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Dynamical Systems and Their Associated Automata

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Abstract

This paper studies problems related to the construction of a robust correspondence between an automaton and a continuous-time dynamical system of the input-output type. Two general methods, based on ideas from topology, are considered. They can be distinguished on the basis of the time scale on which they operate. The slow time scale method utilizes the relationship between the fundamental group of a space and the corresponding deck transformations acting on the covering space. The fast time scale method is based on a suitable topological characterization of pulses and identifies pulses with transitions between the domains of attraction of stable equilibria. As compared with the standard digital electronics paradigm, these results provide a more general conceptual scheme for building robustness into calculating mechanism. The results obtained suggest new ways to interpret neurobiological signal processing.

1 Models for Computing

There have been many attempts to compare analog and digital computing, motivated, at least in part, by the desire to make a clear distinction between natural and machine intelligence. The presently accepted setting for such comparisons dates from the 1940's with the literature including work by scientists as influential as Wiener [1] and von Neumann [2]. Even so, it seems fair to say that most of the major questions remain unanswered. One might argue that difficulties encountered in trying to make precise statements about the relationships between analog and digital computing lie at the heart of what makes biological intelligence mysterious and difficult to study. In contrast with standard textbook approaches to the design of digital systems [3], in our recent papers [4], [5] we have considered a general approach to the problem of obtaining robust designs of computational elements using some ideas from algebraic topology and the theory of dynamical systems. In this paper, we consider a framework capable of handling a wide class of coding schemes, reflecting work in neurobiology that suggests many different types of coding are used.

Considerable work has gone into the study of models of the form

$$\dot{x}(t) = f(x(t), u(t))$$

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with $x(t)$ taking on values in a manifold X and the values of $u(t)$ being restricted to some set $U \subset R^m$. Often one excludes the possibility of restricting u in any way that involves constraints on its time derivative, its integral, etc. In an optimal control context, such restrictions are better handled by redefining the input to be the highest derivative present in the constraints, adding equations of the form $\dot{u}(t) = v(t)$ to the dynamics, and then imposing state-space constraints. Here we consider the possibility of invoking constraints linking $u(t)$, $\dot{u}(t)$, and even higher derivatives. We use the notation $u^{\{k\}} = (u, u^{(1)}, ..., u^{(k)})$ to denote the collection consisting of u and its first k derivatives with respect to time. We denote the k^{th} tangent bundle by T^kU.

With this much notation established, we sketch the two main points of view to be developed. In the interest of readability, we include in the next section the definitions of some of the topological terms used. Most are standard and are explained, for example, in Massey [6] or Bott and Tu [7].

1. The Quasistatic Case. The inputs to the dynamical system are assumed to belong to a connected set $U \subset R^m$ admitting the structure of a differential manifold. The state-space X is a covering space of a manifold X_0 with covering map $\phi : X \to X_0$. The system on X is a lift of a system defined on X_0, with the system on X_0 having input space U and an asymptotically stable equilibrium point x_0 corresponding to an input u_0. Different inputs give rise to different stable equilibria so that a path in U gives rise to a path in X_0. Each closed path in X_0 generates a deck transformation on X such that homotopic paths in X_0 generate the same transformation. These transformations permute the elements of $\phi^{-1}(x_0)$. We take the set $\{x \in X | \phi(x) = x_0\}$ to be the state space of the automaton and identify the inputs with elements of the fundamental group of U.

2. The Pulse-Like Case. The inputs to the dynamical system take on values in R^m subject to certain constraints on the values taken on by them and their time derivatives. We express the constraints as $(u(t), u^{(1)}(t), ..., u^{(k)}(t)) \in K$. The states of the automaton are identified with the stable branches of the set $\{x | f(x, u) = 0\}$. The evolution equation $\dot{x}(t) = f(x(t), u(t))$ is structured in such a way that if u is pulse-like, each pulse transfers the system from the domain of attraction of one asymptotically stable equilibrium point to that of another. The transition function of the automaton is defined by these transitions. We identify the inputs of the automaton with distinct elements of the homotopy classes of K.

In automata theory the passage of time is not marked in an explicit way, but rather it is subordinated to the appearance of a new symbol at the input. In the present context it is necessary to be more explicit. We mark the passage of time in terms of the integral of a closed differential defined on K. We use this to formalize a relationship between systems as defined above and a class of hybrid systems.

2 Automata and Covering Spaces

We begin by fixing our notation for finite automata. By an automaton one understands a five-tuple (V, Z, Y, f, h), with V being the input space, Z the state space, Y the output space, f the state transition function and h the output map. One assumes that Z and Y are finite sets and that f and h are maps $f : Z \times V \rightarrow Z$ and $h : Z \times V \rightarrow Y$. Variables $z \in Z$, $v \in V$ and $y \in Y$ are related by

$$z(k+1) = f(z(k), v(k))$$

$$y(k) = h(x(k), v(k))$$

The interpretation is that the input string $v(0), v(1), ..., v(k)$ is processed by the system, generating an output string $y(0), y(1), ..., y(k)$ which represents the result of the computation. Digital computers, thought of as having a fixed amount of memory and storage capacity, can be modeled in this way.

Perhaps the most elementary procedure for associating an automaton with an input-output dynamical system of the form

$$\dot{x}(t) = f(x(t), u(t)); y(t) = h(x(t))$$

involves the idea of a covering space, and uses the relationship between the fundamental group of a space and the group of deck transformations on a covering space. We need the following ideas. (See, for example, Massey [6].) If X and $\tilde{X}$ are differentiable manifolds and if $\phi : \tilde{X} \rightarrow X$ is a continuous map such that for each x_0 in X there is,in $\tilde{X}$, a neighborhood of each $\phi^{-1}(x_0)$ such that ϕ is one to one and onto some neighborhood of x_0 in X then we say that the pair $(\tilde{X}, \phi)$ define a **covering space** of X with **covering map** ϕ. A continuous function $x : [0,1] \rightarrow X$ such that $x(0) = x(1) = x_0$ defines a closed curve in X. Two closed curves, both starting at a point x_0 and ending at x_0, are said to be **homotopic** if there exists a continuous deformation of one of them into the other.

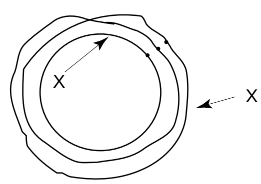

Figure 1. Suggestive representation of a double cover of the circle.

Homotopy defines an equivalence relation on continuous curves starting and ending at x_0. We denote the equivalence class containing the curve γ by $[\gamma]$. Poincaré realized that by using a suitable law of composition, the set of such equivalence classes could be given the structure of a group. The standard notation for this group is $\pi_1(X, x_0)$. (See reference [6] for details.) Given $x_0 \in X$ and $x_{c0} \in X_c$ such that $\phi(x_{c0}) = x_0)$, corresponding to a continuous curve $\gamma \subset X$ there is a unique continuous curve $\tilde{\gamma} \subset \tilde{X}$ such that $\phi(x_c(t)) = x(t)$ for all t. This curve begins and ends in the set of inverse images of x_0. If $\tilde{X}$ is such that any closed curve in it can be continuously deformed into a point then it is intuitive that a closed curve in X is homotopic to a point if and only if the corresponding curve in $\tilde{X}$ is closed. More generally, there is a mapping from the homotopy classes in X to permutations acting on the set of inverse images of x_0. In this way one associates with each element of $\pi(X, x_0)$ a map of $\tilde{X}$ into itself. Such maps are called **deck transformations.** If B is a subset of a topological space X and if $r : X \to B$ is a continuous map, r is said to be a retraction if $r(b) = b$ for all $b \in B$. A jointly continuous family of maps, such that $f(x, 0) = x, f(x, 1) = r(x), f(b, \alpha) = b$, for all $b \in B$ and all $\alpha \in [0, 1]$ is said to define a **deformation retract**.

The idea of a covering space is easily extended to differential equations.

Definition 1. Let X and U be differentiable manifolds and assume that $x(t)$, taking on values in X, and $u(t)$, taking on values in U, are related by

$$\dot{x}(t) = f(x(t), u(t))$$

Let $\tilde{X}$ be a manifold and let $\phi : \tilde{X} \to X$ be such that $(\tilde{X}, \phi)$ is a covering space for X. We will say that a system

$$\dot{x}_c(t) = f_c(x_c(t), u(t))$$

defined on $\tilde{X}$ is a **lift** of the given system if $\phi(x_c(t)) = x(t)$ for all inputs u. The set of possible equilibrium points will be denoted by

$$E = \{x \in \tilde{X} | \exists u \in U s.t. f(x, u) = 0\}$$

Assuming now that f is differentiable with respect to x and u, let $E_+ \subset E$ denote the set of points where the eigenvalues of the Jacobian $\partial f / \partial x$ have real parts that are negative. We call E the $\mathbf{E-set}$ of the system and call E_+ the $\mathbf{E_+ - set}$ of the system.

Example: Embed the circle in R^2 as $x_1^2 + x_2^2 = 1$. Consider a system on this space having scalar input $u(t)$ and evolutionary equation

$$\dot{x}_1(t) = -u(t)x_2(t)$$

$$\dot{x}_2(t) = u(t)x_1(t)$$

Consider the real line as a covering space of the circle with the covering map sending x into $x_1 = \cos x$ and $x_2 = \sin x$. The equation of motion in the covering space is simply

$$\dot{x}(t) = u(t)$$

In this case the $E - set$ for the system on the circle is the whole circle and the $E_+ - set$ is empty. If we add a drift term to get

$$\dot{x}_1(t) = (x_2(t) - u(t))x_2(t)$$

$$\dot{x}_2(t) = (x_2(t) + u(t))x_1(t)$$

then the evolution equation on the covering space is

$$\dot{x}(t) = -\sin x + u(t)$$

The $E - set$ is still the whole circle but now the $E_+ - set$ is the half-circle corresponding to $\pi < x < 2\pi$.

Theorem 1. *Let u and x be related as in Definition 1 with f being differentiable with respect to x and u. Assume that E_+ is connected and that for each $u \in U$ there exists a corresponding equilibrium point $x \in E_+$. Assume, in addition, that $f(x(0), u(0)) = 0$. Then there exists an $\epsilon > 0$ and a map $\phi : (\tilde{X}, \pi_1(U, u(0))) \to \tilde{X}$ such that if $x(0) \in E_+$, $f(x(0), u(0) = 0$, $u(t)$ approaches $u(0)$ as t goes to infinity and $||\dot{u}(t)|| < \epsilon$ then $x_c(\infty) = \phi(x_c(0), [u])$. The number of Nerode equivalence classes associated with the map ϕ is less than or equal to the number of inverse images in $\tilde{X}$ of a point x_0 in X.*

Proof: We may linearize the relationship $f(x, u) = 0$ to get

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u = 0$$

One consequence of the asymptotic stability assumption is that the martix of partial derivatives of f with respect to x is necessarily nonsingular on E_+. A second consequence is that if u changes sufficiently slowly then $x(t)$ will remain close to E_+ and will also change slowly. Thus a closed curve in U generated by a slowly changing input generates a closed curve in X which lifts to a curve in the covering space. The corresponding deck transformation on the covering space then generates a permutation of the points of $\{x \in X | \phi(x) = x_0$.

The above procedure is simple and natural but it is not adequate to represent an arbitrary finite automaton. To see why, it is enough to recall that one may associate with any automaton a set of maps of its state space into itself consisting of those transformations that can be expressed as $\psi(x) = f(f(...(f(x, v_{i_1}), v_{i_2})..., v_{i_{k-1}}), v_{i_k})$ for some choice of the $v's$ and some k. This is a subsemigroup of the semigroup of all maps of Z into Z, with composition being the semigroup operation. It differs only trivially from the Myhill semigroup of automata theory. Clearly it is a finite semigroup because Z is finite. It is well known, any automaton can be realized as

$$m(k + 1) = v(k) * m(k); y(k) = \phi(m(k))$$

with * being composition. On the other hand, the corresponding semigroup for the covering automaton defined by theorem one can always be embedded in a group. This places a restriction on the kinds of automata that can be realized this way.

3 Automata and Coadjoint Orbits I

There is a simple class of systems defined on coadjoint orbits and their covering spaces and which provides a rich set of examples of Theorem 1. In this setting, one can estimate the size of the ϵ that appears there. Instead of considering a full coadjoint orbit, however, we illustrate Theorem 1 in a more restricted setting.

If $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is a set of distinct real numbers, let $Sym(\Lambda)$ denote the set of all real n by n symmetric matrices with spectrum Λ. $Sym(\Lambda)$ admits the structure of a connected Hausdorff manifold of dimension $n(n-1)/2$. The group of n by n orthogonal matrices acts transitively on it, via conjugation, $H \mapsto \Theta^T H \Theta$. There are 2^{n-1} diagonal matrices in $SO(n)$ and their diagonal entries are either plus or minus one, subject to the condition that there is an even number of minus ones. Observe that if $H_0 \in Sym(\Lambda)$ is diagonal, then the values of Θ that satisfy $\Theta^T H_0 \Theta = H_0$ are exactly the diagonal elements of $SO(n)$. Thus we can think of $SO(n)$ as a covering space for $Sym(\Lambda)$, each point in $Sym(\Lambda)$ having 2^{n-1} preimages in $SO(n)$. For $n > 2$ the fundamental group of $SO(n)$ is Z_2. The compact, simply connected group $Spin(n)$ is a double cover. (See, for example, Weyl [8].) Thus we may say that $Sym(\Lambda)$ has $Spin(n)$ as a compact simply connected covering space with 2^n elements in $Spin(n)$ sitting over each point in $Sym(\Lambda)$.

Consider the pair of equations from [9],

$$\dot{H}(t) = [H(t), [H(t), U(t)]]$$

$$\dot{\Theta}(t) = -[H(t), U(t)]\Theta(t)$$

with U and H being symmetric and Θ orthogonal. The first of these can be thought of as an equation on the space of n by n symmetric matrices, the second as an equation on the orthogonal group. Together they evolve in such a way as to keep $\Theta(t)H(t)\Theta^T(t)$ constant and, together, imply the single equation

$$\dot{\Theta}(t) = [\Theta^T(t)Q\Theta(t), U(t)]\Theta(t)$$

We regard these equations as defining an input-output system with input U. They are capable of realizing interesting classes of automata provided that we restrict the choice of U in an appropriate way. More specifically, this system defines a finite automaton that computes a certain topological invariant associated with the input trajectory.

Generic n by n symmetric matrices have unrepeated eigenvalues. We denote by $GSym(n)$ the set of real n by n symmetric matrices without repeated eigenvalues. The following lemma provides a characterization of this space that will be useful in interpreting the theorem that follows.

Lemma 1. *If Λ is without repeated entries then $Sym(\Lambda)$ is a deformation retract of $GSym(n)$.*

Proof: For each element U of $GSym(n)$ there is a unique diagonal matrix $D(U)$ such that D is of the form $\Theta^T U \Theta$ and the diagonal elements of D are in decreasing order. Let Λ be the diagonal matrix in $Sym(\Lambda)$ whose diagonals are in decreasing order. Consider the one parameter family of maps

$$r(\alpha) : GSym(n) \to Sym(\Lambda)$$

defined by

$$r(\alpha) : U \mapsto \Theta(\alpha\Theta^T U \Theta + (1-\alpha)\Lambda\Theta^T$$

For all α between zero and one the convex combination $\alpha D + (1-\alpha)\Lambda$ is without repeated eigenvalues. Thus for each α the above expression defines a map of $GSym(n)$ into itself and for $\alpha = 1$ it is onto $Sym(\Lambda)$.

In stating Theorem 2 we make use of the notation $ad_P(Q) = PQ - QP$ and use $ad_P^{-1}()$ to denote an inverse of this operator.

Theorem 2. *Let Q be an element of $Sym(\Lambda)$ and let $U : [0, \infty) \to GSym(n)$. Consider the system on $SO(n)$*

$$\dot{\Theta}(t) = [\Theta^T(t) Q \Theta(t), U(t)]\Theta(t)$$

There exists $\phi : SO(n) \times \pi_1(GSym(n), U(0)) \to SO(n)$ and an $\epsilon > 0$ such that if

$$||ad_U^{-1}(\dot{U}(t)|| \leq \epsilon$$

then

$$\Theta(\infty) = \phi(\Theta(0), [U])$$

The state-space of this automaton has 2^{n-1} states.

Proof: As discussed in [9], for each element $U \in GSym(n)$ there is a unique asymptotically stable equilibrium point for $\dot{H}(t) = [H(t), [H(t), U(t)]]$. Thus, considering this system, every point in $Sym(\Lambda)$ belongs to E_+ and the linearization that appears in Theorem 1 has a nonsingular Jacobian because the eigenvalues of H are distinct. To show that the automaton has 2^{n-1} states we need to display this number of closed curves in U, distinct in the sense of homotopy. Let P be an arbitrary permutation matrix and consider the one parameter family of orthogonal matrices $P^T \Theta(\theta) P$ with

$$\Theta(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}$$

When θ advances from zero to π the one parameter family in $Sym(\Lambda)$ defined by $H(\theta) = \Theta^T(\theta)\Lambda\Theta(\theta)$ is a closed curve. This closed curve is not homotopic to a point because it induces a nontrivial mapping of the points of $\pi^{-1}(\Lambda)$, sending $\Theta(0)$ into $\Theta(\pi)$. As discussed above there are exactly 2^{n-1} such mappings that are distinct in the sense of homotopy.

To estimate ϵ in Theorem 2, we will make use of particular Riemannian metrics on $SO(n)$ and $Sym(\Lambda)$. First of all, if Ω is skew-symmetric then we let $||\Omega||$ denote the square root of the sum of the squares of the entries in Ω. The distance between $\Theta \in SO(n)$ and $\Psi \in SO(n)$ is the smallest value of $||\Omega||$ relative to all skew-symmetric matrices that satisfy $e^{\Omega}\Theta = \Psi$. There is a corresponding metric on $Sym(\Lambda)$. In terms of this metric, the distance between H_1 and H_2 is the smallest value of $||\Omega||$ relative to all skew- symmetric matrices Ω that satisfy $e^{\Omega} H_1 e^{-\Omega} = H_2$. We use the notation $d_n(H_1, H_2)$ to denote this distance between two elements of $Sym(\Lambda)$. This is sometimes called the normal metric.

The following lemma provides the tools necessary to estimate the size of of the parameter ϵ that appears in Theorem 2.

Lemma 2. *If $H_1(t)$ and $H_2(t)$ are closed curves in $Sym(\Lambda)$ defined for $0 \leq t \leq 1$ and if $d_n(H_1, H_2) < \pi/2$, then the curves are homotopic in $Sym(\Lambda)$. If $H_1(t)$ and $H_2(t)$ are closed curves in $GSym(n)$ defined for $0 \leq t \leq 1$ and if $||\Omega|| + ||\Gamma||$ such that $e^{\Omega} N_1(t) e^{-\Omega}$ and $e^{\Gamma} N_2(t) e^{-\Gamma}$ are diagonal and similarly ordered, is less than $\pi/2$ then H_1 and H_2 are homotopic in $GSym(n)$.*

Proof: If H_1 and H_2 are elements of $Sym(\Lambda)$ and if $d_n(H_1, H_2) < \pi/2$ then the equation

$$e^{\Omega} H_1 e^{-\Omega} = H_2$$

has a unique solution $\Omega(H_1, H_2)$ such the spectral radius of Ω is less than $\pi/2$ and this solution depends continuously on H_1, H_2. To see see that this is the case, note that

$$e^{\Omega}H_1e^{-\Omega} = H_2 = e^{\Psi}H_1e^{-\Psi}$$

This implies that $e^{-\Omega}e^{\Psi}$ is a diagonal matrix in $SO(n)$. Because the shortest distance between two diagonal matrices in $SO(n)$ is at least π, the triangle inequality implies that if $||\Omega|| + ||\Psi|| < \pi$ then the solution is unique. To see that the solution depends smoothly on H_2 it is enough to observe that because the elements of Λ are unrepeated, the Jacobian of the differential is nonsingular at each point.

We use the construction of Ω given in the previous lemma and define the homotopy by $e^{\epsilon\Omega_1(t)}H_1(t)e^{-\epsilon\Omega_1(t)} = H_\epsilon(t)$.

As we have seen above, if $U(t)$ has unrepeated eigenvalues then a continuous path in $GSyn(n)$ lifts in a unique way to a continuous path in $SO(n)$ satisfying $\Theta^TN(t)\Theta = D(t)$ with $D(t)$ being diagonal. If U has unrepeated eigenvalues then it determines a set of 2^{n-1} orthogonal matrices, each with with positive determinant, via the equation $\theta^TU\Theta = D$ with D diagonal and having $d_{ii} > d_{jj}$ for $i > j$. If U depends on a parameter then we can differentiate both sides of this equation to get

$$[\Omega, D] + \Theta^T\dot{U}\Theta = \dot{D}$$

Consequently $ad_D^{-1}(\Theta^T\dot{U}\Theta)$ determines Ω. A further analysis of the $ad_D()$ operator can be used to establish bounds on Ω in terms of the separation between the eigenvalues of U the the magnitude of its derivative..

4 Pulse Space

It appears that in most circumstances neurobiological systems communicate by means of pulses rather than by means of the bilevel signals one finds in digital electronics. In the next section we will describe a general method for associating automata with pulse driven systems but before doing so we need to give a suitable definition of what we mean by a pulse. Informally we think of a pulse as sudden and substantial increase in the value of a function followed by a rapid return to its original value. Often the area under the pulse is an important parameter. The pulse formation process can then be repeated with some delay so as to produce a sequence of pulses. In this section we discuss a method for specifying pulse-like behavior in topological terms.

One can attempt to define what is meant by a pulse by imposing specifications directly on the functions of time that are to be called pulses. However, because of the character of pulse trains and the processes that generate them, it seems to be more efficient to give a characterization in terms of differential inclusions. That is to say, we use inequalities relating u and $\dot{u}$ and possibly higher order derivatives. Figure 2 shows an example of the type of inequality that is useful for this purpose. The figure shown has been constructed from two annular regions in phase space. In the larger of these regions, signals must resemble one cycle of a biased sinewave of a particular frequency. In the other region it must resemble a unbiased sinewave of a second, slower frequency. The former generates the pulse and the latter is responsible for enforcing a refractory period between pulses. In our applications this refractory period give the system a chance to relax toward the nearest asymptotically stable equilibrium point. This **double annulus** model of pulse space must be supplemented with a condition on the second derivative if all possible solutions are to look like pulses.

Notation: Define $s_\epsilon(u)$ as

$$s_\epsilon(u) = \begin{cases} 0, & \text{if } |u| \le \epsilon \\ 1, & \text{if } |u| \ge \epsilon \end{cases}$$

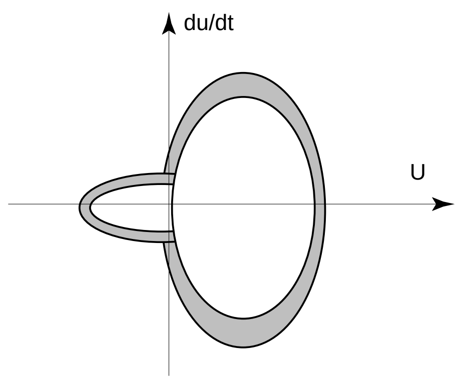

Figure 2. Illustating pulse space as a subset of phase space.

Definition 2: We will say that $u : [0, \infty) \to R$ is an $(\alpha, \beta, \epsilon)-$**pulse train** if

i. $\alpha^2 < \dot{u}^2(t) + \alpha^2(u(t) - 1)^2$

ii. $\epsilon^2\beta^2 < \dot{u}^2(t) + \beta u^2(t)$

iii. either $\dot{u}^2(t) + \alpha^2(u(t) - 1)^2 < \alpha^2(1 + \epsilon)^2$ or $\dot{u}^2(t) + \beta^2 u^2(t) < 2\beta^2\epsilon^2$

iv. $|\ddot{u}(t) + u(t) - 1| s_\epsilon(u(t)) < \epsilon$

Notation: The subset of T^2R that is defined by these conditions will be denoted $P_\epsilon(\alpha, \beta)$. If u is an m-dimensional vector then we write $u^{\{2\}} \in P_\epsilon^m(\alpha, \beta)$ if each component of $u^{\{2\}}$ belongs to $P_\epsilon(\alpha, \beta)$ and no two components of $u(t)$ have an absolute value that exceeds ϵ at any one point in time.

Lemma 3. *If $u^{\{2\}} \in P_\epsilon(\alpha, \beta)$ then*

i. *The minimum period between successive pulses, $T_\epsilon(\alpha, \beta)$, approaches $(2\pi/\alpha) + (\pi/\beta)$ as ϵ goes to zero.*

ii. *If $s_\epsilon(u(a)) = s_\epsilon(u(b))$ then the integral*

$$I(u) = \int_a^b u(t) s_\epsilon(u(t)) dt$$

approaches $(2\pi/\alpha)$ times the number of pulses in $[a,b]$ as ϵ goes to zero.

iii. *If $u(t)$ begins and ends at 0, the time integral of*

$$\frac{1}{2\pi}\dot{\theta} = \frac{1}{2\pi}\frac{(\ddot{u}(t)u(t) - \dot{u}^2(t))}{(u^2(t) + \dot{u}^2(t)}$$

is the number of pulses.

Proof: From the definition of $P_\epsilon(\alpha,\beta)$ we see that for $u(t) > \epsilon$ the pair $(u(t),\dot{u}(t))$ is confined to an annulus in phase space centered at $(1,0)$ and that the thickness of this annulus goes to zero with ϵ. Thus if $u(t)$ exceeds ϵ and its derivative is positive then it must continue to increase until it reaches at least $1-\epsilon$. At such a point, the fourth inequality implies that $\ddot{u}(t)$ is approximately -2 and so $\dot{u}(t)$ is strictly decreasing. The only possible motion in the annulus is then for $u(t)$ to decrease returning to a value near zero. In this portion of the trajectory $|\ddot{u}(t) + \alpha^2 u(t)|$ is small and goes to zero as ϵ goes to zero. Thus the time required to pass around the annulus approaches $2\pi/\alpha$. Near $(\dot{u}(t), u(t)) = (0,0)$, however, the second inequality implies that $u(t)$ must pass around a second smaller annulus of width ϵ. In this annulus $|\ddot{u}(t) + \beta^2 u(t)|$ goes to zero as ϵ goes to zero and so as ϵ goes to zero the time required to pass through a half-circle approaches $2\pi/\beta$. Thus we have the statement on the minimum period. The statement on the area is an immediate consequence of the fact that as ϵ goes to zero the solution approaches a sine wave with average value 1 and there is no effect from the small loop because of the $s_\epsilon(u(t))$ factor. The final assertion is a consequence of the fact that the point $(1,0)$ in the phase space is encircled once per pulse.

We remark that the role of the small annulus is to enforce a refractory period between pulses. Having completed a pulse, u may linger for a long time before starting the process again; the small annulus prevents it from starting a new pulse right away. In fact, there is a minimum latency of about $2\pi/\beta$.

We will use this definition and the claims of the lemma in the next section. These conditions should be thought of as but one example of the type of input constraint mentioned in the first section of the paper. We devote the rest of this section to aspects of the general question.

The k^{th} order tangent bundle T^kU is, of course, a manifold in its own right. Given $K \subset T^kU$ we distinguish between arbitrary curves in K and those curves in K that are of the form $(u, u^1, ..., u^{(k)})$. We will call the latter **lifts**. The set of lifts that come from curves that start and end at a point of the form $(a, 0, ..., 0) \in U^k$ can be composed in a way that is analogous to the way in which curves are composed in arriving at the rule for multiplication in the definition of the fundamental group of a space. The difference is that one can not rescale the parametrization because that would change the value of the derivatives.

In the final assertion of the lemma, we used the derivative of $\tan^{-1}(\dot{u}(t)/u(t))$. In other situations different closed expressions may be needed. The general idea behind the choice of such terms is this. Expressions of the form $\phi(u, u^{(1)}...., u^{(k)})$ that are linear in the highest derivative,may or may not have the property that at each point in $K \subset T^kR^m$ there is a locally defined function $\theta(u, u^{(1)}...., u^{(k-1)})$ such $\phi = d\theta/dt$. If K is not simply connected then $\phi's$ with this property are candidates for generating the first de Rham cohomology class of K. The general situation is clarified by the following lemma.

Lemma 4. *If $K \subset T^k R^m$, if $\phi(u^{\{k\}})$ maps K into the reals, and if*

$$\sum (-1)^i \frac{d^i}{dt^i} \frac{\partial^i \phi}{\partial x^{(i)}} = 0$$

then in any simply connected neighborhood of 0 *one can define a function θ such that θ vanishes at* 0 *and $\phi = d\theta/dt$ in a connected neighborhood of* 0 *with θ being given by*

$$\theta(u^{\{k-1\}}) = \int_0^{u^{\{k-1\}}} \phi(u^{\{k\}}) dt$$

Proof: Pick any path in in a simply connected neighborhood of the origin in K and evaluate the integral along any path joining the origin and the point $u^{\{k-1\}}$. The Euler-Lagrange operator appearing in the lemma defines the linear functional expressing the first order variation of the integral when the path changes. That is,

$$\int_a^b \phi((u+\delta u)^{\{k\}}) dt = \int_a^b \phi((u)^{\{k\}}) + \langle \delta u, \sum (-1)^i \frac{d^i}{dt^i} \frac{\partial^i \phi}{\partial x^{(i)}} \rangle dt + e$$

with e being second order in δu. The fact that the linear functional vanishes identically means that the the integral is independent of path.

Functions satisfying the hypothesis of Lemma 4 will be said to be **closed.**

5 Automata and Pulse Driven Systems

Associating automata with systems whose inputs are pulses requires a different analysis than that which was required in section 3. If the inputs are pulses the state will not simply follow the input in a quasistatic way, always remaining close to equilibrium. Instead, the state follows closely the integral curves associated with a certain input, passing quickly from one equilibrium state to another, or even back to the original equilibrium. The input set will consist of homotopy classes in $K \subset T^k(U)$. We will need to establish a relationship between paths in K and transitions between equilibrium states.

Definition 3: Let X and U be differentiable manifolds. Assume that $x(t)$, taking on values in X, and $u(t)$, taking on values in U, are related by

$$\dot{x}(t) = f(x(t)) + \sum g_i(x) u_i(t))$$

Let E_0 denote the set of points in X where f vanishes and $\partial f/\partial x$ has eigenvalues whose real parts are negative. Assume that there is a positive number ρ such that the domain of attraction of each stable equilibrium point includes a ball of radius ρ. If for each $x_k \in E_0$ and each i the solution of $\dot{x}(t) = g_i(x); x(0) = x_k$, evaluated at $t = 1$ belongs to E_0 we will call the system **control-periodic** .

Examples: The system

$$\dot{x}(t) = -\sin(2\pi\beta x(t)) + (1/\beta) u(t)$$

can be thought of as being defined on the circle or on the real line. In either case it is control periodic for all $\beta > 0$. Vector systems of the form

$$\dot{x}_i(t) = \sum_{i=1}^{n} a_{ij} \sin x_j(t) + 2\pi u_i$$

can be thought of as being defined on the n-torus or on R^n. Again they are control periodic in either setting.

The following example from [3] suggests how the definitions of $P_\epsilon^m(\alpha, \beta)$ and control periodic can be used to describe systems that model automata. Consider

$$\dot{x}(t) = -\sin(2\pi x(t)) + u(t); x(0) = 0$$

We make the change of variables

$$z(t) = x(t) - \int_0^t u(\sigma) d\sigma$$

The differential equation for z is

$$\dot{z}(t) = -\sin(2\pi z(t + 2\pi \int_0^t u(\sigma) d\sigma)$$

We can expand the sine function to display the dependence on the input more explicitly. This gives

$$\dot{z}(t) = -\sin(2\pi z(t)) \cos(2\pi \int_0^t u(\sigma) d\sigma) + \cos(2\pi z(t)) \sin(2\pi \int_0^t u(\sigma) d\sigma)$$

If z is nearly zero, the effect of a quick pulse on this system is to perturb $\dot{z}$ only slightly because the terms

$$a(t) = \cos(2\pi \int_0^t u(\sigma) d\sigma$$

and

$$b(t) = \sin 2\pi \int_0^t u(\sigma) d\sigma)$$

are both bound by one and of short duration.

An analysis of this expression makes it clear that if u consists of a series of narrow pulses having approximately unit area then $x(t)$ will advance by one each time a pulse is received. Suppose that $u^{\{2\}}$ belongs to $P_\epsilon(\alpha, \beta)$. By taking α large, $\beta = 1$ and ϵ small and letting $u = \alpha v$ we insure that u consists of pulses of nearly unit area and short duration. The solution of

$$\dot{x}(t) = -\sin(2\pi x(t)) + \alpha u(t)$$

starting at $x(0) = 0$ and $u(0) = 0$ advances from the domain of attraction of one equilibrium point to that of the next each time a pulse arrives.

In generalizing this example we will define an automaton whose inputs are identified with elements of $P_\epsilon^m(\alpha, \beta)$. To insure that the pulses are sufficiently sharp we chose α and β as in the example.

Theorem 3. *Let u and x be related by a control periodic system*

$$\dot{x}(t) = f(x(t)) + \sum g_i(x)u_i(t); x(0) \in E_0, u(0) = 0$$

with f bounded. Suppose that $u = (\alpha/2\pi)v$ *with* $v^{\{2\}} \in P_\epsilon^m(\alpha,\beta)$*. Then there exists a map* $\phi : E_0 \times \pi_1(P_\epsilon^m(\alpha,\beta)) \to E_0$ *and* α, β *and* ϵ*, all positive real numbers, such that if* $u(t) \to u(0)$ *as* $t \to \infty$ *then* $x_c(\infty) = \phi(x_c(0), [v^{\{2\}}])$*.*

Outline of Proof: The hypothesis implies that the initial state is a stable equilibrium point. For α sufficiently large the effect of the input is either to leave x near its present equilibrium point or else to move it to the neighborhood of another, as dictated by the particular g_i that the pulse interacts with. By suitable choice of the parameter β one can force the time between pulses to be as large as desired. Thus, between pulses, the asymptotic stability of the equilibrium points will bring $x(t)$ as close to the equilibrium point as is necessary to make sure that the effect of the next pulse is such as to place $x(t)$ within the domain of attraction of the appropriate equilibrium point.

6 Automata and Coadjoint Orbits II

As in the case of Theorem 1, this theorem is nicely illustrated by introducing a class of systems on coadjoint orbits. In the interest of brevity we continue with the special case used above. The notation is as in section 3.

Consider systems evolving in $Sym(\Lambda)$ with evolution equation

$$\dot{H}(t) = [H(t), [H(t), N]] + 2\pi[H(t), U(t)]$$

N is a diagonal matrix with unrepeated eigenvalues and $U(t)$ is skew-symmetric. The corresponding equation on the covering space is

$$\dot{\Theta}(t) = [\Theta^T(t)Q\Theta(t), N(t)]\Theta(t) + 2\pi U(t)\Theta(t)$$

Before describing the space K_ϵ we remark that in the two by two case with $N = diag(1, 0)$ the equation for

$$\Theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

can be expressed as

$$\dot{\theta}(t) = -\sin\theta + 2\pi u_{12}(t)$$

Thus this system is control periodic and we can take $K \subset T^k$ to be the to be the set $P_\epsilon(\alpha, \beta)$ as defined above.

Returning now to the general case, we see that if we take U to be tridiagonal and ask that the elements $(u_{i,i+1})$ belong to $P_\epsilon^n(\alpha,\beta)$ then we have a control periodic system.

Theorem 4. *Suppose* $N \in GSym(n)$ *and* $Q \in GSym(n)$*. Assume that* $U = -U^T$ *is tridiagonal and that the vector* $(u_{12}, u_{23}, ..., u_{n-1n})$ *belongs to* $P_\epsilon^{n-1}(\alpha,\beta)$*. Then there exists a map*

$$\phi : SO(n) \times \pi_1(GSym(n), U(0)) \to SO(n)$$

and an $(\alpha, \beta, \epsilon)$ triple such that if $U(t)$ approaches a constant and

$$\dot{\Theta}(t) = [\Theta^T(t)Q\Theta(t), N]\Theta(t) + \alpha U(t)\Theta(t)$$

we have

$$\Theta(\infty) = \phi(\Theta(0), \pi_1(u^{\{2\}}))$$

.

Sketch of Proof: The stable equilibria of this system are the diagonal elements of $SO(n)$. The distance between a diagonal orthogonal matrix with one particular sign pattern and a diagonal orthogonal matrix with a second sign pattern differing only in two places is 2π as measured by the standard Riemannian metric on $SO(n)$. By restricting U to be tridiagonal with only one pair of elements larger than ϵ at any one time, we make sure that the effect of a rapid pulse of area 2π is such as to transfer the system from one equilibrium point to the domain of attraction of another. Further details will not be given here.

It may be noted that the automata defined here are closely related to the automata defined by Theorem 2.

7 Hybrid Systems

The systems described by Theorems 3 and 4 operate as automata if their inputs are restricted in a suitable way. In order to use them in a computational system processing general analog data one would need to have a way of generating these structured inputs from less well structured inputs.

Example. Let α and β be real numbers with $0 < \alpha < \beta$. Consider a twice differentiable function u related to an input v by an equation of the form

$$\ddot{u}(t) + u(t) = f(e(t))v(t)$$

where $e(t) = (u^2(t) + \dot{u}^2(t))$ and

$$\alpha^2 \leq u^2(0) + \dot{u}^2(0) \leq \beta^2$$

Assume that $f(\alpha^2) = f(\beta^2) = 0$ and that f is nonzero between α^2 and β^2. For example, we may assume

$$f(2e) = (2e - \alpha^2)(2e - \beta^2)$$

Under these circumstances $(u(t), \dot{u}(t))$ remains for all time in the annulus

$$\alpha^2 \leq (u^2(t) + \dot{u}^2(t)) \leq \beta^2$$

regardless of the choice of v. Moreover, given any twice differentiable function u that satisfies the given inequality, there exists a v such that (u, v) satisfies the given equation.

Of course the annulus of this example is not exactly the shape required by the theorems above. More generally, we can seek to code an input in such a way as to generate a response such that $u^{\{2\}}$ remains in some set $K \in T^2R$ by generalizing this example. Consider

$$\ddot{u}(t) + a(\dot{u}(t), u(t)) = b(\dot{u}(t), u(t))v(t)$$

If the vector field in the phase space defined by $\ddot{u}(t) + a(\dot{u}(t), u(t)) = 0$ leaves K invariant, and if $b(\dot{u}(t), u(t))$ vanishes on the boundary of K, then this system will act as a coder, mapping arbitrary time functions into elements of K.

Analog computations are usually thought of as proceeding in a continuous fashion, without discrete events, whereas digital computations are thought of as being clocked by means of a "external" timing signal. One aspect of the coding scheme defined by the example is that there is an angle, whose time derivative meets the conditions of Lemma 4, which is monotone increasing and which can be thought of as a way of marking time.

If we combine a coding system of the type just discussed with a dynamical system of the type discussed in Theorem 3 we get an overall system of the form

$$\ddot{u}(t) + a(\dot{u}(t), u(t)) = b(\dot{u}(t), u(t))v(t)$$

$$\dot{x}(t) = f(x(t)) + \sum g_i(x)u_i(t); x(0) \in E_0$$

Such systems can make robust computations on the input data stream represented by v. For the purpose of analysis, however, it may be desirable to simplify this system by replacing the second differential equation by the automaton that is related to it by virtue of theorem 3. This brings us into the realm of hybrid systems as discussed in reference [4].

Consider the model

$$\dot{x}(t) = f(x(t), z\lfloor p \rfloor, u(t), v\lfloor p \rfloor)$$

$$\dot{p}(t) = r(t)$$

$$z\lceil p \rceil = f(z\lfloor p \rfloor, v\lfloor p \rfloor)$$

$$y(t) = c(x(t), u(t))$$

$$w(\lfloor p \rfloor) = h((z\lfloor p \rfloor, v\lfloor p \rfloor)$$

where$\lceil p \rceil$, $\lfloor p \rfloor$ are the ceiling and floor functions defined with the following conventions

$$\lceil p \rceil = \text{smallest integer greater than or equal to p}$$

$$\lfloor p \rfloor = \text{greatest integer smaller than p}$$

The function r is required to be nonnegative so that p is monotone increasing; V and Z are finite sets. The times at which the finite state part of such a system makes a transition from one value of z to another is determined by the times at which $\lceil p \rceil$ changes value. We refer to these as a **hybrid models** . Some of their properties are described in the reference cited.

The mapping of the analog system onto this form is straightforward except, perhaps for the identification of the trigger function r. In the case discussed in in Theorem 3, however, we can take r to be the closed form whose integral marks the advance of the winding number associated with the pulse generation process.

8 Final Remarks

There is a well developed theory of coding for information transmission and storage that was initiated by Shannon [10] and developed extensively in the context of information theory. This theory takes as its starting point the hypothesis that there are a finite number of symbols that can be used to represent information and that there is a noisy channel that will be used to transmit these symbols from one location to another. Important parts of this subject center around the question of how one codes the input so as to make most effective use of the given channel (source coding) and the question of how, and to what extent, one can introduce redundancy so as to make possible the error free transmission of data in spite of the noise that the channel introduces.

Three difficulties associated with the standard model of analog computing, all somehow related to the same problem area, are:

1. The representation of data is completely localized in time and space, making it vulnerable to the effects of noise.

2. The model postulates precise and instantaneous transmission of information in violation of the basic principles of information theory.

3. No distinction is made between computations that are especially sensitive to error in the data or implementation and those that are robust.

In this paper we have tried to address some aspects of these questions by developing a theory of models that are less vulnerable to transmission error as well as the inaccuracies that may enter the devices locally. In the present setup signals are transmitted as pulses characterized by the homotopy classes they define. This provides a conceptual method for both communication and computation, perhaps allowing one to build more substantial bridges between the two.

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