MULTI-DIMENSIONAL
SYMBOLIC DYNAMICAL
SYSTEMS

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1. Shifts of finite type

Let $d \geq 1$, $A$ a finite set (the alphabet), and let $A^{\mathbb{Z}^d}$ be the set of all maps $x : \mathbb{Z}^d \longrightarrow A$. For every subset $F \subset \mathbb{Z}^d$,

$$\pi_F : A^{\mathbb{Z}^d} \longrightarrow A^F$$

is the projection map which restricts each $x \in A^{\mathbb{Z}^d}$ to $F$. For every $n \in \mathbb{Z}^d$ we define a homeomorphism $\sigma_n$ of $A^{\mathbb{Z}^d}$ by

$$(\sigma_n x)_m = x_{m+n}$$

for every $x = (x_m) \in A^{\mathbb{Z}^d}$. The map $\sigma : n \mapsto \sigma^n$ is the shift-action of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$, and a subset $X \subset A^{\mathbb{Z}^d}$ is shift-invariant if $\sigma^n(X) = X$ for all $n \in \mathbb{Z}^d$. A closed, shift-invariant set $X \subset A^{\mathbb{Z}^d}$ is a shift of finite type (SFT) if there exist a finite set $F \subset \mathbb{Z}^d$ and a
subset $P \subset A^F$ such that

$$X = X(F, P) = \{x \in A^Z : \pi_F \cdot \sigma^n(x) \in P$$

for every $n \in \mathbb{Z}^d\}$. \hspace{1cm} \text{(1)}$$

A closed shift-invariant subset $X \subset A^Z$ is a SFT if and only if there exists a finite set $F \subset \mathbb{Z}^d$ such that

$$X = \{x \in A^Z : \pi_F \cdot \sigma^n(x) \in \pi_F(X)$$

for every $n \in \mathbb{Z}^d\}$. \hspace{1cm} \text{(2)}$$

An immediate consequence of this characterization of SFT’s is that the notion SFT is an invariant of topological conjugacy.

If $X \subset A^Z$ is a SFT we may change the alphabet $A$ and assume that

$$F = \{0, 1\}^d \text{ or } F = \{0\} \cup \bigcup_{i=1}^d \{e^{(i)}\},$$

where $e^{(i)}$ is the $i$-th basis vector in $\mathbb{Z}^d$.

Let $X \subset A^Z$ be a SFT. A point $x \in X$ is periodic if its orbit under $\sigma$ is finite. In contrast to the case where $d = 1$, a higher-dimensional SFT $X$ may not
contain any periodic points (we give an example below). This potential absence of periodic points is associated with certain **undecidability problems**: 

1. It is algorithmically undecidable if $X(F, P) \neq \emptyset$ for given $(F, P)$;
2. It is algorithmically undecidable whether an allowed partial configuration can be extended to a point $x \in X(F, P)$.

In dealing with concrete SFT’s undecidability is not really a problem, but it indicates the difficulty of making general statements about higher-dimensional SFT’s.

**For most of this talk we assume that** $d = 2$.

2. **Some examples**

1. *(Chessboards)* Let $n \geq 2$ and $A = \{0, \ldots, n - 1\}$. We interpret $A$ as a set of colours and consider the SFT $X = X^{(n)} \subset \mathbb{Z}^2$ consisting of all configurations in which adjacent lattice points must have different colours.
For $n = 2$, $X^{(2)}$ consists of two points. For $n \geq 3$, $X^{(n)}$ is uncountable.

There is a big difference between $n = 3$ and $n \geq 4$: for $n = 3$ there exist frozen configurations in $X^{(3)}$, which cannot be altered in only finitely many places. These points are the periodic extensions of

\[
\begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 2 & 1 & 0 \\
1 & 0 & 2 & 1 \\
2 & 1 & 0 & 2 \\
0 & 2 & 1 & 0 \\
1 & 0 & 2 & 1 \\
2 & 1 & 0 & 2 \\
\end{array}
\]

(2) (Wang tilings) Let $T$ be a finite nonempty set of distinct, closed $1 \times 1$ squares (tiles) with coloured edges such that no horizontal edge has the same colour as a vertical edge: such a set $T$ is called a collection of Wang tiles. For each $\tau \in T$ we denote by $r(\tau), t(\tau), l(\tau), b(\tau)$ the colours of the right, top, left and bottom edges of $\tau$, and we write $\mathcal{C}(T) = \{r(\tau), t(\tau), l(\tau), b(\tau) : \tau \in T\}$ for the set of colours occurring on the tiles in $T$. A Wang tiling $w$ by $T$ is a covering of $\mathbb{R}^2$ by such that

(i) every corner of every tile in $w$ lies in $\mathbb{Z}^2 \subset \mathbb{R}^2$, 

(ii) two tiles of \( w \) are only allowed to touch along edges of the same colour, i.e. \( r(\tau) = l(\tau') \) whenever \( \tau, \tau' \) are horizontally adjacent tiles with \( \tau \) to the left of \( \tau' \), and \( t(\tau) = b(\tau') \) if \( \tau, \tau' \) are vertically adjacent with \( \tau' \) above \( \tau \).

We identify each such tiling \( w \) with the point

\[
w = (w_n) \in T^\mathbb{Z}^2,\]

where \( w_n \) is the unique element of \( T \) whose translate covers the square \( n + [0, 1]^2 \subset \mathbb{R}^2, \ n \in \mathbb{Z}^2 \). The set \( W_T \subset T^\mathbb{Z}^2 \) of all Wang tilings by \( T \) is obviously a SFT, and is called the Wang shift of \( T \).

Here is an explicit example of a two-dimensional Wang shift: let \( T_D \) be the set of Wang tiles

\[
\text{dom1.eps}
\]

with the colours \( H, h, V, v \) on the solid horizontal, broken horizontal, solid vertical and broken vertical edges. The following picture shows a partial Wang tiling of \( \mathbb{R}^2 \) by \( T_D \) and explains the name ‘domino tiling’ for such a tiling: two tiles meeting along an
edge coloured h or v form a single vertical or horizontal ‘domino’.

\[
\text{dom2.eps} \quad \longrightarrow \quad \text{dom3.eps}
\]

The Wang shift \( W_D \subset T_D^{\mathbb{Z}^2} \) of \( T_D \) is called the \textit{Domino} (or \textit{Dimer}) \textit{Shift}, and is one of the few higher dimensional \textit{SFT’s} for which the dynamics is understood to some extent. The shift-action \( \sigma_{W_D} \) of \( \mathbb{Z}^2 \) on \( W_D \) is topologically mixing, and its topological entropy \( h(\sigma_{W_D}) \) was computed by Kasteleyn:

\[
h(\sigma_{W_D}) = \frac{1}{4} \int_0^1 \int_0^1 (4 - 2 \cos 2\pi s - 2 \cos 2\pi t) \, ds \, dt.
\]

The domino-tilings again have frozen configurations which look like ‘brick walls’.

(3) \textit{(A shift of finite type without periodic points)}

Consider the following set \( T' \) of six polygonal tiles, introduced by Robinson, each of which should be thought of as a \( 1 \times 1 \) square with various bumps and dents.

\[
\text{rob.eps}
\]
We denote by $T$ the set of all tiles which are obtained by allowing horizontal and vertical reflections as well as rotations of elements in $T'$ by multiples of $\frac{\pi}{2}$. Again we consider the set $W_T \subset T^{\mathbb{Z}^2}$ consisting of all tilings of $\mathbb{R}^2$ by translates of elements of $T$ aligned to the integer lattice (as much as their bumps and dents allow). The set $W_T$ is obviously a SFT, and $W_T$ is uncountable and has no periodic points (Robinson). If we allow each (or even only one) of these tiles to occur in two different colours with no restriction on adjacency of colours then we obtain a SFT with positive entropy, but still without periodic points.

The paper by Robinson also contains an explicit set $T$ of Wang tiles for which the extension problem is undecidable.

3. **Wang tiles and shifts of finite type**

**Theorem 1.** Every SFT can be represented (in many different ways) as a Wang tiling.
Proof. Assume that \( F = \{0, 1\}^2 \subset \mathbb{Z}^2 \). We set \( T = \pi_F(X_{(F,P)}) \) and consider each
\[
\tau = \begin{bmatrix} x_{(0,1)} & x_{(1,1)} \\ x_{(0,0)} & x_{(1,0)} \end{bmatrix} \in T
\]
as a unit square with the ‘colours’ \([x_{(0,0)} \ x_{(1,0)}]\) and \([x_{(0,1)} \ x_{(1,1)}]\) along its bottom and top horizontal edges, and \([x_{(1,1)} \ x_{(0,0)}]\) and \([x_{(0,1)} \ x_{(1,0)}]\) along its left and right vertical edges. With this interpretation we obtain a one-to-one correspondence between the points \( x = (x_n) \in X \) and the Wang tilings \( w = (w_n) = (\pi_F \cdot \sigma^n(x)) \in T^{\mathbb{Z}^2} \).

This correspondence allows us to regard each SFT as a Wang shift and vice versa. However, the correspondence is a bijection only up to topological conjugacy: if we start with a SFT \( X \subset A^{\mathbb{Z}^2} \) with \( F = \{0, 1\}^2 \), view it as the Wang \( W_T \subset T^{\mathbb{Z}^2} \) with \( T = \pi_F(X) \), and then interpret \( W_T \) as a SFT as above, we do not end up with \( X \), but with the 2-block representation of \( X \).

Definition 2. Let \( A \) be a finite set and \( X \subset A^{\mathbb{Z}^2} \) a SFT, \( T \) a set of Wang tiles and \( W_T \) the associated
Wang shift. We say that $W_T \text{ represents } X$ if $W_T$ is topologically conjugate to $X$. Two Wang shifts $W_T$ and $W_{T'}$ are equivalent if they are topologically conjugate as $SFT$’s.

Since any given infinite $SFT$ $X$ has many different representations by Wang shifts one may ask whether these different representations of $X$ have anything in common. The answer to this question turns out to be related to a measure of the ‘complexity’ of the $SFT$ $X$. For this we need to introduce the tiling group associated with a Wang shift.

Let $T$ be a collection of Wang tiles and $W_T \subset T$ the Wang shift of $T$.

Following Conway, Lagarias and Thurston we write

$$\Gamma(T) = \langle C(T) | t(\tau)l(\tau) = r(\tau)b(\tau), \ \tau \in T \rangle$$

for the free group generated by the colours occurring on the edges of elements in $T$, together with the relations $t(\tau)l(\tau) = r(\tau)b(\tau), \ \tau \in T$. The countable, discrete group $\Gamma(T)$ is called the tiling group of $T$ (or of the Wang shift $W_T$). From the definition of
\( \Gamma(T) \) it is clear that the map \( \theta : \Gamma(T) \to \mathbb{Z}^2 \), given by

\[
\theta(b(\tau)) = \theta(t(\tau)) = (1, 0), \\
\theta(l(\tau)) = \theta(r(\tau)) = (0, 1),
\]

for every \( \tau \in T \), is a group homomorphism whose kernel is denoted by

\( \Gamma_0(T) = \ker(\theta) \).

Suppose that \( E \subset \mathbb{Z}^2 \) is a finite set, and that \( w \in T^{\mathbb{Z}^2 \setminus E} \) is a Wang-tiling of \( \mathbb{Z}^2 \setminus E \). When can we complete \( w \) to a Wang-tiling of \( \mathbb{Z}^2 \) (possibly after enlarging \( E \) by a finite amount)? After a finite enlargement we may assume that \( E \) is a rectangle:

\[
\text{tiles.eps}
\]

The words in \( \Gamma(T) \) obtained by reading off the colours along the edges of the two holes coincide because of the tiling relations:

\[
\begin{align*}
& r_1^{-1}r_2^{-1}r_3^{-1}b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}l_3l_2l_1t_4t_3t_2t_1
\end{align*}
\]
\begin{equation*}
= r_1^{-1} r_2^{-1} r_3^{-1} b_1^{-1} b_2^{-1} b_3^{-1} b_4^{-1} t_2 t'_2 t_3 t_2 t_1
\end{equation*}

In particular, if the hole can be closed, then the word must be the identity.

If \( X \subset A^{\mathbb{Z}^2} \) is a SFT and \( W_T \) a Wang representation of \( X \) then the tiling group \( \Gamma(T) \) gives an obstruction to the weak closing of holes (i.e. the closing of holes after finite enlargement) for points \( x \in A^{\mathbb{Z}^2} \setminus E \), where \( E \subset \mathbb{Z}^2 \) is a finite set. However, different Wang-representations of \( X \) may give different answers.

**Example 3.** Let \( X \) be the 3-coloured chessboard, and let \( T \) be the set of Wang tiles

\[ \text{chess1.eps} \]

with the colours\[
h_0 = \begin{cases} \text{chess21.eps} & \text{chess22.eps} \\
\text{chess23.eps} & \text{chess24.eps} \end{cases} \]

\[ v_0 = \begin{cases} \text{chess24.eps} \end{cases} \]

\[ \text{chess25.eps} \]

\[ \text{chess26.eps} \]

on the horizontal and vertical edges. Then \( W_T \) represents \( X \). The tiling group \( \Gamma(T) \) is of the form

\[ \Gamma(T'_C) = \{h_i, v_i, i = 0, 1, 2\} \]
\[ v_1 h_0 = v_2 h_0 = h_1 v_0 = h_2 v_0, \]
\[ v_2 h_1 = v_0 h_1 = h_2 v_1 = h_0 v_1, \]
\[ v_0 h_2 = v_1 h_2 = h_0 v_2 = h_1 v_2 \}

Since \( h_0 = h_1 = h_2, v_0 = v_1 = v_2 \) and \( h_0 v_0 = v_0 h_0, \)
\( \Gamma(T) \cong \mathbb{Z}^2, \) and every hole appears closable.

With a different representation of \( X \) as a Wang shift we obtain more information. Let \( T' \) be the set of Wang tiles

\[
\begin{array}{cccccccccc}
1 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\
2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 2 & 0 & 2 & 0 & 2 & 1 & 2 \\
1 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 1 \\
0 & 1 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\
\end{array}
\]

with the colours \( h_{ij} = [i \ j] \) on the horizontal and \( v_i^j = [j \ i] \) on the vertical edges, where \( i, j \in \{0, 1, 2\} \) and \( i \neq j \). Then \( W_{T'} \) represents \( X \).

There exists a group homomorphism \( \phi : \Gamma(T') \rightarrow \mathbb{Z} \) with

\[
\phi(h_{01}) = \phi(h_{12}) = \phi(h_{20}) = \phi(v_0^1) = \phi(v_1^2) = \phi(v_2^0) = 1,
\]
\[
\phi(h_{10}) = \phi(h_{21}) = \phi(h_{02}) = \phi(v_1^0) = \phi(v_2^1) = \phi(v_0^2) = -1.
\]
This homomorphism detects that the hole with the edge

\[
\begin{array}{ccc}
1 & 2 & 1 \\
2 & ? & 0 \\
0 & 1 & 2 \\
\end{array}
\]

cannot be closed, no matter how much it is enlarged initially.

This example raises the alarming possibility that more and more complicated Wang-representations of a SFT $X$ will give more and more combinatorial information about $X$. Remarkably, this is not the case.

**Theorem 4.** In all currently understood topologically mixing examples of $\mathbb{Z}^2$-SFT’s there exists a Wang-representation $W_T$ of $X$ which contains all the combinatorial information obtainable from all possible Wang-representations of $X$.

In order to make this statement comprehensible one has to express it in terms of the continuous cohomology of $X$. 
4. COHOMOLOGICAL RIGIDITY

Let $X \subset A^{\mathbb{Z}^2}$ be a SFT and $G$ a discrete group with identity element $1_G$. A map $c: \mathbb{Z}^2 \times X \longrightarrow G$ is a cocycle for the shift-action $\sigma$ of $\mathbb{Z}^2$ on $X$ if $c(\mathbf{n}, \cdot): X \longrightarrow G$ is continuous for every $\mathbf{n} \in \mathbb{Z}^2$ and

$$c(\mathbf{m} + \mathbf{n}, x) = c(\mathbf{m}, \sigma^n(x))c(\mathbf{n}, x)$$

for all $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$. One can interpret this equation as path-independence.

A cocycle $c: \mathbb{Z}^2 \times X \longrightarrow G$ is a homomorphism if $c(\mathbf{n}, \cdot)$ is constant for every $\mathbf{n} \in \mathbb{Z}^2$, and $c$ is a coboundary if there exists a continuous map $b: X \longrightarrow G$ such that

$$c(\mathbf{n}, x) = b(\sigma^n(x))^{-1}b(x)$$

for all $x \in X$ and $\mathbf{n} \in \mathbb{Z}^2$. Two cocycles $c, c': \mathbb{Z}^2 \times X \longrightarrow G$ are cohomologous with continuous transfer function $b: X \longrightarrow G$, if

$$c(\mathbf{n}, x) = b(\sigma^n x)^{-1}c'(\mathbf{n}, x)b(x)$$

for all $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$.

Every Wang representation $W_T$ of $X$ defines a tiling cocycle $c_T: \mathbb{Z}^2 \times W_T \longrightarrow \Gamma(T)$ (or, equivalently,
$c_T: \mathbb{Z}^2 \times X \longrightarrow \Gamma(T))$ by

$$c_T((1, 0), w) = b(w_0), \ c_T((0, 1), w) = l(w_0)$$

for every Wang tiling $w \in W_T \subset T^{\mathbb{Z}^2}$, and by using the cocycle equation to extend $c_T$ to a map $\mathbb{Z}^2 \times W_T \longrightarrow \Gamma(T)$ (the relations $t(\tau)l(\tau) = r(\tau)b(\tau)$, $\tau \in T$, in the tiling group are precisely what is needed to allow such an extension).

**Definition 5.** A cocycle $c^*: \mathbb{Z}^2 \times X \longmapsto G^*$ with values in a discrete group $G^*$ is *fundamental* if the following is true: for every discrete group $G$ and every cocycle $c: \mathbb{Z}^2 \times X \longmapsto G$ there exists a group homomorphism $\theta: G^* \longmapsto G$ such that $c$ is cohomologous to the cocycle $\theta \cdot c^*: \mathbb{Z}^2 \times X \longmapsto G$, defined by

$$\theta \cdot c^*(n, x) = \theta(c^*(n, x))$$

for every $n \in \mathbb{Z}^2$ and $x \in X$.

A more precise form of Theorem 4 can be stated as follows.
Theorem 6. In all currently understood mixing examples of topologically mixing $\mathbb{Z}^2$-SFT’s there exists a fundamental cocycle.

In these examples the fundamental cocycle can be calculated explicitly.

Although one can make analogous definitions for classical (one-dimensional) SFT’s, they never have fundamental cocycles. The existence of fundamental cocycles is a (not very well understood) rigidity phenomenon specific to certain $\mathbb{Z}^d$-actions.

5. Isomorphism Rigidity

Let $d \geq 1$, and let $X$ be a compact abelian group with normalized Haar measure $\lambda_X$. A $\mathbb{Z}^d$-action $\alpha : \mathbb{N} \mapsto \alpha^n$ by continuous automorphisms of $X$ is called an algebraic $\mathbb{Z}^d$-action on $X$. An algebraic $\mathbb{Z}^d$-action $\alpha$ on $X$ is expansive if there exists an open neighbourhood $\mathcal{U}$ of the identity $0_X$ in $X$ with $\bigcap_{n \in \mathbb{Z}^d} \alpha^{-n}(\mathcal{U}) = \{0_X\}$, and irreducible if every closed $\alpha$-invariant subgroup $Y \subsetneq X$ is finite.
Suppose that $\alpha$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$. An $\alpha$-invariant probability measure $\mu$ on the Borel field $\mathcal{B}_X$ of $X$ is ergodic if $\mu\left(\bigcup_{n \in \mathbb{Z}^d} \alpha^{-n}(B)\right) \in \{0, 1\}$ for every $B \in \mathcal{B}_X$, and mixing if

$$\lim_{n \to \infty} \mu(B \cap \alpha^{-n}(B')) = \mu(B)\mu(B')$$

for all $B, B' \in \mathcal{B}_X$. The action $\alpha$ is ergodic or mixing if $\lambda_X$ is ergodic or mixing.

Let $\alpha_1, \alpha_2$ be algebraic $\mathbb{Z}^d$-actions on compact abelian groups $X_1$ and $X_2$, respectively. A Borel bijection $\phi : X_1 \hookrightarrow X_2$ is a measurable conjugacy of $\alpha_1$ and $\alpha_2$ if

$$\lambda_{X_1} \phi^{-1} = \lambda_{X_2}$$

and

$$\phi \cdot \alpha_1^n(x) = \alpha_2^n \cdot \phi(x)$$

for every $n \in \mathbb{Z}^d$ and $\lambda_{X_1}$-a.e. $x \in X_1$.

A continuous group isomorphism $\phi : X_1 \hookrightarrow X_2$ is an algebraic conjugacy of $\alpha_1$ and $\alpha_2$ if it satisfies the second equation above for every $n \in \mathbb{Z}^d$ and $x \in X_1$. 
The actions $\alpha_1, \alpha_2$ are *measurably* (resp. *algebraically*) *conjugate* if there exists a measurable (resp. algebraic) conjugacy between them.

Finally we call a map $\phi: X_1 \mapsto X_2$ *affine* if there exist a continuous group isomorphism $\psi: X_1 \mapsto X_2$ and an element $x' \in X_2$ such that

$$\phi(x) = \psi(x) + x'$$

for every $x \in X_1$.

In this part of the talk we restrict ourselves to irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact, zero-dimensional groups. Every such action can be shown to be a *SFT*. In our earlier discussion of *SFT*'s we were interested in topological conjugacy invariants. Here we turn to measurable conjugacy and exhibit another rigidity phenomenon specific to $\mathbb{Z}^d$-actions with $d > 1$. Consider the following examples.

**Example 7.** The shift automorphisms

$$(\sigma x)_n = x_{n+1}$$
on the compact abelian groups
\[ X = (\mathbb{Z}/4\mathbb{Z})^\mathbb{Z}, \]
\[ Y = ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}))^\mathbb{Z}. \]
are measurably (even topologically) conjugate, but
the groups \( X \) and \( Y \) are not algebraically isomorphic. Neither of the \( \mathbb{Z} \)-actions defined by \( \sigma \) on \( X \) and \( Y \) is irreducible. However, the shift \( \sigma \) on \( Z = (\mathbb{Z}/3\mathbb{Z})^\mathbb{Z} \) is irreducible.

**Example 8.** Denote by \( \sigma \) the shift-action of \( \mathbb{Z}^d \) on
\[ \tilde{X} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^d} \]
given by
\[ (\sigma_m x)_n = x_{m+n} \]
for every \( x = (x_n) \in \tilde{X} \) and \( m \in \mathbb{Z}^d \).

For every nonempty finite set \( E \subset \mathbb{Z}^d \) we denote by
\( X_E \subset \tilde{X} \) the closed shift-invariant subgroup consisting of all \( x \in \tilde{X} \) whose coordinates sum to 0 in every translate of \( E \) in \( \mathbb{Z}^d \). If \( E \) has at least two points
then \( X_E \) is uncountable and the restriction \( \sigma_E \) of \( \sigma \)
to \( X_E \) is an expansive algebraic \( \mathbb{Z}^d \)-action.

For \( d = 2 \) and the subset
\[ E = \{(0,0), (1,0), (0,1)\} \subset \mathbb{Z}^2, \]
the $\mathbb{Z}^2$-action $\sigma_E$ on $X_E$ is called Ledrappier’s example: $\sigma_E$ is mixing and expansive, but not mixing of order 3 (for every $n \geq 0$, $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0$).

We also consider the subsets

$$E_1 = \{(0, 0), (1, 0), (2, 0), (1, 1), (0, 2)\},$$

$$E_2 = \{(0, 0), (2, 0), (0, 1), (1, 1), (0, 2)\},$$

$$E_3 = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (0, 2)\}.$$

of $\mathbb{Z}^2$. The shift-actions $\sigma_i = \sigma_{E_i}$ of $\mathbb{Z}^2$ on $X_i = X_{E_i}$ are mixing, irreducible and expansive.

For every $n \in \mathbb{Z}^2$, the automorphisms $\sigma^n_i$ are measurably conjugate, and none of the standard invariants of measurable conjugacy distinguish between these actions.

The shift-action $\sigma_4$ of $\mathbb{Z}^2$ on $X_{E_4}$ with

$$E_4 = \{(0, 0), (2, 0), (0, 2)\}$$

is also mixing and expansive, but reducible: the map $\phi : X_{E_4} \to X_E$, given by

$$\phi(x)_{(m,n)} = x_{m,n} + x_{(m+1,n)} + x_{(m,n+1)},$$

is a shift-commuting surjective group homomorphism with kernel $X_E$. 

20
As we saw above, single automorphisms of compact abelian groups can be measurably conjugate without being algebraically conjugate. For \( d \geq 2 \) the situation is different.

**Theorem 9** (Kitchens-S). *Let \( d > 1 \), and let \( \alpha_1 \) and \( \alpha_2 \) be measurably conjugate irreducible mixing algebraic \( \mathbb{Z}^d \)-actions on compact zero-dimensional abelian groups \( X_1 \) and \( X_2 \), respectively. Then any measurable conjugacy \( \phi: X_1 \rightarrow X_2 \) of \( \alpha_1 \) and \( \alpha_2 \) is \( \lambda_{X_1} \)-a.e. equal to an affine map.*

From this theorem one can conclude that the \( \mathbb{Z}^2 \)-actions \( \sigma_i \) above are not measurably conjugate.

At this stage of the game, the assumption of irreducibility is still crucial. However, Theorem 9 may well be true for measurably conjugate mixing algebraic \( \mathbb{Z}^d \)-actions on compact zero-dimensional abelian groups *provided that these actions have zero entropy* (otherwise one has Bernoulli factors which prevent any form of isomorphism rigidity).