-v} \circ \phi^{-u} \circ \psi^t \circ \phi^s)$. Define $\beta : I^4 \to M$ by

$$\beta(s,t,u,v) = (\psi^{-v} \circ \phi^{-u} \circ \psi^t \circ \phi^s)(m_0).$$

Define $g : I^4 \to \mathbb{R}$ by $g(s,t,u,v) = f(\beta(s,t,u,v))$. Replacing ϵ by a smaller positive number, if necessary, we may assume that, for all $t \in [0,\epsilon)$, that $\sqrt{t} \in I$. Then, for all $t \in [0,\epsilon)$, we have $\alpha(t) = \beta(\sqrt{t},\sqrt{t},\sqrt{t},\sqrt{t})$, which implies that $f(\alpha(t)) = g(\sqrt{t},\sqrt{t},\sqrt{t},\sqrt{t})$.

We wish to show that, for $t \in [0,\epsilon)$, we have:

$$f(\alpha(t)) \quad = \quad (f(m_0)) \quad + \quad ([X,Y]_{m_0} f)t \quad + \quad o(t).$$

It suffices to show that, for $t \in I$, we have:

$$g(t,t,t,t) \quad = \quad (f(m_0)) \quad + \quad ([X,Y]_{m_0} f)t^2 \quad + \quad o(t^2).$$

Since $g(0,0,0,0) = f(m_0)$, by Taylor's Theorem, it suffices to show both that

$$(D/dt)_{t=0}(g(t,t,t,t)) = 0 \qquad \text{and that} \qquad (D^2/dt^2)_{t=0}(g(t,t,t,t)) = 2[X,Y]_{m_0} f.$$

For all integers $i \in [1,4]$, let $\partial_i := \partial/\partial x_i$. By the chain rule,

$$(D/dt)(g(t,t,t,t)) = \sum_{i=1}^{4} (\partial_i g)(t,t,t,t).$$

Similarly, for all integers $i \in [1,4]$,

$$(D/dt)((\partial_i g)(t,t,t,t)) = \sum_{j=1}^{4} (\partial_j \partial_i g)(t,t,t,t).$$

Then

$$(D^2/dt^2)(g(t,t,t,t)) = \frac{D}{dt} \sum_{i=1}^{4} (\partial_i g)(t,t,t,t) = \sum_{i=1}^{4} \sum_{j=1}^{4} (\partial_j \partial_i g)(t,t,t,t).$$

Thus, we have

$$(D/dt)_{t=0}(g(t,t,t,t)) = \sum_{i=1}^{4} (\partial_i g)(0,0,0,0)$$

and

$$\begin{aligned}(D^2/dt^2)_{t=0}(g(t,t,t,t)) &= \sum_{i=1}^{4} \sum_{j=1}^{4} (\partial_j \partial_i g)(0,0,0,0) \\ &= \left[\sum_{i=1}^{4} (\partial_i^2 g)(0,0,0,0) \right] + 2 \left[\sum_{1 \le i < j \le 4} (\partial_i \partial_j g)(0,0,0,0) \right].\end{aligned}$$

For all $s,t,u,v \in I$, we have

$$\begin{aligned}&(\partial_1 g)(s,0,0,0) = (Xf)(\beta(s,0,0,0)), \\ &(\partial_2 g)(s,t,0,0) = (Yf)(\beta(s,t,0,0)), \\ &(\partial_3 g)(s,t,u,0) = (-Xf)(\beta(s,t,u,0)) \quad \text{and} \\ &(\partial_4 g)(s,t,u,v) = (-Yf)(\beta(s,t,u,v)).\end{aligned}$$

Then, for all $s, t, u, v \in I$, we have

$$\begin{aligned}
(\partial_1^2 g)(s,0,0,0) &= (XXf)(\beta(s,0,0,0)),\\
(\partial_1\partial_2 g)(s,0,0,0) &= (XYf)(\beta(s,0,0,0)),\\
(\partial_2^2 g)(s,t,0,0) &= (YYf)(\beta(s,t,0,0)),\\
(\partial_1\partial_3 g)(s,0,0,0) &= (-XXf)(\beta(s,0,0,0)),\\
(\partial_2\partial_3 g)(s,t,0,0) &= (-YXf)(\beta(s,t,0,0)),\\
(\partial_3^2 g)(s,t,u,0) &= (XXf)(\beta(s,t,u,0)),\\
(\partial_1\partial_4 g)(s,0,0,0) &= (-XYf)(\beta(s,0,0,0)),\\
(\partial_2\partial_4 g)(s,t,0,0) &= (-YYf)(\beta(s,t,0,0)),\\
(\partial_3\partial_4 g)(s,t,u,0) &= (XYf)(\beta(s,t,u,0)),\\
(\partial_4^2 g)(s,t,u,v) &= (YYf)(\beta(s,t,u,v)).
\end{aligned}$$

Then

$$(D/dt)_{t=0}(g(t,t,t,t)) = \sum_{i=1}^{4} (\partial_i g)(0,0,0,0) = ((X+Y-X-Y)f)(\beta(0,0,0,0)) = 0.$$

Moreover,

$$(D/dt)_{t=0}(g(t,t,t,t)) = \sum_{i=1}^{4} (\partial_i g)(0,0,0,0)$$

and

$$\begin{aligned}
(D^2/dt^2)_{t=0}(g(t,t,t,t)) &= \left[\sum_{i=1}^{4} (\partial_i^2 g)(0,0,0,0)\right] + 2\left[\sum_{1\le i<j\le 4} (\partial_i\partial_j g)(0,0,0,0)\right]\\
&= ((XX + 2XY + YY - 2XX - 2YX + XX\\
&\qquad - 2XY - 2YY + 2XY + YY)f)(\beta(0,0,0,0)).
\end{aligned}$$

That is, $(D/dt)_{t=0}(g(t,t,t,t)) = ((2XY - 2YX)f)(m_0) = 2[X,Y]_{m_0} f$. **QED**

Definition. Let M be a manifold and let $X, Y \in \mathrm{VF}(M)$. We say that X **commutes** with Y if $[X,Y] = 0$.

EXERCISE 27B: Let M be a manifold and let $X, Y \in \mathrm{VF}(M)$. Let $\phi : U \to M$ and $\psi : V \to M$ be the maximal flows of X and Y, respectively. Show that X commutes with Y iff: for all $m \in M$, there exists an $\epsilon > 0$ such that, for all $s, t \in (-\epsilon, \epsilon)$, we have:

(1) $m \in \mathrm{dom}(\psi^t \circ \phi^s)$;
(2) $m \in \mathrm{dom}(\phi^s \circ \psi^t)$; and
(3) $(\psi^t \circ \phi^s)(m) = (\phi^s \circ \psi^t)(m)$.

For all integers $n \ge 0$, let $e_1^n, \ldots, e_n^n$ be the standard basis of $\mathbb{R}^n$, let $e_0^n := 0 \in \mathbb{R}^n$ and let Δ^n denote the closed convex hull in $\mathbb{R}^n$ of $e_0^n, \ldots, e_n^n$; the set Δ^n will be called the **standard n-simplex**. A parametric n-simplex in a

Definition. Let $n \geq 0$ be an integer and let M be a manifold. A **parametric n-simplex in** M is a function $\Delta^n \to M$ which has a smooth extension in $\mathbb{R}^n$ to M. Let $A := C^\infty_{\mathbb{R}^n}(\Delta^n, M)$ be the set of all parametric n-simplices in M. Let $S_n M := \mathbb{R}[A]$ be the vector space of all formal $\mathbb{R}$-linear combinations of parametric n-simplices in M. An element of $S^n M$ is called an **n-chain** in M.

Definition. Let $n \geq 0$ be an integer and let M be a manifold. The set of differential n-forms on M is denoted $\Omega^n M := \Gamma[\wedge^n (TM)^*]$. Let $\omega \in \Omega^n M$. The map $f \mapsto \int_f \omega : A \to \mathbb{R}$ extends by linearity to a map $S_n M \to \mathbb{R}$, and the image of $\sigma \in S_n M$ under this map is denoted $\int_\sigma \omega$.

If V is a vector space, if $n \geq 1$ is an integer and if $v_1, \ldots, v_n \in V$, then $v_1 \wedge \cdots \wedge v_n$ denotes the image of $v_1 \otimes \cdots \otimes v_n \in \otimes^n V$ under the canonical map $\otimes^n V \to \wedge^n V$.

Definition. Let V be a vector space and let $n \geq 1$ be an integer. Let $\omega \in \wedge^n V^* = (\wedge^n V)^*$. Let $p := (v_1, \ldots, v_n) \in V^n$ be an n-parallelpiped. Then $\langle \omega, p \rangle$ denotes $\omega(v_1 \wedge \cdots \wedge v_n)$.

When $n = 0$, we have $\wedge^n V = \mathbb{R}$, and we will define $V^0 := \mathbb{R}$. (That is, the set of 0-parallelpipeds in V is the set of real numbers.) For $\omega \in \wedge^0 V$ and $p \in V^0$, we define $\langle \omega, p \rangle = \omega \cdot p$, where $\cdot$ denotes ordinary multiplication of real numbers.

Recall that an n-form is to be thought of as measuring the size of n-parallelpipeds in tangent spaces. A parametric n-simplex gives rise to a field of n-paralellpipeds, paramaterized by the simplex. Evaluating the size of each and then integrating against Lebesgue measure on the simplex, gives a real number. The preceding definition simply extends this by linearity from parametric n-simplices to n-chains.

Let $h : \mathbb{R} \to \mathbb{R}$ be smooth. Let $V := \partial/\partial x \in \mathrm{VF}(\mathbb{R})$. Let $\eta \in \Omega^1 \mathbb{R}$ be defined by: for all $q \in \mathbb{R}$, we have: $\langle \eta_q, V_q \rangle = h(q)$. This 1-form η is often denoted $h\,dx$. Note that $\Delta^1 = [0, 1]$. Let $f : \Delta^1 \to \mathbb{R}$ have a smooth extension in $\mathbb{R}$ to $\mathbb{R}$. Note, for all $p \in \mathbb{R}$, that $f_*(V_p) = (f'(p))(V_{f(p)})$. Let H be an antiderivative of h. Then

$$\begin{aligned} \int_f \eta &= \int_0^1 \langle \eta_{f(p)}, (f'(p))(V_{f(p)}) \rangle \, dp \\ &= \int_0^1 [h(f(p))][f'(p)] \, dp = H(f(1)) - H(f(0)). \end{aligned}$$

Definition. Let $n \geq 1$ be an integer. Given $v_1, \ldots, v_n \in \Delta^n$, if $f : \Delta^{n-1} \to \Delta^n$ is the unique affine map satisfying

$$f(e_0^{n-1}) = v_1, \qquad \ldots, \qquad f(e_{n-1}^{n-1}) = v_n,$$

then we denote f by $[v_1, \ldots, v_n]$.

Definition. Let $n \geq 1$ be an integer and let M be a manifold. Let $\sigma : \Delta^n \to M$ be a parametric n-simplex in M. Let

$$k_0 := [e_1^n, \ldots, e_n^n], \quad k_1 := [e_0^n, e_2^n, \ldots, e_n^n], \quad \ldots, \quad k_n := [e_0^n, \ldots, e_{n-1}^n].$$

Then we define $\partial\sigma := \sum_{j=0}^{n}(-1)^j(\sigma\circ k_j) \in S_{n-1}M$.

Let $n \geq 1$ be an integer and let M be a manifold. Let $A := C^\infty_{\mathbb{R}^n}(\Delta^n, M)$ be the set of all parametric n-simplices in M. Recall that $S_nM = \mathbb{R}[A]$. Then we extend the map $\partial : A \to S_{n-1}M$ by $\mathbb{R}$-linearity to $\partial : S_nM \to S_{n-1}M$.

Next, we state the Fundamental Theorem of Calculus in a fancy way:

Theorem. For any $\omega \in \Omega^0\mathbb{R}$, there exists $\eta \in \Omega^1\mathbb{R}$ such that, for all $\sigma \in S_1\mathbb{R}$, we have

$$\int_{\partial\sigma} \omega = \int_\sigma \eta.$$

Proof: Let $M := \mathbb{R}$. We have $\wedge^0(TM)^* = M \times \mathbb{R}$, so $\Omega^0 M = \Gamma(M\times\mathbb{R})$. For all $F \in C^\infty(M)$, let $\widetilde{F} \in \Gamma(M\times\mathbb{R})$ be defined by $\widetilde{F}(p) = (p, F(p))$. Then

$$F \quad\mapsto\quad \widetilde{F} \quad:\quad C^\infty(M) \quad\to\quad \Gamma(M\times\mathbb{R})$$

is a bijection. Choose $H \in C^\infty(M)$ such that $\widetilde{H} = \omega$. Let $h := H'$ be the ordinary calculus derivative of H. Define $\eta \in \Omega^1\mathbb{R}$ by, for all $q \in \mathbb{R}$, we have: $\langle \eta_q, V_q\rangle = h(q)$. Let $\sigma \in S_1\mathbb{R}$. We wish to show that $\int_{\partial\sigma}\widetilde{H} = \int_\sigma \eta$.

We may assume that σ is a parametric 1-simplex in M. That is $\sigma : [0,1] \to M$ has a smooth extension in $\mathbb{R}$ to M. Let $V := \partial/\partial x \in \mathrm{VF}(M)$. As calculated before,

$$\begin{aligned}\int_\sigma \eta &= \int_0^1 \langle \eta_{\sigma(p)}, (\sigma'(p))(V_{\sigma(p)})\rangle\, dp\\ &= \int_0^1 [h(\sigma(p))][\sigma'(p)]\, dp = H(\sigma(1)) - H(\sigma(0)).\end{aligned}$$

We leave it as an unassigned exercise to trace through definitions and show that this is equal to $\int_{\partial\sigma}\widetilde{H}$. **QED**

We can now generalize the Fundamental Theorem of Calculus to arbitrary manifolds (not just $\mathbb{R}$) and to arbitrary forms (not just 0-forms):

Stokes' Theorem. Let M be a manifold and let $n \geq 0$ be an integer. Then there is a unique R-linear map

$$d : \Omega^n M \to \Omega^{n+1}M$$

such that, for all $\omega \in \Omega^n M$, for all $\sigma \in S_{n+1}M$, we have

$$\int_{\partial\sigma}\omega = \int_\sigma d\omega.$$

Let's assume that Stokes' Theorem is true and set ourselves to the task of figuring out how to compute this map d. To this end it will help to have parametric boxes to work with, because their geometry is perhaps a bit more intuitive.

Let $n \geq 0$ be an integer. A **box** in $\mathbb{R}^n$ is a product of n intervals.

Definition. Let M be a manifold and let $n \geq 0$ be an integer. A **parametric n-box in M** consists of

(1) a compact box $B \subseteq \mathbb{R}^n$; and
(2) a map $B \to M$ which has a smooth extension in $\mathbb{R}^n$ to M.

Let $n \geq 0$ be an integer. Let $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ and assume, for all integers $i \in [1, n]$, that $a_i < b_i$. Let $B := [a_1, b_1] \times \cdots \times [a_n, b_n]$. For all integers $i \in [1, n]$, let

$$B_i := [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n].$$

Let M be a manifold. Let $\tau : B \to M$ be a parametric n-box. For any integer $i \in [1, n]$, we define $\partial_i^- \tau, \partial_i^+ \tau : B_i \to M$ by

$$(\partial_i^- \tau)(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) = \tau(s_1, \ldots, s_{i-1}, a_i, s_{i+1}, \ldots, s_n)$$
$$(\partial_i^+ \tau)(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) = \tau(s_1, \ldots, s_{i-1}, b_i, s_{i+1}, \ldots, s_n).$$

Definition. Let M be a manifold and let $n \geq 0$ be an integer. Let A denote the set of all parametric n-boxes in M. We define $S_n^{\text{bx}} M := \mathbb{R}[A]$ to be formal finite $\mathbb{R}$-linear combinations of elements of A. Elements of $S_n^{\text{bx}} M$ will be called **n-box chains** in M.

Let M be a manifold and let $n \geq 0$ be an integer. Let A denote the set of all parametric n-boxes in M. For any $\tau \in A$, we define

$$\partial \tau := \sum_{j=1}^{n} (-1)^{j-1} (\partial_j^+ \tau - \partial_j^- \tau) \quad \in \quad S_{n-1}^{\text{bx}} M.$$

We extend the map $\tau \mapsto \partial \tau : A \to S_{n-1}^{\text{bx}} M$ by $\mathbb{R}$-linearity to a map $\tau \mapsto \partial \tau : S_n^{\text{bx}} M \to S_{n-1}^{\text{bx}} M$

Any rectangle can be cut along the diagonal to make two triangles. The overlap is just the diagonal, which has measure zero. More generally, this can be extened to parametric boxes: For any parametric n-box, there is a sum of parametric n-simplices that covers the parametric box in a nonoverlapping way.

This can then be extended by $\mathbb{R}$-linearity to n-box chains: For any n-box chain β, there is an n-chain σ_β that covers the box chain in a nonoverlapping way. This association $\beta \mapsto \sigma_\beta$ can be carried out for each n in such a way that we have the formula: $\partial \sigma_\beta = \sigma_{\partial \beta}$

This intuition can be developed to yield the following rigorous statement:

Lemma. Let M be a manifold and let β be an n-box chain on M. Then there is an n-chain σ on M such that, for every $\omega \in \Omega^n(M)$ and every $\omega' \in \Omega^{n-1}(M)$, we have

$$\int_\beta \omega = \int_\sigma \omega \qquad \text{and} \qquad \int_{\partial \beta} \omega' = \int_{\partial \sigma} \omega'.$$

This lemma, combined with Stokes' Theorem gives us:

Box chain form of Stokes' Theorem. Let M be a manifold and let $n \geq 0$ be an integer. Then, for all $\omega \in \Omega^n M$, for all $\beta \in S^{\rm bx}_{n+1}M$, we have

$$\int_{\partial\beta} \omega = \int_\beta d\omega.$$

TIMEOUT TO DISCUSS MATERIAL RELEVANT TO PRELIMS:

Let M be a manifold and let E and F be vector bundles over M, with bundle maps $p : E \to M$ and $q : F \to M$. For all $m \in M$, let E_m be the vector space $p^{-1}(m)$, and let F_m be the vector space $q^{-1}(m)$. Let

$$E \times_M F := \{(e, f) \in E \times F \,|\, p(e) = q(f)\}.$$

This is a closed submanifold of $E \times F$, and we give $E \times_M F$ the inherited sumanifold structure (*i.e.*, the unique manifold structure such that the inclusion $E \times_M F \to E \times F$ is an immersion). Define $\pi : E \times_M F \to M$ by $\pi(e, f) = p(e) = q(f)$. For all $m \in M$, we have $\pi^{-1}(m) = E_m \times F_m$, and we give $\pi^{-1}(m)$ the product vector space structure. Then $E \times_M F$, together

(1) with $\pi : E \times_M F \to M$;
(2) with the manifold structure on $E \times_M F$; and
(3) with the vector space structures on the fibers of π

form a vector bundle over M. This vector bundle is denoted $E \times_M F$.

Let V be a vector space and let $k, l \geq 0$ be integers. Let $\Phi : (\wedge^k V) \times (\wedge^l V) \to \wedge^{k+l} V$ be the map defined by

$$\Phi(\, v_1 \wedge \cdots \wedge v_k \,,\, v_{k+1} \wedge \cdots \wedge v_{k+l} \,) \qquad = \qquad v_1 \wedge \cdots \wedge v_{k+l}.$$

Then, for all $\alpha \in \wedge^k V$, for all $\beta \in \wedge^l V$, we define $\alpha \wedge \beta := \Phi(\alpha, \beta)$.

Let E be a vector bundle over a manifold M and let $k, l \geq 0$ be integers. Then $(\alpha, \beta) \mapsto \alpha \wedge \beta : (\wedge^k E) \times_M (\wedge^l E) \to \wedge^{k+l} E$ is a morphism in the category of vector bundles. For all $\sigma \in \Gamma(\wedge^k E)$ and $\tau \in \Gamma(\wedge^l E)$, we define $\sigma \wedge \tau \in \Gamma(\wedge^{k+l} E)$ by $(\sigma \wedge \tau)_m = \sigma_m \wedge \tau_m$. Thus $(\sigma, \tau) \mapsto \sigma \wedge \tau$ is a map

$$\Gamma(\wedge^k E) \qquad \times \qquad \Gamma(\wedge^l E) \qquad \to \qquad \Gamma(\wedge^{k+l} E).$$

In the special case where $E = (TM)^*$, we get a map

$$(\sigma, \tau) \qquad \mapsto \qquad \sigma \wedge \tau \qquad : \qquad \Omega^k M \quad \times \quad \Omega^l M \qquad \to \qquad \Omega^{k+l} M.$$

Let E be a vector bundle over a manifold M. For any $\sigma \in \Gamma(E)$, for any $f \in C^\infty(M)$, define $f\sigma \in \Gamma(E)$ by $(f\sigma)_m = f(m) \cdot \sigma_m$, where $\cdot$ denotes scalar multiplication in the vector space E_m. Thus $\Gamma(E)$ is a module over the ring $C^\infty(M)$. In the special case where, for some integer $k \geq 0$, $E = \wedge^k(TM)^*$, we see that $\Omega^k(M)$ is a module over $C^\infty(M)$.

Let E be a vector bundle over a manifold M. Given $\sigma \in \Gamma(E)$ and $\tau \in \Gamma(E^*)$, we define $\tau(\sigma) \in C^\infty(M)$ by $(\tau(\sigma))(m) = \tau_m(\sigma_m)$.

Now fix an integer $n \geq 0$. Let $\partial/\partial x_1, \ldots, \partial/\partial x_n$ be the standard framing of $\mathbb{R}^n$. Let $\mathbf{0}, \mathbf{1} \in C^\infty(M)$ be defined by $\mathbf{0}(m) = 0$ and $\mathbf{1}(m) = 1$. Let $dx_1, \ldots, dx_n \in \Gamma((TM)^*) = \Omega^1 M$ be defined by, for all integers $i, j \in [1, n]$,

$$dx_i(\partial/\partial x_j) = \begin{cases} \mathbf{0}, & \text{if } i \neq j, \\ \mathbf{1}, & \text{if } i = j. \end{cases}$$

The forms $dx_1, \ldots, dx_n$ are called the **standard constant 1-forms** on $\mathbb{R}^n$.

Fact. Let $k, n \geq 0$ be integers. Let

$$I := \{(i_1, \ldots, i_k) \in \mathbb{Z}^k \,|\, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

For all $i = (i_1, \ldots, i_k) \in I$, let $dx_i := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then the module $\Omega^k \mathbb{R}^n$ over $C^\infty(\mathbb{R}^n)$ is free with basis $\{dx_i \,|\, i \in I\}$.

In other words, for all $\omega \in \Omega^k \mathbb{R}^n$, there exists a unique $(f_i)_{i \in I} \in (C^\infty(\mathbb{R}^n))^I$ such that

$$\omega \quad = \quad \sum_{i \in I} f_i \, dx_i.$$

Thus, to define the map $d : \Omega^k \mathbb{R}^n \to \Omega^{k+1} \mathbb{R}^n$ of Stokes' Theorem, it suffices, for all $f \in C^\infty(M)$, for all $i \in I$, to compute $d(f\, dx_i)$.

For all $f \in C^\infty(M)$, we define

$$df \quad := \quad \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \quad \in \quad \Omega^1(M).$$

Fact. For all $f \in C^\infty(M)$, for all $i \in I$, we have $d(f\, dx_i) = df \wedge dx_i$.

For example, let dx, dy, dz denote the standard constant 1-forms on $\mathbb{R}^3$. Keep in mind that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. We leave it as an exercise for the interested reader to see that

$$\begin{aligned} d(x^2 y\, dx \wedge dy + (\cos(xyz))\, dx \wedge dz + y\, dy \wedge dz) &= -[(\sin(xyz))(xz)]\, dy \wedge dx \wedge dz \\ &= [xz(\sin(xyz))]\, dx \wedge dy \wedge dz. \end{aligned}$$

Let M be a manifold. For all integers $k < 0$, we define $\Omega^k M = 0$.

Fact. Let M be a manifold and let $k \geq 0$ be an integer. Let $\omega \in \Omega^k M$. Let $\mathbf{0} \in \Omega^{k+2} M$ denote the zero section of $\wedge^{k+2}((TM)^*)$. That is, let $\mathbf{0}$ be the zero element of the vector space $\Omega^{k+2} M$. Then $d(d\omega) = \mathbf{0}$.

A **graded vector space** is a bi-infinite sequence of vector spaces. The category of graded vector spaces is denoted $\{\text{vector spaces}\}^{\mathbb{Z}}$.

The preceding fact asserts that $\Omega^\bullet M$ is a cochain complex. The de Rham Theorem (below) implies, for all manifolds M, that $H^\bullet(M;R)$ and $H^\bullet(\Omega^\bullet M)$ are ("naturally") isomorphic graded vector spaces.

The de Rham Theorem. The functors

$$M \quad \mapsto \quad H^\bullet(M;R) \quad : \quad \{\text{manifolds}\} \quad \to \quad \{\text{vector spaces}\}^{\mathbb{Z}}$$

and

$$M \quad \mapsto \quad H^\bullet(\Omega^\bullet M) \quad : \quad \{\text{manifolds}\} \quad \to \quad \{\text{vector spaces}\}^{\mathbb{Z}}$$

are equivalent.

Let M be a manifold, let $k \geq 0$ be an integer and let $\omega \in \Omega^k M$. We say that ω is **closed** if $d\omega = 0$. We say that ω is **exact** if $\omega \in d(\Omega^{k-1}M)$. Then

$$H^k(\Omega^\bullet M) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}.}$$

Because $d^2 = 0$, we see that any exact form is closed. Because of the de Rham Theorem, we see, for any contractible manifold, that any closed form is exact. Thus, the Poincaré Lemma (below) is a special case of the de Rham Theorem, but is much easier to prove and is typically proved *on the way to proving* the de Rham Theorem.

Poincaré Lemma. Let $k, n \geq 0$ be integers. Let U be the open unit ball in $\mathbb{R}^n$. Then every closed k-form on U is exact.

That is, for any $\omega \in \Omega^k U$, there is a solution η of the differential equation $d\eta = \omega$ iff we have $d\omega = 0$. Here's an application of this fact:

Corollary. Let U denote the unit ball in $\mathbb{R}^3$. Let $V \in \mathrm{VF}(U)$. Then there is a solution f to $\nabla f = V$ iff $\nabla \times V = 0$.

Proof: Let $\partial/\partial x, \partial/\partial y, \partial/\partial z$ be the restriction to U of the standard framing of $\mathbb{R}^3$. Let dx, dy, dz be the restriction to U of the standard constant 1-forms on $\mathbb{R}^3$.

Choose $f, g, h \in C^\infty(U)$ such that $V = f(\partial/\partial x) + g(\partial/\partial y) + h(\partial/\partial z)$. Let $\omega := f\,dx + g\,dy + h\,dz$. We leave it as an unassigned exercise to show that there is a solution f to $\nabla f = V$ iff there is a solution η to $d\eta = \omega$. Moreover, we leave it as an unassigned exercise to show that $\nabla \times V = 0$ iff $d\omega = 0$. Thus the corollary follows directly from the Poincaré Lemma. **QED**

Definition. Let $d \geq 0$ be an integer and let $p \in [0, d]$ be an integer. Let $q := d - p$. Let M be a d-dimensional manifold and let $L \subseteq M$. Then L is a **trivial p-dimensional leaflike submanifold** of M if there is a diffeomorphism $\Phi : M \to \mathbb{R}^p \times \mathbb{R}^q$ and there is a countable subset $C \subseteq \mathbb{R}^q$ such that $\Phi(L) = \mathbb{R}^p \times C$.

Definition. Let $d \geq 0$ be an integer and let $p \in [0, d]$ be an integer. Let $q := d - p$. Let M be a d-dimensional manifold and let $L \subseteq M$. Then L is a **p-dimesional leaflike submanifold** of M if, for all $l \in L$, there exists a neighborhood M_0 of l in M such that $L \cap M_0$ is a trivial p-dimensional leaflike submanifold of M_0.

A subset L of a manifold M is a **leaflike submanifold** if there exists an integer $p \in [0, \dim(M)]$ such that L is a p-dimensional leaflike submanifold.

Fact. Let L be a leaflike submanifold of a manifold M. Then there is a unique maximal atlas on L with respect to which the inclusion map $L \to M$ is an immersion.

This maximal atlas on L is called the **inherited manifold structure** on L. Its topology is called the **leaflike topology** on L. WARNING: The leaflike topology on L may not be the topology on L inherited from M.

If R and S are equivalence relations on the sets A and B, respectively, then the equivalence relation $R \times S$ on $A \times B$ is defined by:

$$((a,b),(a',b')) \in R \times S \qquad \text{iff} \qquad (a,a') \in R \text{ and } (b,b') \in S.$$

Let A be a set. The **transitive equivalence relation** T_A on A is $T_A = A \times A$. Under this equivalence relation all points of A are equivalent to all other points of A. An equivalence relation R on A is said to be **countable** if every equivalence class of R is countable.

If R is an equivalence relation on a set S if T is a set and if $f : S \to T$ is a bijection, then $f_*(R)$ is the equivalence relation on T defined by: $(t, t') \in f_*(R)$ iff $(f^{-1}(t), f^{-1}(t')) \in R$.

If R is an equvalence relation on a set S and if $S_0 \subseteq S$, then $R|S_0$ denotes the equivalence relation on S_0 defined by: $R|S_0 := R \cap (S_0 \times S_0)$.

Definition. Let $\pi : E \to M$ be a vector bundle. For all $m \in M$, let E_m denote the vector space $\pi^{-1}(m)$. Let $S \subseteq E$. Then we say that S is a **vector subbundle** of E if:

(1) S is a closed submanifold of E;
(2) for all $m \in M$, $S_m := S \cap E_m$ is a vector subspace of E_m; and
(3) with the manifold structure on S inherited from E and with the vector space structures on the fibers $\{S_m \mid m \in M\}$, the map $\pi|S : S \to M$ becomes a vector bundle.

Definition. Let M be a manifold. A **distribution** on M is a subbundle of TM.

Intuitively, on chooses, from each tangent space of M, a subspace. Moreover, all of these subspaces of the same dimension, and the subspaces must vary smoothly as one moves from point to point in the manifold M.

Recall that the rank of a vector bundle is the dimension of its fibers, so the **rank** of a distribution is the dimension of the fibers, *i.e.*, of the various subspaces.

If E is a vector bundle on a manifold M, then, for all $m \in M$, the fiber of E over m is typically denoted E_m. Thus, if Δ is a distribution on a manifold M, then, for all $m \in M$, we have $\Delta_m = \Delta \cap (T_m M)$.

We now begin the topic of integration of distributions. Fix integers $d, p, q \geq 0$ such that $d = p + q$. Let M be a d-dimensional manifold and let $\mathcal{F} \subseteq M \times M$ be an equivalence relation on M.

Definition. We say that $\mathcal{F}$ is a **trivial p-dimensional prefoliation** on M if there exists a diffeomorphism $\Phi : M \to \mathbb{R}^p \times \mathbb{R}^q$ and there exists a countable equivalence relation R on $\mathbb{R}^q$ such that $\Phi_*(\mathcal{F}) = T_{\mathbb{R}^p} \times R$. (Recall that $T_{\mathbb{R}^p}$ is the transitive equivalence relation on $\mathbb{R}^p$.)

Definition. We say that $\mathcal{F}$ is a **p-dimensional prefoliation** on M if, for all $m \in M$, there exists an open neighborhood M_0 of m in M such that $\mathcal{F}|M_0$ is a trivial p-dimensional prefoliation on M_0. In this case, the equivalence classes of $\mathcal{F}$ are called **leaves** of $\mathcal{F}$.

Note that, in this case, the leaves of $\mathcal{F}$ are leaflike submanifolds, and so have a leaflike topology.

Definition. We say that $\mathcal{F}$ is a **p-dimensional foliation** on M if $\mathcal{F}$ is p-dimensional prefoliation on M and if every leaf of $\mathcal{F}$ is connected in the leaflike topology.

Now let M be any manifold. An equivalence relation $\mathcal{F}$ on M is said to be a **prefoliation** on M if there exists an integer $p \in [0, \dim(M)]$ such that $\mathcal{F}$ is a p-dimensional prefoliation on M. An equivalence relation $\mathcal{F}$ on M is said to be a **foliation** on M if there exists an integer $p \in [0, \dim(M)]$ such that $\mathcal{F}$ is a p-dimensional foliation on M.

Definition. Let M be a manifold and let $\mathcal{F}$ be a prefoliation on M. Let $\mathcal{L}$ be the set of leaves of $\mathcal{F}$. Then $T\mathcal{F} := \cup\{TL \mid L \in \mathcal{L}\} \subseteq TM$.

Remark. Let M be a manifold and let $\mathcal{F}$ be a prefoliation on M. Then $T\mathcal{F}$ is a distribution on M, *i.e.*, is a vector subbundle of TM.

Fact. Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations on a manifold M and assume that $T\mathcal{F} = T\mathcal{F}'$. Then $\mathcal{F} = \mathcal{F}'$.

The preceding Fact is not true if "foliation" is replaced by "prefoliation".

Definition. Let Δ be a distribution on a manifold M and let L be a leaflike submanifold of M. We say that L is an **integral submanifold** of Δ if L is connected (in the leaflike topology) and if, for all $l \in L$, we have $T_l L = \Delta_l$. Let $\mathcal{L}$ be the set of integral submanifolds of M. We say that L is a **maximal integral submanifold** of Δ if both $L \in \mathcal{L}$ and, for all $L' \in \mathcal{L}$, we have: $L \subseteq L' \implies L = L'$.

Fact. Let M be a manifold and let Δ be a distribution on M. Then, for every integral submanifold $L_0 \in \mathcal{L}$, there is a maximal integral submanifold L of Δ such that L_0 is an open subset of L (in the leaflike topology). Moreover, for all $m \in M$, there is at most one maximal integral submanifold L of Δ such that $m \in L$.

Remark. Let M be a manifold, let $\mathcal{F}$ be a foliation on M and let $\Delta = T\mathcal{F}$. Then the set of maximal integral submanifolds of Δ is exactly the set of leaves of $\mathcal{F}$.

Definition. A distribution Δ on a manifold M is said to be **integrable** if there is a foliation

$\mathcal{F}$ on M such that $T\mathcal{F} = \Delta$.

Question: Are all distributions integrable?

Definition. Let Δ be a distribution on a manifold M. For $V \in \mathrm{VF}(M)$, we say that V is **in** Δ if, for all $m \in M$, we have $V_m \in \Delta_m$. We say that Δ is **involutive** if, for all V, W in Δ, we have that $[V, W]$ is in Δ.

The Frobenius Theorem. Let Δ be a distribution on a manifold M. Then Δ is integrable iff Δ is involutive.

It helps to have the following fairly easy test for involutivity:

Fact. Let M be a manifold and let $k \geq 1$ be an integer. Let $V^1, \ldots, V^k \in \mathrm{VF}(M)$. For all $m \in M$, let $\Delta_m \subseteq T_m M$ be the span of $V^1_m, \ldots, V^k_m$. Assume, for all $m, m' \in M$, that $\dim(\Delta_m) = \dim(\Delta_{m'})$. Then Δ is a distribution on M of rank $\leq k$. Moreover, Δ is involutive iff, for all integers $i, j \in [1, k]$, we have that $[V^i, V^j]$ is in Δ.

Let M be a manifold, let $k \geq 1$ be an integer and let $V^1, \ldots, V^k \in \mathrm{VF}(M)$. Let Δ be a distribution on M. If, for all $m \in M$, $\Delta_m \subseteq T_m M$ is the span of $V^1_m, \ldots, V^k_m$, then we say simply that Δ is **spanned** by $V^1, \ldots, V^k$.

We can now answer our question: Let $M := \mathbb{R}^3$. Let $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ be defined by $x(s, t, u) = s$, by $y(s, t, u) = t$ and by $z(s, t, u) = u$. Let $\partial_x, \partial_y, \partial_z$ be the standard framing of $\mathbb{R}^3$. Let Δ be the distribution spanned by $V := \partial_x$ and $W := (x^2 + 1)\partial_y + (x^2 + 2)\partial_z$. Then V is in Δ, W is in Δ and $[V, W] = 2x\partial_y + 2x\partial_z$. Let $p := (1, 0, 0)$ and let $\iota : T_p\mathbb{R}^3 \to \mathbb{R}^3$ be the standard identification. Then $\iota(V_p) = (1, 0, 0)$, $\iota(W_p) = (0, 2, 3)$ and $\iota([V, W]_p) = (0, 2, 2)$. Thus $[V, W]_p$ is not in the span of V_p and W_p, so $[V, W]$ is not in Δ. Then Δ is not involutive, so, by the Frobenius Theorem, we see that Δ is not integrable.

A last topic for review for the prelims is *Sard's Theorem.*

Definition. Let M and N be manifolds and let $f : M \to N$ be a smooth map. A point $n \in N$ is said to be a **regular value** for f if, for all $m \in f^{-1}(n)$, we have that $(df)_m : T_m M \to T_n N$ is surjective. We make the convention that, if $f^{-1}(n) = \emptyset$, then n is considered to be a regular value of f.

It is a consequence of the Implicit Function Theorem that, for any regular value n of f, the fiber $f^{-1}(n)$ over n either is empty or is a closed submanifold of M.

Definition. A subset X' of a topological space X is said to be **meager** in X if there exists a countable collection $X_1, X_2, \ldots$ of closed subsets of X such that $X \subseteq X_1 \cup X_2 \cup \cdots$ and such that, for all integers $i \geq 1$, we have that $\mathrm{Int}_X(X_i) = \emptyset$.

Definition. Let M be a d-dimensional manifold. A subset M' of M is said to be **null** if, for any chart $\phi : \mathbb{R}^d \to M$, we have $\phi^{-1}(M')$ has zero Lebesgue measure.

Sard's Theorem. Let M and N be manifolds and let $f : M \to N$ be a smooth map. Let R be the set of regular values of f. Then $N \backslash R$ is meager and null.

END OF TIMEOUT TO DISCUSS MATERIAL RELEVANT TO PRELIMS.

TIMEOUT TO DISCUSS RELEVANCE OF THE FROBENIUS THEOREM:

Let's consider looking for solutions of Laplace's Equation $\Delta f = 0$ on $\mathbb{R}^2$. That is: Let ∂_1, ∂_2 be the standard framing of $\mathbb{R}^2$. Let $\mathcal{F}$ be the set of all smooth functions $\mathbb{R}^2 \to \mathbb{R}$. $\mathcal{S} := \{f \in \mathcal{F} \,|\, (\partial_1^2 f) + (\partial_2^2 f) = 0\}$. We wish to describe a strategy for "finding" the solution set $\mathcal{S}$ for Laplace's Equation.

For all $f \in \mathcal{F}$, define $J^2 f : \mathbb{R}^2 \to \mathbb{R}^8$ by

$$(J^2 f)(p) = (p, f(p), (\partial_1 f)(p), (\partial_2 f)(p), (\partial_1^2 f)(p), (\partial_1 \partial_2 f)(p), (\partial_2^2 f)(p)).$$

Let $\mathcal{G}$ be the set of all smooth functions $\mathbb{R}^2 \to \mathbb{R}^8$. For $g \in \mathcal{G}$, we will say that g is **holonomic** if there exists $f \in \mathcal{F}$ such that $g = J^2 f$. For all integers $i \in [1, 8]$, let $\pi_i : \mathbb{R}^8 \to \mathbb{R}$ denote projection onto the ith coordinate. For all $g \in \mathcal{G}$, for all integers $i \in [1, 8]$, let $g_i := \pi_i \circ g : \mathbb{R}^2 \to \mathbb{R}$. Let $\pi_* : \mathbb{R}^8 \to \mathbb{R}^2$ be defined by $\pi(q) = (\pi_1(q), \pi_2(q))$; this is projection onto the first two coordinates. For all $g \in \mathcal{G}$, let $g_* := \pi_* \circ g : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity map.

We leave it as an exercise for the reader to verify, for all $g \in \mathcal{G}$, that g is holonomic iff $g_* = \mathrm{id}$, $\partial_1 g_3 = g_4$, $\partial_2 g_3 = g_5$, $\partial_1 g_4 = g_6$, $\partial_2 g_4 = g_7$ and $\partial_2 g_5 = g_8$. Note that these conditions imply that

$$\partial_1 g_5 = \partial_1 \partial_2 g_3 = \partial_2 \partial_1 g_3 = \partial_2 g_4 = g_7.$$

Then $f \mapsto J^2 f$ gives a bijection between $\mathcal{S}$ and

$$\mathcal{S}' := \{g \in \mathcal{G} \,|\, g \text{ is holonomic and } g_6 + g_8 = 0\}.$$

We therefore wish to "find" $\mathcal{S}'$.

For all $g \in \mathcal{G}$, $dg : T\mathbb{R}^2 \to T\mathbb{R}^8$ and, using standard identifications, this dg identifies with a map $Dg : \mathbb{R}^4 \to \mathbb{R}^{16}$. One checks, for all $g \in \mathcal{G}$, for all $p \in \mathbb{R}^2$, for all $v = (v_1, v_2) \in \mathbb{R}^2$, that

$$(Dg)(p, v) = (g(p), v_1 \cdot (\partial_1 g_1)(p) + v_2 \cdot (\partial_2 g_1)(p), \ldots, v_1 \cdot (\partial_1 g_8)(p) + v_2 \cdot (\partial_2 g_8)(p)),$$

where $\cdot$ denotes ordinary multiplication on $\mathbb{R}$.

For all $g \in \mathcal{G}$, we have: g is holonomic iff, for all $p = (p_1, p_2) \in \mathbb{R}^2$, for all $v = (v_1, v_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}(Dg)(p, v) = \big(&p_1, p_2, g_3(p), \cdots, g_8(p),\\ &v_1, v_2,\\ &v_1 \cdot g_4(p) + v_2 \cdot g_5(p),\\ &v_1 \cdot g_6(p) + v_2 \cdot g_7(p),\\ &v_1 \cdot g_7(p) + v_2 \cdot g_8(p),\\ &v_1 \cdot (\partial_1 g_7)(p) + v_2 \cdot (\partial_2 g_7)(p),\\ &v_1 \cdot (\partial_1 g_8)(p) + v_2 \cdot (\partial_2 g_8)(p)\big).\end{aligned}$$

For all $q = (q_1, \cdots, q_8) \in \mathbb{R}^8$, for all $w = (w_1, \cdots, w_8) \in \mathbb{R}^8$, let us say that (q, w) is **holonomic** if

$$\begin{aligned} w_3 &= w_1 q_4 + w_2 q_5\\ w_4 &= w_1 q_6 + w_2 q_7\\ w_5 &= w_1 q_7 + w_2 q_8. \end{aligned}$$

Then, for all $g \in \mathcal{G}$, g is holonomic iff both $g_* = \text{id}$ and, for all $p, v \in \mathbb{R}^2$, $(Dg)(p,v)$ is holonomic.

Under the standard identification $T\mathbb{R}^8 \longleftrightarrow \mathbb{R}^{16}$, let $\Delta \subseteq T\mathbb{R}^8$ correspond to the set of holonomic elements of $\mathbb{R}^{16}$. Then, for all $g \in \mathcal{G}$, g is holonomic iff both $g_* = \text{id}$ and $\text{im}(dg) \subseteq \Delta$.

Let's say that a subset $L \subseteq \mathbb{R}^8$ is **attached** if there exists $h : \mathbb{R}^2 \to \mathbb{R}^6$ such that L is the graph of h, *i.e.*, such that $L = \{(p, h(p)) \,|\, p \in \mathbb{R}^2\}$. If so, then L is a closed 2-dimensional submanifold of $\mathbb{R}^8$. Note, for all $f \in \mathcal{F}$, that $\text{im}(J^2 f)$ is attached. That is, for all holonomic $g \in \mathcal{G}$, we see that $\text{im}(g)$ is attached. Moreover, for all holonomic $g \in \mathcal{G}$, we have $T(\text{im}(g)) = \text{im}(dg) \subseteq \Delta$.

Let $\mathcal{L}$ denote the set of all closed 2-dimensional submanifolds L of $\mathbb{R}^8$ such that L is attached and $TL \subseteq \Delta$. Thus, for all holonomic $g \in \mathcal{G}$, we have $\text{im}(g) \in \mathcal{L}$.

Conversely, let $L \in \mathcal{L}$. We claim that there is a holonomic $g \in \mathcal{G}$ such that $L = \text{im}(g)$. By definition of "attached", we see that L is the graph of some function $h : \mathbb{R}^2 \to \mathbb{R}^6$. Let $\rho : \mathbb{R}^6 \to \mathbb{R}$ be projection onto the third coordinate, defined by $\rho(t_1, \ldots, t_6) = t_3$. Let $f := \rho \circ h : \mathbb{R}^2 \to \mathbb{R}$ and let $g := J^2 f$. We leave it as an unassigned exercise to show that $\text{im}(g) = L$.

Thus there is a one-to-one correspondence $g \mapsto \text{im}(g)$ between holonomic $g \in \mathcal{G}$ and attached, closed 2-dimensional submanifolds L of $\mathbb{R}^8$ such that $TL \subseteq \Delta$.

Recall that we wish to describe $\mathcal{S}' := \{g \in \mathcal{G} \,|\, g \text{ is holonomic and } g_6 + g_8 = 0\}$. Let $M := \{(q_1, \ldots, q_8) \in \mathbb{R}^8 \,|\, q_6 + q_8\} = 0$. Then

$$\mathcal{S}' = \{g \in \mathcal{G} \,|\, g_* = \text{id}, \text{im}(g) \subseteq M, \text{im}(dg) \subseteq \Delta\}.$$

Note that M is a submanifold of $\mathbb{R}^8$. Let $\Delta_M := \Delta \cap (TM)$, so Δ_M is a distribution on M. Then $g \mapsto \text{im}(g)$ defines a one-to-one correspondence between $\mathcal{S}'$ and

$$\mathcal{L}_M := \{L \in \mathcal{L} \,|\, L \subseteq M\}.$$

We therefore wish to "compute" $\mathcal{L}_M$.

If it so happens that Δ_M is involutive, then, by the Frobenius Theorem, there is a unique foliation $\mathcal{F}_M$ such that $T\mathcal{F}_M = \Delta_M$. In this case, let $\mathcal{L}'_M$ denote the set of leaves of $\mathcal{F}_M$. Then $\mathcal{L}_M$ is just the union, over all $L' \in \mathcal{L}'_M$ of the set of attached, closed 2-dimensional submanifolds of L'.

So, to "solve" Laplace's equation, one must find $\mathcal{F}_M$, then $\mathcal{L}'_M$, and then, for all $L' \in \mathcal{L}'_M$ find the attached, closed 2-dimensional submanifolds of L'.

The beauty of this is that it is exceedingly flexible. For example, if, instead of Laplace's equation, one wants all solutions to the nonlinear equation $\partial_1^2 f + (\partial_2^2 f)^2 = 0$, then the only change we must make is to replace M by $\{(q_1, \ldots, q_8) \in \mathbb{R}^8 \,|\, q_6 + q_8^2\} = 0$. Moreover, there is no reason to restrict to second-order PDE. There is, in fact, a natural generalization to solutions of systems of higher-order PDE on functions $\mathbb{R}^a \to \mathbb{R}^b$. Moreover the strategy can be adjusted to study boundary data, or to analyze systems of PDE on manifolds, including a whole host of problems that arise from differential geometry.

END OF TIMEOUT TO DISCUSS RELEVANCE OF THE FROBENIUS THEOREM.

Now let $k \geq 1$ be an integer, let $\partial_1, \ldots, \partial_k$ be the standard framing of $\mathbb{R}^k$ and let $dx_1, \ldots, dx_k$ be the standard constant 1-forms on $\mathbb{R}^k$. Let $\mathbf{0} : \mathbb{R}^k \to \mathbb{R}$ be the constant zero function defined by $\mathbf{0}(p) = 0$. Let $\mathbf{1} : \mathbb{R}^k \to \mathbb{R}$ be the constant one function defined by $\mathbf{1}(p) = 1$.

Then, for all integers $i, j \in [1, k]$, if $i \neq j$, then $(dx_i)(\partial_j) = \mathbf{0}$. Moreover, for all integers $i \in [1, k]$, we have $(dx_i)(\partial_i) = \mathbf{1}$.

Let $n \in [0, k]$ be an integer. Let $I := \{(i_1, \ldots, i_n) \in \mathbb{Z}^n \,|\, 1 \leq i_1 < i_2 < \cdots < i_n \leq k\}$. For all $i = (i_1, \ldots, i_n) \in I$, let $dx_i := dx_{i_1} \wedge \cdots \wedge dx_{i_n}$. For all $i = (i_1, \ldots, i_n) \in I$, let $\partial_i := (\partial_{i_1}, \ldots, \partial_{i_n})$, so ∂_i is a field of n-parallelpipeds, one in each tanget space of $\mathbb{R}^k$.

For all $i, j \in I$, if $i \neq j$, then $(dx_i)(\partial_j) = \mathbf{0}$. Moreover, for all $i \in I$, we have $(dx_i)(\partial_i) = \mathbf{1}$.

Recall that $\Omega^n \mathbb{R}^k$ is a module over $C^\infty(\mathbb{R}^k)$.

Fact. The $(C^\infty(\mathbb{R}^k))$-module $\Omega^n \mathbb{R}^k$ is free with basis $\{dx_i \,|\, i \in i\}$.

That is to say, for every $\phi \in \Omega^n \mathbb{R}^k$, there is a unique I-tuple $(f_i)_{i \in I} \in (C^\infty(\mathbb{R}^k))^I$ of C^∞ functions on $\mathbb{R}^k$ such that $\phi = \sum_{i \in I} f_i \, dx_i$.

Fix $a \in I$ and $f \in C^\infty(\mathbb{R}^k)$. Let $\omega := f \, dx_a$, let

$$\eta_1 := (\partial_1 f) \, dx_1 \wedge dx_a, \qquad \ldots, \qquad \eta_k := (\partial_k f) \, dx_k \wedge dx_a$$

and let $\eta := \eta_1 + \cdots + \eta_k$. We would like to explain why, in order to have Stokes' Theorem, we must have $d\omega = \eta$. Note that Stokes' Theorem implies the box chain form of Stokes' Theorem. We will use the box chain form of Stokes' Theorem to show that $(d\omega)_0 = \eta_0 \in \wedge^{n+1}(T_0\mathbb{R}^k)^*$, and leave it to the reader to verify, using a similar argument, that $d\omega$ and η agree other points of $\mathbb{R}^k$, and not just at $0 \in \mathbb{R}^k$.

Let $b := (1, \ldots, n+1)$ and let $\partial_b := (\partial_1, \ldots, \partial_{b+1})$, so ∂_b is a field of $(n+1)$-parallelpipeds on the tangent spaces of $\mathbb{R}^k$. Then $\langle d\omega, \partial_b \rangle \in C^\infty(\mathbb{R}^k)$ and $\langle \eta, \partial_b \rangle \in C^\infty(\mathbb{R}^k)$. For any $p \in \mathbb{R}^k$, for any $f \in C^\infty(\mathbb{R}^k)$, let $f|_p := f(p)$. We will show that $\langle d\omega, \partial_b \rangle\big|_0 = \langle \eta, \partial_b \rangle\big|_0$,

We will then leave it as an unassigned exercise to modify the proof and show that a similar result holds if b is replaced by any element of

$$\{(i_1, \ldots, i_{n+1}) \in \mathbb{Z}^n \,|\, 1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq k\}.$$

Once this unassigned exercise is completed, the reader can use that to then show that $(d\omega)_0 = \eta_0 \in \wedge^{n+1}(T_0\mathbb{R}^k)^*$.

For all $\epsilon = (\epsilon_0, \ldots, \epsilon_n) \in (0, \infty)^{n+1}$, let $B_\epsilon := [0, \epsilon_0] \times \cdots \times [0, \epsilon_n] \subseteq \mathbb{R}^{n+1}$, let $v_\epsilon := \epsilon_0 \cdots \epsilon_n$ and define $\sigma_\epsilon : B_\epsilon \to \mathbb{R}^k$ by $\sigma_\epsilon(s_0, \ldots, s_n) = (s_0, \ldots, s_n, 0, \ldots, 0)$. For all integers $a \geq 0$, let 0^a be the zero element of $\mathbb{R}^a$. Then

$$\frac{1}{v_\epsilon} \int_{\sigma_\epsilon} d\omega \qquad = \qquad \frac{1}{v_\epsilon} \int_{B_\epsilon \times \{0^{k-n-1}\}} \langle d\omega, \partial_b \rangle \qquad \longrightarrow \qquad \langle d\omega, \partial_b \rangle\big|_{0^k},$$

as $\epsilon \to 0^{n+1}$.

For all $\epsilon = (\epsilon_0, \ldots, \epsilon_n) \in (0,\infty)^{n+1}$, let

$$\widetilde{B}_\epsilon := [0,\epsilon_1] \times \cdots \times [0,\epsilon_n] \subseteq \mathbb{R}^n,$$

let $\widetilde{v}_\epsilon := \epsilon_1 \cdots \epsilon_n$, define $\widetilde{\sigma}_\epsilon^- : \widetilde{B}_\epsilon \to \mathbb{R}^k$ by $\sigma_\epsilon(s_0, \ldots, s_n) = (0, s_1, \ldots, s_n, 0, \ldots, 0)$ and define $\widetilde{\sigma}_\epsilon^+ : \widetilde{B}_\epsilon \to \mathbb{R}^k$ by $\sigma_\epsilon(s_0, \ldots, s_n) = (\epsilon_0, s_1, \ldots, s_n, 0, \ldots, 0)$. Then, for all $\epsilon \in (0,\infty)^{n+1}$, we have, from the alternating sum definition of the boundary of a box:

$$\partial\sigma_\epsilon = (\partial_1^+\sigma_\epsilon - \partial_1^-\sigma_\epsilon) - \cdots = (\widetilde{\sigma}_\epsilon^+ - \widetilde{\sigma}_\epsilon^-) - \cdots,$$

and we leave it to the reader to think about what belongs in the elipses, but there will be one term for every face of an $(n+1)$-dimensional box.

Let $\widetilde{b} := (2, \ldots, n+1) \in I$. Then $\partial_{\widetilde{b}} = (\partial_2, \ldots, \partial_{n+1})$. For all $\epsilon \in (0,\infty)^{n+1}$, we have

$$\int_{\partial\sigma_\epsilon} \omega = \left[\left(\int_{\widetilde{\sigma}_\epsilon^+} \omega\right) - \left(\int_{\widetilde{\sigma}_\epsilon^-} \omega\right)\right] - \cdots,$$

we also have

$$\int_{\widetilde{\sigma}_\epsilon^+} \omega = \int_{\{\epsilon_0\}\times\widetilde{B}_\epsilon\times\{0^{k-n-1}\}} \langle\omega, \partial_{\widetilde{b}}\rangle$$

and we also have

$$\int_{\widetilde{\sigma}_\epsilon^-} \omega = \int_{\{0\}\times\widetilde{B}_\epsilon\times\{0^{k-n-1}\}} \langle\omega, \partial_{\widetilde{b}}\rangle.$$

For all $\epsilon \in (0,\infty)^{n+1}$, we have

$$\frac{1}{\widetilde{v}_\epsilon}\int_{\{\epsilon_0\}\times\widetilde{B}_\epsilon\times\{0^{k-n-1}\}} \langle\omega, \partial_{\widetilde{b}}\rangle \to \langle\omega, \partial_{\widetilde{b}}\rangle\big|_{(\epsilon_0, 0^{k-1})}$$

and we have

$$\frac{1}{\widetilde{v}_\epsilon}\int_{\{0\}\times\widetilde{B}_\epsilon\times\{0^{k-n-1}\}} \langle\omega, \partial_{\widetilde{b}}\rangle \to \langle\omega, \partial_{\widetilde{b}}\rangle\big|_{0^k}.$$

For any $p, q \in \mathbb{R}^k$, for any $f \in C^\infty(\mathbb{R}^k)$, let $f|_p^q := (f(q)) - (f(p))$. For all $\epsilon_0 > 0$, we have $v_\epsilon = \epsilon_0\widetilde{v}_\epsilon$, and, together with the other observations in this paragraph, we get

$$\frac{1}{v_\epsilon}\int_{\partial\sigma_\epsilon} \omega = \frac{1}{\epsilon_0}\left[\langle\omega, \partial_{\widetilde{b}}\rangle\big|_{0^k}^{(\epsilon_0, 0^{k-1})}\right].$$

We now assume without proof that both

$$I^+ := \lim_{\epsilon\to 0^{n+1}} \frac{1}{v_\epsilon}\int_{\widetilde{\sigma}_\epsilon^+} \omega \qquad \text{and} \qquad I^- := \lim_{\epsilon\to 0^{n+1}} \frac{1}{v_\epsilon}\int_{\widetilde{\sigma}_\epsilon^-} \omega$$

exist. The results of the last paragraph then imply that

$$I^+ - I^- = \lim_{\epsilon_0\to 0} \frac{1}{\epsilon_0}\left[\langle\omega, \partial_{\widetilde{b}}\rangle\big|_{0^k}^{(\epsilon_0, 0^{k-1})}\right] = \left[\partial_1(\langle\omega, \partial_{\widetilde{b}}\rangle)\right]\big|_{0^k}.$$

Since $\omega = f\,dx_a$, we get $\langle \omega, \partial_{\tilde{b}} \rangle = f \langle dx_a, \partial_{\tilde{b}} \rangle$. As $\partial_{\tilde{b}} = (\partial_2, \ldots, \partial_{n+1})$ and as $\partial_b = (\partial_1, \ldots, \partial_{n+1})$, we get $\langle dx_a, \partial_{\tilde{b}} \rangle = \langle dx_1 \wedge dx_a, \partial_b \rangle$. Note that this function is constant. Putting all this together, we have

$$I^+ - I^- = \big((\partial_1 f) \cdot \langle dx_1 \wedge dx_a, \partial_b \rangle \big)\big|_{0^k} = \big(\langle \eta_1, \partial_b \rangle \big)\big|_{0^k}.$$

Assuming the box chain form of Stokes' Theorem to be true, we conclude that

$$\frac{1}{v_\epsilon} \int_{\partial \sigma_\epsilon} \omega \quad = \quad \frac{1}{v_\epsilon} \int_{\sigma_\epsilon} d\omega \quad \longrightarrow \quad \big(\langle d\omega, \partial_b \rangle \big)\big|_{0^k},$$

as $\epsilon \to 0^{n+1}$. However, we also have that

$$\frac{1}{v_\epsilon} \int_{\partial \sigma_\epsilon} \omega \quad \longrightarrow \quad [I^+ - I^-] - \cdots \quad = \quad \langle \eta_1, \partial_b \rangle\big|_{0^k} - \cdots,$$

as $\epsilon \to 0^{n+1}$. Recall that $\eta = \eta_1 + \cdots + \eta_k$. We leave it to the reader to verify that, when the last elipsis is evaluated, one has

$$\frac{1}{v_\epsilon} \int_{\partial \sigma_\epsilon} \omega \quad \longrightarrow \quad \langle \eta, \partial_b \rangle\big|_{0^k},$$

as $\epsilon \to 0^{n+1}$. Then

$$\langle d\omega, \partial_b \rangle\big|_{0^k} \quad = \quad \langle \eta, \partial_b \rangle\big|_{0^k},$$

as desired.

Carrying the above logic through with general commuting vector fields on a general manifold, one verifies that, quite generally, for Stokes' Theorem to be true, we must have:

Proposition. Let M be a manifold, let $n \geq 0$ be an integer and let $\omega \in \Omega^n M$. Let $X_0, \ldots, X_n \in \mathrm{VF}(M)$ be pairwise-commuting. Then $(d\omega)(X_0, \ldots, X_n) \in C^\infty(M)$ equals:

$$[X_0(\omega(X_1, \ldots, X_n))] - [X_1(\omega(X_0, X_2, \ldots, X_n))] + \cdots \pm [X_n(\omega(X_0, \ldots, X_{n-1}))].$$

Note that, when $M = \mathbb{R}^k$ and letting $X_0, \ldots, X_n$ range across all the $n$tuples of vector fields taken from the standard framing of $\mathbb{R}^k$, then we can use the preceding proposition to prove the formula: $d(f\,dx_a) = df \wedge dx_a$.

When the vector fields $X_0, \ldots, X_n$ do not pairwise-commute, the preceding proposition does not apply, but there is a version of it that does, and we will describe that next. We remark that, all through differential topology and differential geometry, this situation occurs: There's a formula for a tensor in terms of its effect on a tuple of commuting vector fields, and an analogous (longer) formula can always be worked out without assuming the commutation.

Let's first consider the formula for $(d\omega)(X, Y)$, when ω is a 1-form and when $X, Y \in \mathrm{VF}(M)$. From the preceding proposition, if $[X, Y] = 0$, then we get

$$(d\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X))$$

. If X and Y are not assume to commute, then we have:

Lemma. Let M be a manifold and let $\omega \in \Omega^1 M$. Let $X, Y \in \mathrm{VF}(M)$. Then

$$(d\omega)(X,Y) = [X(\omega(Y))] - [Y(\omega(X))] - [\omega([X,Y])].$$

Proof: Let $k := \dim(M)$. We may assume that $M = \mathbb{R}^k$. Let $\partial_1, \ldots, \partial_k$ be the standard framing of $\mathbb{R}^k$. We may assume, for some $f, g \in C^\infty(\mathbb{R}^k)$ and for some integers $i, j \in [1, k]$, that $X = f\partial_1$ and $Y = g\partial_j$. Then

$$(d\omega)(X,Y) = fg \cdot (d\omega)(\partial_i, \partial_j) = fg \cdot (\partial_i(\omega(\partial_j)) - \partial_j(\omega(\partial_i))).$$

We have

$$X(\omega(Y)) = f\partial_i(g \cdot \omega(\partial_j)) = [(f\partial_i g) \cdot \omega(\partial_j)] + [fg \cdot (\partial_i(\omega(\partial_j)))]$$

and

$$Y(\omega(X)) = g\partial_j(f \cdot \omega(\partial_i)) = [(g\partial_j f) \cdot \omega(\partial_i)] + [fg \cdot (\partial_j(\omega(\partial_i)))]$$

and

$$\omega([X,Y]) = \omega(f(\partial_i g)\partial_j - g(\partial_j f)\partial_i) = [(f\partial_i g) \cdot \omega(\partial_j)] - [(g\partial_j f) \cdot \omega(\partial_i)].$$

Then

$$[X(\omega(Y))] - [Y(\omega(X))] - [\omega([X,Y])] = fg \cdot (\partial_i(\omega(\partial_j)) - \partial_j(\omega(\partial_i))) = (d\omega)(X,Y),$$

as desired. **QED**

We leave it to the reader to carry out similar computations for forms of higher degree. We simply record the final result:

Proposition. Let M be a manifold, let $n \geq 0$ be an integer and let $\omega \in \Omega^n M$. Let $X_0, \ldots, X_n \in \mathrm{VF}(M)$. Then $(d\omega)(X_0, \ldots, X_n) \in C^\infty(M)$ is equal to the sum of

$$[X_0(\omega(X_1, \ldots, X_n))] - [X_1(\omega(X_0, X_2, \ldots, X_n))] + \cdots \pm [X_n(\omega(X_0, \ldots, X_{n-1}))].$$

and

$$\sum_{i<j} (-1)^{i+j} \cdot \omega([X_i, X_j], X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n).$$

If M is a manifold and $\omega \in \Omega^1(M)$, then we define $\ker\, \omega := \{v \in TM \mid \omega(v) = 0\}$. This set is a union of subspaces of the tangent spaces of M and each subspace as codimension at most 1.

Let M be a manifold and let $\omega_1, \ldots, \omega_n \in \Omega^1 M$. Let $\Delta := (\ker\, \omega_1) \cap \cdots \cap (\ker\, \omega_n)$. For all $m \in M$, let $\Delta_m := \Delta \cap T_m M$; then Δ_m is a subspace of $T_m M$. Note that Δ is a distribution on M iff for all $m, m' \in M$, we have $\dim(\Delta_m) = \dim(\Delta_{m'})$.

From the preceding lemma, we can derive the differential forms form of the Frobenius Theorem:

Theorem. Let M be a manifold and let $\omega_1, \ldots, \omega_n \in \Omega^1 M$. Let $\Delta := (\ker \omega_1) \cap \cdots \cap (\ker \omega_n)$. Assume that Δ is a distribution on M. Assume, for all integers $i \in [1, n]$, that there exist $\eta_1^i, \ldots, \eta_n^i \in \Omega^1 M$ such that

$$d\omega_i = \sum_j \omega_j \wedge \eta_j^i.$$

Then Δ is integrable.

A restatement of this last theorem is: "If the ideal generated by the 1-forms $\omega_1, \ldots, \omega_n$ is a differential ideal (*i.e.*, an ideal that is closed under exterior differentiation $\omega \mapsto d\omega$), and if the kernel of the ideal is a distribution, then that distribution is integrable.

Proof: Let $X, Y \in \mathrm{VF}(M)$, and assume that X and Y are both in Δ. By the Frobenius Theorem, we wish to show that $[X, Y]$ is in Δ. That is, we wish to show, for all integers $i \in [1, n]$, that $\omega_i([X, Y]) = 0$.

For all integers $i \in [1, n]$, we have $\omega_i(X) = \omega_i(Y)$. For all integers $i, j \in [1, n]$, $(\omega_i \wedge \eta_j^i)(X, Y)$ is equal to the determinant of

$$\begin{bmatrix} \omega_i(X) & \omega_i(Y) \\ \eta_j^i(X) & \eta_j^i(Y) \end{bmatrix},$$

so $(\omega_i \wedge \eta_j^i)(X, Y) = 0$.

For all integers $i \in [1, n]$, we have $d\omega_i = \sum_j \omega_j \wedge \eta_j^i$, so $(d\omega_i)(X, Y) = 0$. Then, for all integers $i \in [1, n]$, we have

$$0 = (d\omega_i)(X, Y) = [X(\omega_i(Y))] - [Y(\omega_i(X))] - [\omega_i([X, Y])] = -[\omega_i([X, Y])],$$

so $\omega_i([X, Y]) = 0$, as desired. **QED**

We remark that the converse of the last theorem is also true: If $\omega_1, \ldots, \omega_n \in \Omega^1 M$ and if $(\ker \omega_1) \cap \cdots \cap (\ker \omega_n)$ is an integrable distribution, then the ideal generated by $\omega_1, \ldots, \omega_n$ is differential. In fact, that is the "easy" direction of the differential forms form of the Frobenius Theorem. The "hard" direction is the preceding theorem.

The last section of these notes concerns the proof of the Frobenius Theorem in its original form:

The Frobenius Theorem. Let Δ be a distribution on a manifold M. Then Δ is integrable iff Δ is involutive.

Before we give the proof, we develop some notation and terminology.

Let R be an equivalence relation on a set S. For all $s \in S$, we let $R(s)$ denote the equivalence class containing s, given by $R(s) := \{s' \in S \mid (s, s') \in R\}$. For any $S_0 \subseteq S$, we let $R(S_0)$ denote the union of all equivalence classes meeting S_0; this set is called the **R-saturation** of S_0 and is defined by $R(S_0) := \bigcup_{s \in S_0} R(s)$.

Let R and R' be equivalence relations on a set S. We say that R has **countable index** in R' if: for all $s \in S$, there exists a countable subset $S_0 \subseteq S$ such that $R'(s) = R(S_0)$. (That is, if every equivalence class of R' is a countable union of equivalence classes of R.)

The following is an exercise in set theory, which we leave to the interested reader:

Set Theory Fact. Let X be a set and let $X_1, X_2, \ldots \subseteq X$. Assume that $\bigcup_{i=1}^{\infty} X_i = X$. For all integers $i \geq 1$, let R_i be an equivalence relation on X_i. Let R be the equivalence relation on X generated by $\bigcup_{i=1}^{\infty} R_i$. Assume, for all integers $i, j \geq 1$, for all $x \in X_i$, there is a countable subset $C \subseteq X_j$ such that $(R_i(x)) \cap X_j \subseteq R_j(C)$. Then, for all integers $i \geq 1$, R_i has countable index in $R|X_i$.

That is, if each equivalence class of R_i meets only countably many equivalence classes of R_j, then each equvalence class of R meets only countably many equivalence classes of any R_i.

For any integer $p \geq 0$, let $\mathrm{tran}_p := \mathbb{R}^p \times \mathbb{R}^p$ denote the transitive equivalence relation on $\mathbb{R}^p$. For any integer $q \geq 0$, let $\mathrm{triv}_q := \{(x,x) \,|\, x \in \mathbb{R}^q\}$ denote the trivial equivalence relation on $\mathbb{R}^q$.

We can now given an equivalent definition for the term "prefoliation":

Definition. Let M be a manifold, let $d := \dim(M)$ and let $p \in [0, d]$ be an integer. Let $\mathcal{F}$ be an equivalence relation on M. We say that $\mathcal{F}$ is a **p-dimensional prefoliation** on M if, for all $m \in M$, there exists a chart $\phi : \mathbb{R}^d \to M$ such that $m \in \phi(\mathbb{R}^d)$ and such that $\mathrm{tran}_p \times \mathrm{triv}_q$ has countable index in $\phi^*(\mathcal{F})$. In this case, the equivalence classes of $\mathcal{F}$ are called **leaves**.

Recall that any leaf of a prefoliation is a leaflike manifold and so has a unique maximal atlas under which the inclusion map is an immersion. This maximal atlas on L is called the **inherited manifold structure** on L. Its topology is called the **leaflike topology** on L. Recall that a prefoliation is a **foliation** if all of its leaves are connected in the leaflike topology. The following three results are not hard:

Foliation Fact 1. Let M be a manifold. Let $p \in [0, \dim(M)]$ be an integer. Let $\mathcal{F}$ be a p-dimensional prefoliation on M and let $\mathcal{F}'$ be an equivalence relation on M. Suppose that $\mathcal{F}$ has countable index in $\mathcal{F}'$. Then $\mathcal{F}'$ is a p-dimensional prefoliation on M.

Foliation Fact 2. Let M be a manifold. Let $p \in [0, \dim(M)]$ be an integer. Let $\mathcal{F}$ be an equivalence on M and let $\mathcal{F}'$ be a p-dimensional prefoliation on M. Suppose that $\mathcal{F}$ has countable index in $\mathcal{F}'$. Suppose, for all $m \in M$, that $\mathcal{F}(m)$ is an open subset of $\mathcal{F}'(m)$, in the leaflike topology. Then $\mathcal{F}$ is a p-dimensional prefoliation on M.

Recall that, for any prefoliation $\mathcal{F}$ on a manifold M, the **tangent bundle** $T\mathcal{F}$ of $\mathcal{F}$ is the union, over all leaves L of $\mathcal{F}$, of TL.

Foliation Fact 3. Let M be a manifold. Let $p \in [0, \dim(M)]$ be an integer. Let $\mathcal{F}$ and $\mathcal{F}'$ be p-dimensional prefoliations on M. Assume that $\mathcal{F}$ has countable index in $\mathcal{F}'$. Then

$T\mathcal{F} = T\mathcal{F}'$.

Corollary. Let M be a manifold. Let $\mathcal{F}'$ be a prefoliation on M. Then there is a foliation $\mathcal{F}$ on M such that $T\mathcal{F} = T\mathcal{F}'$.

Proof: Choose $p \in [0, \dim(M)]$ such that $\mathcal{F}'$ is a p-dimensional foliation on M. Define $\mathcal{F}$ by: $(m, m') \in \mathcal{F}$ iff there is a leaf L of $\mathcal{F}'$ such that m and m' are in the same connected component of L, in the leaflike topology on L. By Foliation Fact 2, $\mathcal{F}$ is a p-dimensional prefoliation on M. On the other hand, the leaves of $\mathcal{F}$ are connected, in the leaflike topology, so $\mathcal{F}$ is a p-dimensional foliation. Finally, by Foliation Fact 3, $T\mathcal{F} = T\mathcal{F}'$. **QED**

Foliation Fact 4. Let M be a manifold and let $\mathcal{F}$ be a prefoliation on M. Let Y be a manifold and let $\phi : Y \to M$ be a smooth map. Assume that $(d\phi)(TY) \subseteq T\mathcal{F}$. Then there is a countable $C \subseteq M$ such that $\phi(Y) \subseteq \mathcal{F}(C)$.

Again, we leave this as an unassigned exercise for the interested reader, but we comment that, in fact, from $(d\phi)(TY) \subseteq T\mathcal{F}$, one can prove that each connected component of Y is mapped into a singe connected component (in the leaflike topology) of a single leaf of $\mathcal{F}$. Thus, as Y has only countably many connected components, it follows that the image of Y is contained in countably many leaves of $\mathcal{F}$, which is exactly what Foliation Fact 4 asserts.

If $X^1, \ldots, X^p$ are vector fields on a manifold M, then

$$\operatorname{span}\{X^1, \ldots, X^p\} := \{c_1 X^1_m + \cdots + c_p X^p_m \,|\, m \in M, c_1, \ldots, c_p \in \mathbb{R}\}$$

is the **span** of $X^1, \ldots, X^p$.

Remark. Let $p, d \geq 0$ be integers such that $p \leq d$. Let $q := d - p$ and let $\mathcal{G} := \mathrm{tran}_p \times \mathrm{triv}_q$. Let M be a d-dimensional manifold and let $X^1, \ldots, X^p$ be pairwise-commuting vector fields on M. Let $m \in M$ and assume that $X^1_m, \ldots, X^p_m$ are linearly independent elements of $T_m M$. Then there is an open neighborhood M_0 of m in M an a diffeomorphism $\psi : \mathbb{R}^d \to M_0$ such that $(d\phi)(T\mathcal{G}) = \operatorname{span}\{X^1|M_0, \ldots, X^p|M_0\}$.

Sketch of proof: Let $\partial^1, \ldots, \partial^d$ be the standard framing of $\mathbb{R}^d$.

We leave it as an unassigned exercise to prove that there are:

(1) an open neighborhood U of 0 in $\mathbb{R}^d$;
(2) an open neighborhood V of m in M; and
(3) a diffeomorphism $\phi : U \to V$

such that

(A) $\phi(0) = m$; and
(B) for all integers $i \in [1, p]$, $\phi_*(\partial^i|U) = X^i|V$.

Choose $\epsilon > 0$ such that $(-\epsilon, \epsilon)^d \subseteq U$. Let $U_0 := (-\epsilon, \epsilon)^d$. Choose a diffeomorphism $\alpha : \mathbb{R} \to (-\epsilon, \epsilon)$. Define $\chi : \mathbb{R}^d \to U_0$ by $\chi(t_1, \ldots, t_d) = (\alpha(t_1), \ldots, \alpha(t_d))$. Let $\mathcal{G}_0 := \mathcal{G}|U_0$. Then $(d\chi)(T\mathcal{G}) = T\mathcal{G}_0$.

Let $\psi := \phi \circ \chi : \mathbb{R}^d \to M$. Let $M_0 := \phi(U_0) = \psi(\mathbb{R}^d)$. By (B), we have $(d\phi)(T\mathcal{G}_0) = \operatorname{span}\{X^1|M_0, \ldots, X^p|M_0\}$. Then

$$(d\psi)(T\mathcal{G}) = ((d\phi) \circ (d\chi))(T\mathcal{G}) = (d\phi)(T\mathcal{G}_0) = \operatorname{span}\{X^1|M_0, \ldots, X^p|M_0\}. \qquad \mathbf{QED}$$

The Frobenius Theorem. Let Δ be a distribution on a manifold M. Then Δ is integrable iff Δ is involutive.

Proof of the Frobenius Theorem: Let $d := \dim(M)$. Let p be the rank of Δ. Let $q := d - p$. Let $\mathcal{G} := \mathrm{tran}_p \times \mathrm{triv}_q$. Let $\partial^1, \ldots, \partial^d$ be the standard framing of $\mathbb{R}^d$.

We begin with the proof of "only if". Let $\mathcal{F}$ be a foliation on M such that $T\mathcal{F} = \Delta$. We wish to show that Δ is involutive.

We may assume that $M = \mathbb{R}^d$ and that $\mathcal{G}$ has finite index in $\mathcal{F}$. Then, by Foliation Fact 3, $T\mathcal{F} = T\mathcal{G}$. That is, $\Delta = \mathrm{span}\{\partial^1, \ldots, \partial^p\}$. For all integers $i, j \in [1, p]$, we have $[\partial^i, \partial^j] = 0$, so $[\partial^i, \partial^j]$ is in Δ. Thus Δ is involutive. This concludes the proof of "only if". It remains to proof "if".

Assume that Δ is involutive. By the preceding corollary, we wish to show that there is a prefoliation $\mathcal{F}$ on M such that $T\mathcal{F} = \Delta$.

Claim 1: For all $m \in M$, there exists a neighborhood U of m in M and there exist pairwise-commuting $X^1, \ldots, X^p \in \mathrm{VF}(U)$ such that $\Delta|U = \mathrm{span}\{X^1, \ldots, X^p\}$.

Proof of Claim 1: We may assume that M is an open neighborhood of 0 in $\mathbb{R}^d$ and that $m = 0 \in \mathbb{R}^d$. By rotating, we may assume that $\Delta_0 = (T\mathcal{G})_0$. Then, for all integers $i \in [1, p]$, we have $\partial^i_0 \in \Delta_0$. Let $\mathcal{H} := \mathrm{span}\{\partial^{p+1}, \ldots, \partial^d\}$. Then, no nontrivial linear combination of $\partial^1_0, \ldots, \partial^p_0$ is in $T\mathcal{H}$.

Let U be a neighborhood of 0 in M such that, for all $u \in U$, no nontrivial linear combination of $\partial^1_u, \ldots, \partial^p_u$ is in $T\mathcal{H}$. For all integers $i \in [1, p]$, we choose vector fields $X^i \in \mathrm{VF}(U)$ in $(T\mathcal{G})|U$ and $Y^i \in \mathrm{VF}(U)$ in $(T\mathcal{H})|U$ such that $\partial^i|U = X^i + Y^i$.

For any $u \in U$, since no nontrival linear combination of $\partial^1_u, \ldots, \partial^p_u$ is in $T\mathcal{H}$, it follows that no nontrivial linear combination of $X^1_u, \ldots, X^p_u$ is equal to zero. We conclude that $\Delta|U = \mathrm{span}\{X^1, \ldots, X^p\}$. We must show that $X^1, \ldots, X^p$ are pairwise commuting.

We leave it as an unassigned exercise to verify, for any integer $i \in [1, d]$, for any vector field $Q \in \mathrm{VF}(U)$ in $(T\mathcal{H})|U$, that $[(\partial_i|U), Q]$ is again in $(T\mathcal{H})|U$. Then, as $(T\mathcal{H})|U$ is involutive, it follows, for any integers $i, j \in [1, d]$, for any vector fields $P, Q \in \mathrm{VF}(U)$ in $(T\mathcal{H})|U$, that $[(\partial_i|U) + P, (\partial_j|U) + Q]$ is in $(T\mathcal{H})|U$.

Thus, for any integers $i, j \in [1, p]$, we see that $[X^i, X^j] = [(\partial^i|U) - Y^i, (\partial^j|U) - Y^j]$ is in $(T\mathcal{H})|U$; on the other hand, as Δ is involutive, we see that $[X^i, X^j]$ is in $(T\mathcal{G})|U$. However $(T\mathcal{G}) \cap (T\mathcal{H})$ is the image of the zero section $\mathbb{R}^d \to T\mathbb{R}^d$, so, for all integers $i, j \in [1, p]$, we conclude that $[X^i, X^j] = 0$, as desired. *End of proof of Claim 1.*

By Claim 1 and the preceding remark, for all $m \in M$, there exists a neighborhood M_0 of m in M and a diffeomorphism $\psi : \mathbb{R}^d \to M_0$ such that $(d\phi)(T\mathcal{G}) = \Delta|M_0$. So, since M is Lindelöf, choose a countable open cover $M_1, M_2, \ldots$ of M and diffeomorphisms

$$\psi_1 : \mathbb{R}^d \to M_1, \qquad \psi_2 : \mathbb{R}^d \to M_2, \qquad \ldots$$

such that, for all integers $i \geq 1$, we have: $(d\phi_i)(T\mathcal{G}) = \Delta|M_i$.

For all integers $i \geq 1$, let $\mathcal{F}_i := (\phi_i)_*(\mathcal{G})$, so $\mathcal{F}_i$ is a p-dimensional foliation on M_i. For all integers $i \geq 1$, we have $T\mathcal{F}_i = (d\phi_i)(T\mathcal{G}) = \Delta|M_i$.

Claim 2: For all integers $i, j \geq 1$, for all $m \in M$, there is a countable $C \subseteq M_j$ such that $(\mathcal{F}_i(m)) \cap M_j \subseteq \mathcal{F}_j(C)$.

Proof of Claim 2: Let $Y := (\mathcal{F}_i(m)) \cap M_j$. Since Y is open in $\mathcal{F}_i(m)$, in the leaflike topology on $\mathcal{F}_i(m)$, it follows that Y is a manifold. Let $\phi : Y \to M_j$ be the inclusion map. Then $(d\phi)(TY) \subseteq T\mathcal{F}_j$. So, by Foliation Fact 4 (but with M replaced by M_j and $\mathcal{F}$ replaced by $\mathcal{F}_j$), we are done. *End of proof of Claim 2.*

Now let $\mathcal{F}$ be the equivalence relation on M generated by $\bigcup_{i=1}^{\infty} \mathcal{F}_i$. By the Set Theory Fact above, we conclude, for all integers $i \geq 1$, that $\mathcal{F}_i$ has countable index in $\mathcal{F}|M_i$.

Then, by Foliation Fact 1, we have that $\mathcal{F}|M_i$ is a p-dimensional prefoliation on M_i. So, since M_i is an open cover of M, we see that $\mathcal{F}$ is a prefoliation on M.

By Foliation Fact 3, we see, for all integers $i \geq 1$, that $T(\mathcal{F}_i|M) = T\mathcal{F}_i$. Then, for all integers $i \geq 1$, we get $T(\mathcal{F}_i|M) = \Delta|M_i$. So, since M_i is an open cover of M, we see that $T\mathcal{F} = \Delta$, completing the proof. **QED**

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