MODULE THEORY OF LINEAR SYSTEMS

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Abstract

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ROTA'S THEOREM
UNIVERSALITY OF THE SHIFT

$$S_-(x_0, x_1, \ldots,) = (x_1, x_2, \ldots,)$$

$$(x_0, x_1, \ldots,) \longmapsto \sum_{i=0}^{\infty} \frac{x_i}{z^{i+1}}$$

$$\begin{array}{cccc} & x & \longrightarrow & (x, Ax, A^2x, \ldots) & \\ A & \downarrow & & \downarrow & S_- \\ & Ax & \longrightarrow & (Ax, A^2x, \ldots) & \end{array}$$

$$\begin{array}{cccc} & V & \longrightarrow & M \subset z^{-1}V[[z^{-1}]] & \\ A & \downarrow & & \downarrow & S_- \\ & V & \longrightarrow & M \subset z^{-1}V[[z^{-1}]] & \end{array}$$

$$\begin{array}{ccc} v_0 + \cdots + v_s z^s & \longmapsto & v_0 + Av_1 + \cdots + A^s v_s \\ S_+ \downarrow & & \downarrow A \\ z(v_0 + \cdots + v_s z^s) & \longmapsto & A(v_0 + Av_1 + \cdots + A^s v_s) \end{array}$$

$$\begin{array}{ccc} V[z] & \overset{\Phi}{\longrightarrow} & V \\ S_+ \downarrow & & \downarrow A \\ V[z] & \overset{\Phi}{\longrightarrow} & V \end{array}$$

$$\operatorname{Ker} \Phi = (zI - A)V[z]$$

$$A \simeq S_+ | (V[z]/(zI - A)V[z])$$

SUBMODULE REPRESENTATION

$$\operatorname{Ker}\Phi = (zI - A)V[z]$$

$$V \simeq V[z]/\operatorname{Ker}\Phi = V[z]/(zI - A)V[z]$$

$$M \subset V[z] \Leftrightarrow M = DV[z]$$

D **unique up to a right unimodular factor**

$$M \subset z^{-1}V[[z^{-1}]] \Leftrightarrow M = X^D$$

D **unique up to a left unimodular factor**

Reduction to column (row) proper form.

POLYNOMIAL MODELS

$$D(z) \in F^{m\times m}[z], \det D(z) \neq 0$$

$$\pi_D : F^m[z] \longrightarrow F^m[z]$$
$$\pi_D p = D\pi_- D^{-1} p$$

$$X_D = \operatorname{Im} \pi_D = \{f \in F^m[z] | f = Dh, \quad h \in z^{-1}F^m[[z^{-1}]]\}$$

$$S_D p = \pi_D z p(z)$$

$$\dim X_D = \deg \det D$$

GENERIC SCALAR POLYNOMIAL MODELS

$$d(z) = z^n + d_{n-1}z^{n-1} + \cdots + d_0 = \Pi_{j=1}^n (z - \alpha_j)$$

$$X_d = \{p | \deg p < n\}$$

$$p_i(z) = \Pi_{j\neq i}^n (z - \alpha_j) = \frac{d(z)}{z - \alpha_i}$$

$$(S_d p_i)(z) = \alpha_i p_i(z)$$

RATIONAL MODELS

$$D(z) \in F^{m\times m}[z], \det D(z) \neq 0$$

$$\pi^D : z^{-1}F^p[[z^{-1}]] \longrightarrow z^{-1}F^p[[z^{-1}]]$$
$$\pi^D h = \pi_- D^{-1}\pi_+ Dh$$

$$X^D = \operatorname{Im}\pi^D$$

$$S^D h = \pi_- z h(z)$$

$$d(z) = z^n + d_{n-1}z^{n-1} + \cdots + d_0 = \Pi_{j=1}^n (z-\alpha_j)$$

$$X^d = \{\frac{p}{d} | \deg p < n\}$$

$$h_i(z) = \frac{1}{z-\alpha_i}$$

$$(S^d h_i)(z) = \pi_- \frac{z}{z-\alpha_i} = \pi_- \frac{z-\alpha_i+\alpha_i}{z-\alpha_i} = \alpha_i h_i(z)$$

$$X_D = DX^D$$

FACTORIZATIONS AND INVARIANT SUBSPACES

$$M \subset X_D$$

$$S_D M \subset M \Leftrightarrow M = E X_F; \quad D = EF$$

$$M \subset X^D$$

$$S^D M \subset M \Leftrightarrow M = X^F; \quad D = EF$$

Direct connection between polynomial matrices and geometry. Linear system theory and factorization theory are equivalent.

DUALITY

$$f, g \in F^p((z^{-1}))$$

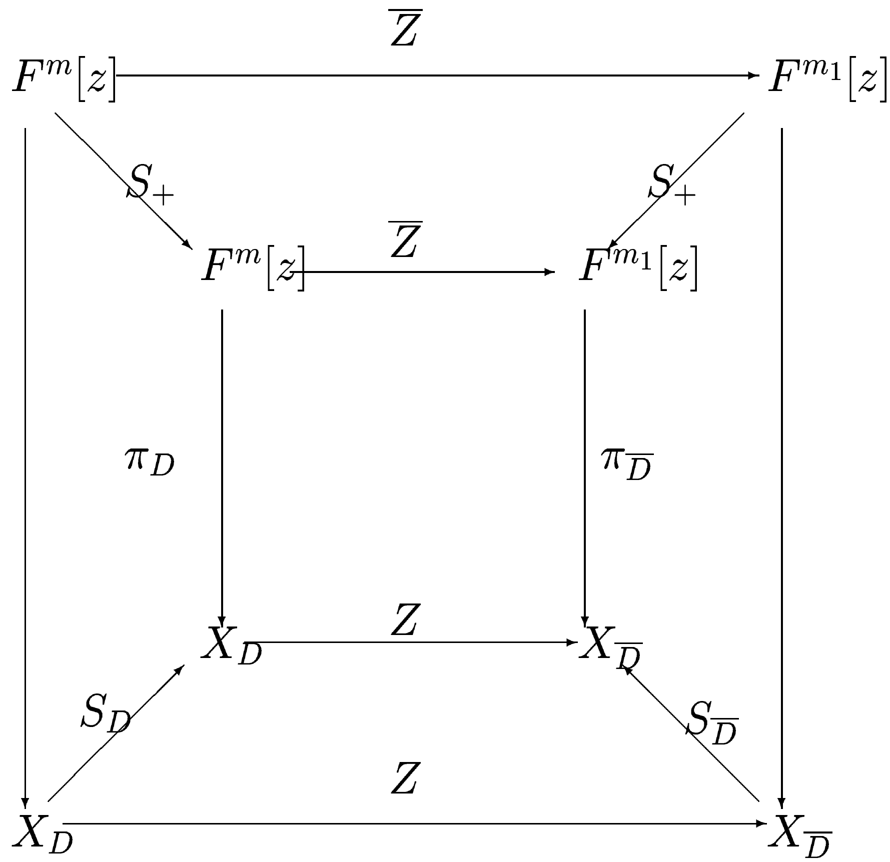
$$[g, f] = \Sigma_{i=-\infty}^{\infty} \tilde{g}_{-i-1} f_i$$

$$(F^p[z])^{\perp} = F^p[z]$$

$$X_D \simeq F^p[z]/DF^p[z]$$

$$(X_D)^* \simeq (F^p[z]/DF^p[z])^* \simeq^{\perp} (DF^p[z]) = X^{\tilde{D}}$$

COMMUTANT LIFTING THEOREM



$$\overline{Z}f = Mf, \quad M \in F^{m_1 \times m}[z]$$

$$\overline{Z}\mathrm{Ker}\,\pi_D \subset \mathrm{Ker}\,\pi_{D_1}$$

$$MD = D_1 N$$

$$Zf = \pi_{D_1} Mf$$

SPECTRAL MAPPING THEOREM

$$KerZ = FX_G$$

$D = FG$ **and** G **is a g.c.r.d. of** D **and** E

$$ImZ = \overline{E}X_{\overline{G}}$$

$\overline{D} = \overline{FG}$ **and** $\overline{F}$ **is a g.c.l.d. of** $\overline{D}$ **and** $\overline{E}$

Z **surjective** $\Leftrightarrow$ $\overline{E}$ **and** $\overline{D}$ **are left coprime**

$\exists A, B$ **for which** $\overline{D}B - \overline{E}A = I$

Z **injective** $\Leftrightarrow$ D **and** E **are right coprime**

$\exists \overline{A}, \overline{B}$ **for which** $\overline{B}D - \overline{A}E = I$

HANKEL OPERATORS

$$\boxed{\Sigma_j\, u_j z^j \mapsto \Sigma_j\, A^j B u_j}$$

$$\boxed{\xi \mapsto C(zI - A)^{-1}\xi = \Sigma_{i=1}^{\infty}\, \frac{CA^{i-1}\xi}{z^i}}$$

$$\Sigma_j\, u_j z^j \mapsto \Sigma_{i=1}^{\infty}\, \frac{CA^{i-1}\, \Sigma_j\, A^j B u_j}{z^i} = \sum_{i=1}^{\infty} \frac{1}{z^i} \sum_j CA^{i+j-1} B u_j$$

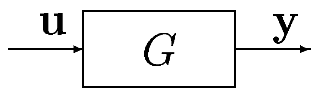
$$= \Sigma_{i=1}^{\infty}\, \frac{y_i}{z^i}$$

$$y_i = \Sigma_j\, CA^{i+j-1} B u_j = \Sigma_j\, G_{i+j} u_j$$

$$G_i = CA^{i-1}B$$

$$H_G f = \pi_- G f$$

LINEAR SYSTEMS



$$x_{n+1} = Ax_n + Bu_n$$

$$y_n = Cx_n + Du_n$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$G(s) = D + C(sI - A)^{-1}B = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} RAR^{-1} & RB \\ \hline CR^{-1} & D \end{array}\right)$$

$$\begin{aligned} G &= G_0 + \frac{G_1}{s} + \frac{G_2}{s^2} + \cdots \\ &= NM^{-1} = \overline{M}^{-1}\overline{N} \\ &= VT^{-1}U + W \end{aligned}$$

THE FUHRMANN SHIFT REALIZATION

$$G = VT^{-1}U + W.$$

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\begin{cases} A = S_T \\ B\xi = \pi_T U\xi, \\ Cf = (VT^{-1}f)_{-1} \\ D = G(\infty). \end{cases}$$

The realization is reachable

$\Leftrightarrow$

T **and** U **are left coprime**

The realization is observable

$\Leftrightarrow$

T **and** V **are right coprime.**

A nonsingular polynomial matrix D**,**
unique up to a right unimodular factor,
is equivalent to a reachable pair (A, B)**,**
unique up to similarity.

STRICT SYSTEM EQUIVALENCE (FSE)

$$G = VT^{-1}U + W = \overline{V}\,\overline{T}^{-1}\overline{U} + \overline{W}$$

The shift realizations are isomorphic

$$\Leftrightarrow$$

$$\begin{pmatrix} \overline{M} & 0 \\ X & I \end{pmatrix} \begin{pmatrix} T & U \\ -V & W \end{pmatrix} = \begin{pmatrix} \overline{T} & \overline{U} \\ -\overline{V} & \overline{W} \end{pmatrix} \begin{pmatrix} M & -Y \\ 0 & I \end{pmatrix}$$

$$\begin{cases} \overline{M}, \overline{T} & \textbf{left coprime} \\ T, M & \textbf{right coprime} \end{cases}$$

DOUBLY COPRIME FACTORIZATIONS

$$G = NM^{-1} = \overline{M}^{-1}\overline{N}$$

$$\begin{pmatrix} \overline{V} & -\overline{U} \\ -\overline{N} & \overline{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\delta \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \delta \begin{pmatrix} \overline{V} & -\overline{U} \\ -\overline{N} & \overline{M} \end{pmatrix} = \delta(G)$$

$$G = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \textbf{ proper, rational } \& \delta(G_1) = n$$

$$\delta(G) = n \Leftrightarrow ImH_{G_2} \subset ImH_{G_1}$$

$$G_1 = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \text{ minimal}$$

$$\exists B_2, C_2; \quad G_2 = \left(\begin{array}{c|c} A_1 & B_2 \\ \hline C_1 & D_2 \end{array}\right)$$

HANKEL OPERATORS

$$G \ \textbf{RATIONAL}\ p \times m.$$

$$H_G : F^m[z] \longrightarrow z^{-1}F^p[[z^{-1}]]$$

$$H_G f = \pi_-(Gf), \quad for\ f \in F^m[z].$$

$$S_- H_G = H_G S_+$$

$$S_+ Ker H_G \subset Ker H_G$$

$$S_- Im H_G \subset Im H_G$$

$$Ker H_G = D F^m[z]$$

$$Im H_G = X^E$$

$$G = ND^{-1} = E^{-1}M$$

$$(\textbf{CoprimeFactorizations})$$

$$MD = EN \qquad X_D \simeq X_E$$

LEFT WIENER-HOPF FACTORIZATIONS AND TOEPLITZ OPERATORS

$$G \in F_{pr}(z)^{p\times p}$$

$$G(z) = \Gamma(z)\Delta(z)U(z)$$

$U(z)$ **polynomial unimodular**

$\Gamma(z)$ **proper rational unimodular**

$$\Delta(z) = \mathrm{diag}\,(z^{\mu_1}, \ldots, z^{\mu_p}), \mu_1 \geq \cdots \geq \mu_p$$

$$\mathcal{T}_G : F^p[z] \longrightarrow F^p[z]$$

$$\mathcal{T}_G f = \pi_+ G f$$

$\mathcal{T}_G$ **is injective** $\Leftrightarrow \mu_i \geq 0$

$\mathcal{T}_G$ **is surjective** $\Leftrightarrow \mu_i \leq 0$

RATIONAL MODELS

$$(C, A) \in F^{p\times n} \times F^{n\times n} \textbf{ observable}$$

$$C(zI - A)^{-1} = T(z)^{-1}H(z)$$

$$X_{(C,A)} = \{C(zI - A)^{-1}\xi | \xi \in F^n\} \subset F_-^p(z)$$

$$X^T = \{h \in F_-^p(z) | \pi_- Th = 0\}$$

$$\boxed{X_{(C,A)} = X^T}$$

$$T(z)C(zI - A)^{-1}\xi = H(z)\xi \Rightarrow X \subset X^T.$$

$$\xi \mapsto C(zI - A)^{-1}\xi \textbf{ injective} \Rightarrow \dim X = n$$

$$T(z) = U(z)\Delta(z)\Gamma(z) \textbf{ right W-H factorization}$$

$$\Delta(z) = \operatorname{diag}(z^{\nu_1}, \ldots, z^{\nu_p}),$$
$$h \in X^T \Leftrightarrow \Gamma h \in X^{U\Delta} \Rightarrow \dim X^T = \dim X^{U\Delta}$$

$$U\Delta h \in F^p[z] \Leftrightarrow \Delta h \in F^p[z] \Rightarrow \dim X^T = \dim X^\Delta = \Sigma_{i=1}^p \nu_i = n$$

MODULE STRUCTURE

$$\begin{cases} \phi : F^n \longrightarrow X^T \\ \boxed{\xi \mapsto C(zI - A)^{-1}\xi} \end{cases}$$

$$\boxed{S_- X_{(C,A)} \subset X_{(C,A)}}$$

$$A^T = S^T = S_- | X_{(C,A)}$$
$$C^T h = (h)_{-1}$$

$$A^T \phi(\xi) = \pi_- z C(zI - A)^{-1}\xi = C(zI - A)^{-1} A\xi \phi(A\xi)$$
$$C^T \phi(\xi) = (C(zI - A)^{-1}\xi)_{-1}$$

REALIZATION THEORY

$$G = ND^{-1} = \Sigma_{i=1}^{\infty} \frac{G_i}{z^{i+1}} = C(zI - A)^{-1}B$$

ABSTRACT

$$\begin{array}{ccc} F^m[z] & \overset{H_G}{\longrightarrow} & z^{-1}F^p[[z^{-1}]] \\ \searrow & & \nearrow \\ & F^m[z]/KerH_G & \end{array}$$

CHOOSE REPRESENT.

$$\begin{array}{ccc} F^m[z] & \overset{H_G}{\longrightarrow} & z^{-1}F^p[[z^{-1}]] \\ \pi_D \searrow & & \nearrow \\ & X_D & \end{array}$$

$$\begin{array}{rcl} A & = & S_D \\ B\xi & = & \pi_D\xi \\ Cf & = & (ND^{-1}f)_{-1} \end{array}$$

CHOOSE BASIS

CONCRETE

Matrix Representation

REALIZATION THEORY

$$g = pq^{-1}$$

$$\begin{array}{|rcl|}\hline A & = & S_q \\ B\xi & = & \pi_q \xi \\ Cf & = & (pq^{-1}f)_{-1} \\ \hline\end{array}$$

CONTROLLABILITY REALIZATION

Standard basis: $\{1, z, \ldots, z^{n-1}\}$

$$A = \begin{pmatrix} 0 & . & . & . & -q_0 \\ 1 & & & & . \\ & . & & & . \\ & & . & & . \\ & & & 1 & -q_{n-1} \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, C = \begin{pmatrix} g_1 & . & . & . & g_n \end{pmatrix}$$

CONTROLLER REALIZATION

Control basis: $\{e_1, \ldots, e_n\}$

$$A = \begin{pmatrix} 0 & 1 & & & \\ & & . & & \\ & & & . & \\ & & & & 1 \\ -q_0 & . & . & . & -q_{n-1} \end{pmatrix}, B = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} p_0 & . & . & . & p_{n-1} \end{pmatrix}$$

REALIZATION THEORY

$$g = q^{-1}p$$

$$\begin{array}{rcl} A & = & S_q \\ B\xi & = & \pi_q p\xi \\ Cf & = & (q^{-1}f)_{-1} \end{array}$$

OBSERVER REALIZATION

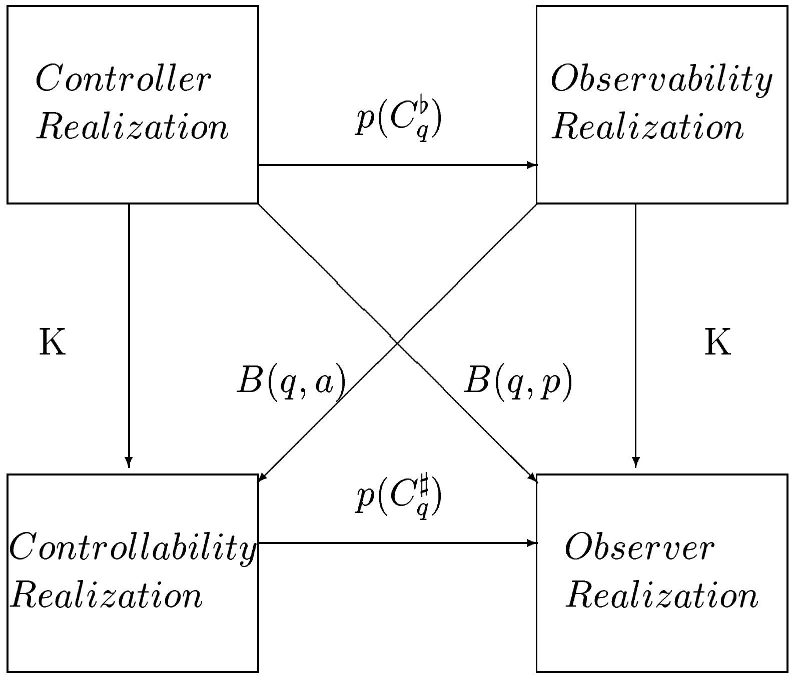
Standard basis: $\{1, z, \ldots, z^{n-1}\}$

$$A = \begin{pmatrix} 0 & . & . & . & -q_0 \\ 1 & & & & . \\ & . & & & . \\ & & . & & . \\ & & & 1 & -q_{n-1} \end{pmatrix}, B = \begin{pmatrix} p_0 \\ . \\ . \\ . \\ p_{n-1} \end{pmatrix}, C = \begin{pmatrix} 0 & . & . & 0 & 1 \end{pmatrix}$$

OBSERVABILITY REALIZATION

Control basis: $\{e_1, \ldots, e_n\}$

$$A = \begin{pmatrix} 0 & 1 & & & \\ & & . & & \\ & & & . & \\ & & & & 1 \\ -q_0 & . & . & . & -q_{n-1} \end{pmatrix}, B = \begin{pmatrix} g_1 \\ . \\ . \\ . \\ g_{n-1} \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & . & . & 0 \end{pmatrix}$$



$$C_q^\sharp = \begin{pmatrix} 0 & . & . & . & -q_0 \\ 1 & & & & . \\ & . & & & . \\ & & . & & . \\ & & & 1 & -q_{n-1} \end{pmatrix} \quad C_q^\flat = \begin{pmatrix} 0 & 1 & & & \\ & & . & & \\ & & & . & \\ & & & & 1 \\ -q_0 & . & . & . & -q_{n-1} \end{pmatrix}$$

$$K = [I]_{co}^{st} = \begin{pmatrix} q_1 & . & . & . & q_{n-1} & 1 \\ . & . & . & . & 1 & \\ . & . & . & . & & \\ . & . & . & & & \\ q_{n-1} & 1 & & & & \\ 1 & & & & & \end{pmatrix}$$

CONTINUED FRACTION REALIZATION

$$q_{-1} = q, \quad q_0 = p$$
$$q_{i+1}(z) = a_{i+1}(z)q_i(z) - \beta_i q_{i-1}(z), \quad \deg q_{i+1} < \deg q_i$$

$$Q_{-1} = 0, \quad Q_0 = 1$$
$$Q_{k+1}(z) = a_{k+1}(z)Q_k(z) - \beta_k Q_{k-1}(z)$$

$$g(z) = \frac{\beta_0}{a_1(z) - \cfrac{\beta_1}{a_2(z) - \cfrac{\beta_2}{a_3(z) - \ddots \cfrac{\beta_{r-2}}{a_{r-1}(z) - \cfrac{\beta_{r-1}}{a_r(z)}}}}}$$

$$\begin{aligned} A &= S_q \\ B\xi &= \pi_q \xi \\ Cf &= (pq^{-1}f)_{-1} \end{aligned}$$

CONTINUED FRACTION REALIZATION

LANCZOS Basis:

$$\{1, z, \ldots, z^{n_1-1}, Q_1, zQ_1, \ldots, z^{n_2-1}Q_1, \ldots, Q_{r-1}, \ldots, z^{n_{r-1}-1}Q_{r-1}\}$$

$$A = \begin{pmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & . & & \\ & . & . \; . & & \\ & & . \; . & & A_{r-1\,r} \\ & & & A_{r\,r-1} & A_{rr} \end{pmatrix}$$

$$A_{ii} = \begin{pmatrix} 0 & . & . & . & -a_0^{(i)} \\ 1 & & & & . \\ & . & & & . \\ & & . & & . \\ & & & 1 & -a_{n_i-1}^{(i)} \end{pmatrix}, \; i = 1, \ldots r,$$

$$A_{i+1\,i} = \begin{pmatrix} 0 & . & . & 0 & 1 \\ . & & & & 0 \\ . & & & & . \\ . & & & & . \\ 0 & . & . & . & 0 \end{pmatrix}$$

$$A_{i\,i+1} = \beta_{i-1} A_{i+1\,i}$$

$$b = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & . & . & \beta_0 & 0 & . & . & 0 \end{pmatrix}$$

LYAPUNOV BALANCED REALIZATION

$$g = d^{-1}n \in H_{-}^{\infty}$$

$$H_{\frac{n}{d}} : X^{d^*} \longrightarrow X^d$$

$$\sigma_1 > \cdots > \sigma_n$$

$$\{\frac{p_i}{d^*}, \frac{p_i^*}{d}\}$$

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$$

$$A\Sigma + \Sigma\tilde{A} = -B\tilde{B}$$

$$\tilde{A}\Sigma + \Sigma A = -\tilde{C}C.$$

BALANCED REALIZATION

Schmidt basis: $\{p_1^*, \ldots, p_n^*\}$

$$A = \left(\frac{\epsilon_j b_i b_j}{\lambda_i + \lambda_j}\right), B = (b_1, \ldots, b_n), C = (\epsilon_1 b_1, \ldots, \epsilon_n b_n)$$

OUTPUT INJECTION vs. STATE FEEDBACK

$$C(zI - A)^{-1} = T(z)^{-1}H(z)$$

$$T(z)C = H(z)(zI - A)$$

$$\exists\phi : F^n \longrightarrow X^T$$

$$\xi \mapsto C(zI - A)^{-1}\xi$$

$$\overline{\phi} : F^n \longrightarrow X_T$$
$$\xi \mapsto H(z)\xi$$

$$(zI - A)^{-1}B = H(z)D(z)^{-1}$$

$$BD(z) = (zI - A)H(z)$$

$$\exists\phi : X_D \longrightarrow F^n$$

$$\boxed{\Sigma_i f_i z^i \mapsto \Sigma_i A^i B f_i}$$

1. Use row spaces
2. Use transposed matrices.

INVARIANT SUBSPACES AND FACTORIZATIONS

$$(C, A) \in F^{p\times n} \times F^{n\times n} \textbf{ observable}$$

$$C(zI - A)^{-1} = T(z)^{-1}H(z)$$

$$F^n = \mathcal{V} \oplus \mathcal{W}$$

$$(C, A) = \left(\begin{pmatrix} C_1 & C_2 \end{pmatrix}, \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \right)$$

$$(C_1, A_1) \in F^{p\times n_1} \times F^{n_1\times n_1} \textbf{ observable}$$
$$C_1(zI - A_1)^{-1} = T_1(z)^{-1}H_1(z)$$

$$C(zI - A)^{-1} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} (zI - A_1)^{-1} & -(zI - A_1)^{-1}A_3(zI - A_2)^{-1} \\ 0 & (zI - A_2)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} C_1(zI - A_1)^{-1} & -C_1(zI - A_1)^{-1}A_3(zI - A_2)^{-1} + C_2(zI - A_2)^{-1} \end{pmatrix}$$

$$= T_1^{-1}T_2^{-1}H(z)$$

$$\boxed{T = T_2T_1,\ X^{T_1} \subset X^T}$$

$$h \in X_{T_1} \Rightarrow Th = T_2(T_1h) \in F^p[z]$$

CONTROLLED INVARIANT SUBSPACES

$$A : F^n \longrightarrow F^n, \quad B : F^m \longrightarrow F^n$$
$$\dot{x} = Ax + Bu$$

$\mathcal{V} \subset F^n$ is **controlled invariant** if $\exists F : F^n \longrightarrow F^m$

$$(A + BF)\mathcal{V} \subset \mathcal{V}$$

$\mathcal{V} \subset F^n$ is **controlled invariant**

$$\Leftrightarrow$$
$$A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B$$

CONDITIONED INVARIANT SUBSPACES

$$A : F^n \longrightarrow F^n, \quad C : F^n \longrightarrow F^m$$

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$$

$\mathcal{V} \subset F^n$ is **conditioned invariant** if $\exists H : F^p \longrightarrow F^n$

$$(A + HC)\mathcal{V} \subset \mathcal{V}$$

$\mathcal{V} \subset F^n$ is **conditioned invariant**

$$\Leftrightarrow$$
$$A(\mathcal{V} \cap \operatorname{Ker} C) \subset \mathcal{V}$$

CONDITIONED INVARIANT SUBSPACES - PREVIEW

$$(C,A)\in F^{p\times n}\times F^{n\times n}\ \textbf{observable}$$

$$C(zI-A)^{-1}=T(z)^{-1}H(z)\Rightarrow$$

$$\begin{cases} T(z)C &= H(z)(zI-A)\\ H(z)KC &= H(z)KC\end{cases}$$

$$(T(z)+H(z)K)C=H(z)(zI-A+KC)$$

$$C(zI-A+KC)^{-1}=(T(z)+H(z)K)^{-1}H(z)=\overline{T}(z)^{-1}H(z)$$

$$(A-KC)\mathcal{V}\subset\mathcal{V}\Leftrightarrow S^{\overline{T}}\overline{\mathcal{V}}\subset\overline{\mathcal{V}}$$

$$\overline{\mathcal{V}}=\{C(zI-A+KC)^{-1}\xi|\xi\in\mathcal{V}\}=X^{\overline{T}_1}$$

$$X_{\overline{T}}=\overline{T}X^{\overline{T}}\Rightarrow\overline{T}X^{\overline{T}_1}=\overline{T}_2\overline{T}_1X^{\overline{T}_1}=\overline{T}_2X_{\overline{T}_1}$$

$$\boxed{X_T\text{"}=\text{"}X_{\overline{T}}}$$

$$\mathcal{V}=\overline{T}_2X_{\overline{T}_1}=X_T\cap T_2F^p[z]=X_T\cap M$$

$$\mathcal{V}=X_T\cap M,\quad zM\subset M$$
$$f\in\mathcal{V}\cap\operatorname{Ker}C\Rightarrow S_Tf=zf\in X_T\cap M=\mathcal{V}.$$

GEOMETRIC CHARACTERIZATION OF CONTROLLED INVARIANT SUBSPACES

$$G = D_l^{-1} E_l = E_r D_r^{-1}$$

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

$$\begin{cases} A = S^{D_r} \\ B\xi = \pi_- D_r^{-1} \xi, \\ Ch = (E_r h)_{-1} \\ D = G(\infty). \end{cases}$$

$\mathcal{V} \subset X^{D_r}$ is **controlled invariant**

$$\Leftrightarrow$$

$$\mathcal{V} = \pi^{D_r} L$$

$L \subset z^{-1} F^m[[z^{-1}]]$ **a submodule**, $L = X^E$

NO UNIQUENESS

GEOMETRIC CHARACTERIZATION OF CONDITIONED INVARIANT SUBSPACES

$$G = D_l^{-1} E_l = E_r D_r^{-1}$$

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

$$\begin{cases} A = S_{D_l} \\ B\xi = \pi_{D_l} E_l \xi \\ Cp = (D_l^{-1} p)_{-1} = \lim_{z \to \infty} z D(z)^{-1} p(z) \\ D = G(\infty). \end{cases}$$

$\mathcal{V} \subset X_{D_l}$ is **conditioned invariant**

$$\Leftrightarrow$$

$$\mathcal{V} = X_{D_l} \cap M$$

$M \subset F^p[z]$ a **submodule**, $M = TF^p[z]$

STABILIZABILITY AND DETECTABILITY SUBSPACES

$$F = \mathbf{R}, \mathbf{C}$$

$$A : F^n \longrightarrow F^n, \quad B : F^m \longrightarrow F^n$$

$$\dot{x} = Ax + Bu$$

$\mathcal{V} \subset F^n$ is **inner stabilizable** if $\exists F : F^n \longrightarrow F^m$
$(A + BF)\mathcal{V} \subset \mathcal{V}$ and $A + BF|_{\mathcal{V}}$ is stable

$\mathcal{V} \subset F^n$ is **outer stabilizable** if $\exists F : F^n \longrightarrow F^m$
$(A + BF)\mathcal{V} \subset \mathcal{V}$ and $A + BF|_{X/\mathcal{V}}$ is stable

$$A : F^n \longrightarrow F^n, \quad C : F^n \longrightarrow F^p$$

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$$

$\mathcal{V} \subset F^n$ is **inner detectable** if $\exists H : F^p \longrightarrow F^n$
$(A + HC)\mathcal{V} \subset \mathcal{V}$ and $A + HC|_{\mathcal{V}}$ is stable

$\mathcal{V} \subset F^n$ is **outer detectable** if $\exists H : F^p \longrightarrow F^n$
$(A + HC)\mathcal{V} \subset \mathcal{V}$ and $A + HC|_{X/\mathcal{V}}$ is stable

NOTE: **Controllability of (A, B) implies outer (anti)stabilizability.**
Observability of (C, A) implies inner (anti)detectability.

ALGEBRAIC CHARACTERIZATION

$$G = D_l^{-1} E_l = E_r D_r^{-1}$$

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

$$\begin{cases} A = S_{D_l} \\ B\xi = \pi_{D_l} E_l \xi, \\ Cp = (pD_l^{-1})_{-1} = \lim_{z\to\infty} zD_l(z)^{-1}p(z) \\ D = G(\infty). \end{cases}$$

$\mathcal{V} \subset X_{D_l}$ is **outer antidetectable**

$\Leftrightarrow$

$\mathcal{V} = X_{D_l} \cap E_+ \mathbf{C}^m[z]$

E_+ **antistable**

& right Wiener-Hopf indices of $E_+^{-1} D_l$ **nonnegative**

$\mathcal{V} \subset X_{D_l}$ is **outer detectable**

$\Leftrightarrow$

$\mathcal{V} = X_{D_l} \cap \mathbf{C}^m[z] E_-$

E_- **stable**

& right Wiener-Hopf indices of $E_-^{-1} D_l$ **nonnegative**

ALGEBRAIC CHARACTERIZATION

$$G = D_l^{-1} E_l = E_r D_r^{-1}$$
$$G = \left(\begin{array}{c|c} D & C \\ \hline B & A \end{array}\right)$$

$$\begin{cases} A = S^{D_r} \\ B\xi = \pi_- D_r^{-1} \xi, \\ Cf = (E_r f)_{-1} \\ D = G(\infty). \end{cases}$$

$\mathcal{V} \subset X^{D_r}$ is **inner stabilizable**
& left Wiener-Hopf indices of $D_r E_-^{-1}$ **nonnegative**

$\Leftrightarrow$

$\mathcal{V} = \pi^{D_r} X^{E_-}$

E_- **stable**

$\mathcal{V} \subset X^{D_r}$ is **inner antistabilizable**

$\Leftrightarrow$

$\mathcal{V} = \pi^{D_r} X^{E_+}$

E_+ **antistable**
& left Wiener-Hopf indices of $D_r E_+^{-1}$ **nonnegative**