

On the Phase Trajectories of the Turbo-Decoding Algorithm

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What is New Here

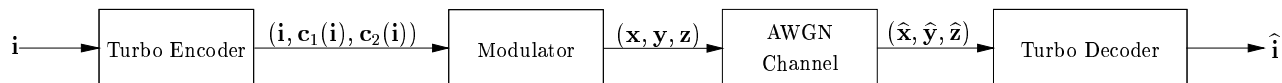
- Based on the characteristics of phase trajectories, the entire SNR range can be subdivided into three regions:
 - A low SNR region
 - The waterfall region
 - A high SNR region
- The turbo-decoding algorithm has two main types of fixed points:
 - Indecisive fixed points
 - Unequivocal fixed points

Outline

- Preliminaries
- Fixed Points at Asymptotic SNRs
- Fixed Points at Practical SNRs
- Continuation Principle
- Bifurcation of Fixed Points
- Conclusions

Preliminaries

System Model:



Richardson's Formulation of Turbo Decoding:

$$Q_1 \leftarrow \pi_{P_1}(P_0 + Q_2) - (P_0 + Q_2)$$

$$Q_2 \leftarrow \pi_{P_2}(P_0 + Q_1) - (P_0 + Q_1)$$

Q_1 : Extrinsic information contributed by \tilde{y}

Q_2 : Extrinsic information contributed by \tilde{z}

P_0 : Posterior log-density induced by \tilde{x}

P_1 : Posterior log-density induced by \tilde{y}

P_2 : Posterior log-density induced by \tilde{z}

Fixed Points at Asymptotically Low SNRs

Theorem

Given any ϵ between 0 and 1, there exists a $\sigma^2(\epsilon) > 0$ such that for $\sigma^2 > \sigma^2(\epsilon)$, the turbo-decoding algorithm has a unique fixed point with probability more than $1 - \epsilon$.

Sketch of Proof:

1. There exists a set $\mathcal{U} \subset \Phi \times \Phi \times \Phi$ such that if $(P_0, P_1, P_2) \in \mathcal{U}$, then the turbo-decoding algorithm has a unique fixed point.
Richardson '98
2. As $\text{SNR} \rightarrow 0$, $P_0 \rightarrow 0$, $P_1 \rightarrow 0$, and $P_2 \rightarrow 0$.
3. $(0, 0, 0) \in \mathcal{U}$, and the set \mathcal{U} is open.

Notes:

- The unique fixed point corresponding to $(0, 0, 0)$ is $(0, 0)$. Continuity of the unique fixed point implies that at low SNRs, the fixed point is close to the point $(0, 0)$.

Fixed Points at Asymptotically High SNRs

Lemma

Under some sufficient conditions, for a turbo decoder starting from the unbiased initialization, extrinsic information for the i -th information bit in the l -th iteration can be bounded as follows:

$$q_1^{(l)}(\mathbf{b}^i) < (\hat{p}_1 + \alpha_1 \hat{p}_2) (1 + (\alpha_1 \alpha_2) + \cdots + (\alpha_1 \alpha_2)^{l-1}) + (\alpha_1 \alpha_2)^l$$
$$q_2^{(l)}(\mathbf{b}^i) < (\hat{p}_2 + \alpha_2 \hat{p}_1) (1 + (\alpha_1 \alpha_2) + \cdots + (\alpha_1 \alpha_2)^{l-1}) + (\alpha_1 \alpha_2)^l$$

Theorem

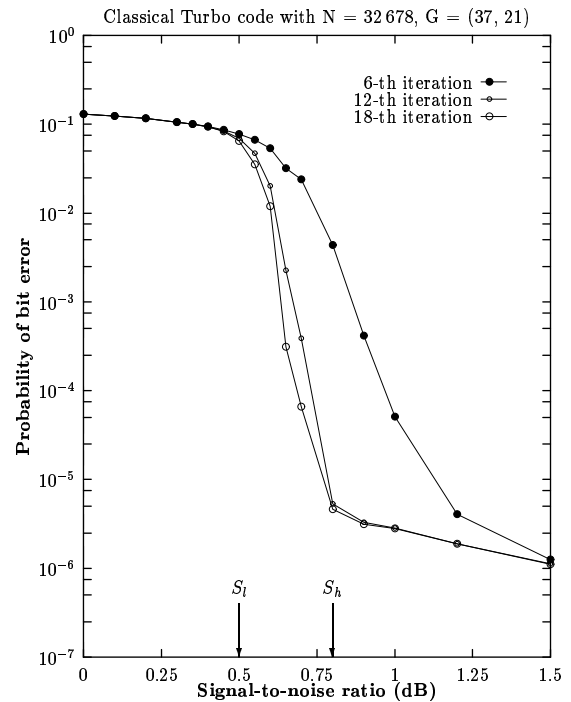
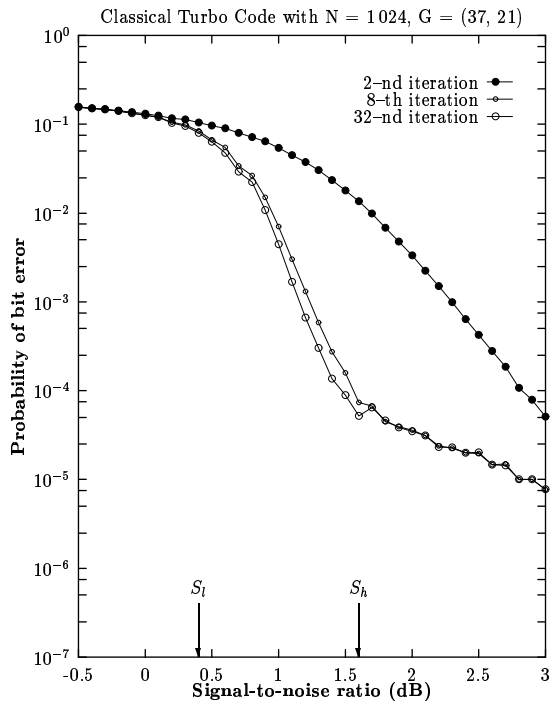
Given any ϵ between 0 and 1, there exists a $\sigma^2(\epsilon) > 0$ such that for $0 < \sigma^2 < \sigma^2(\epsilon)$, with probability more than $1 - \epsilon$, the turbo-decoding algorithm has at least one fixed point that corresponds to the correct decisions on the information bits.

Moreover, starting from the unbiased initialization, turbo decoding can converge only to one of these fixed points.

Notes:

- Proof of this theorem indicates that the extrinsic information of these fixed points will take large positive and large negative values. Furthermore, hard-decisions implied by the extrinsic information will correspond to the transmitted codeword.

Fixed Points at Practical SNRs

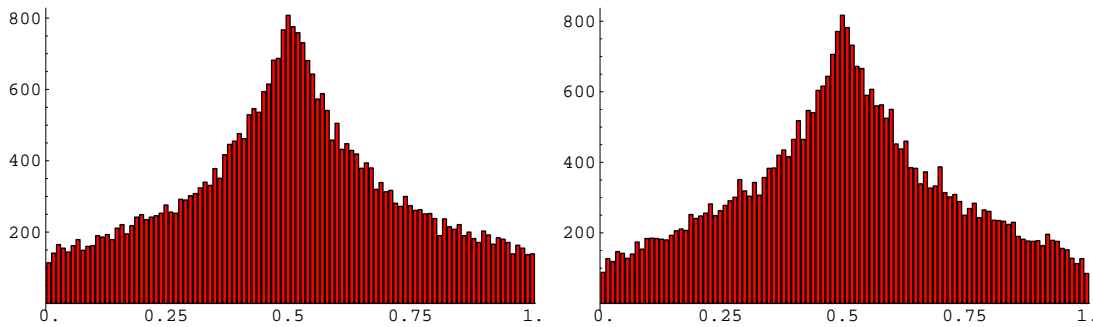


Notes:

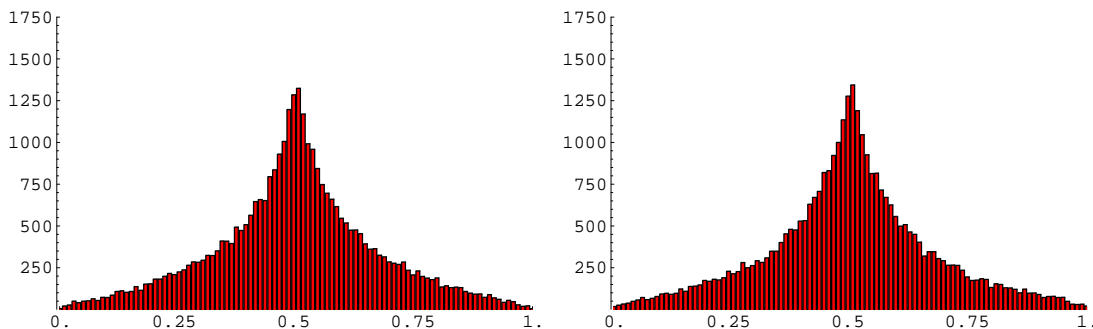
1. In all simulated cases, the turbo-decoding algorithm converged beyond the two threshold values S_l and S_h which coincide with the waterfall region.
2. For SNRs between S_l and S_h , decoding algorithm may not converge.

Histograms of $\Pr_{Q_1^*}(b_i = 0)$ and $\Pr_{Q_2^*}(b_i = 0)$ (SNRs lower than S_l)

At SNR = 0.35 dB



At SNR = 0.0 dB

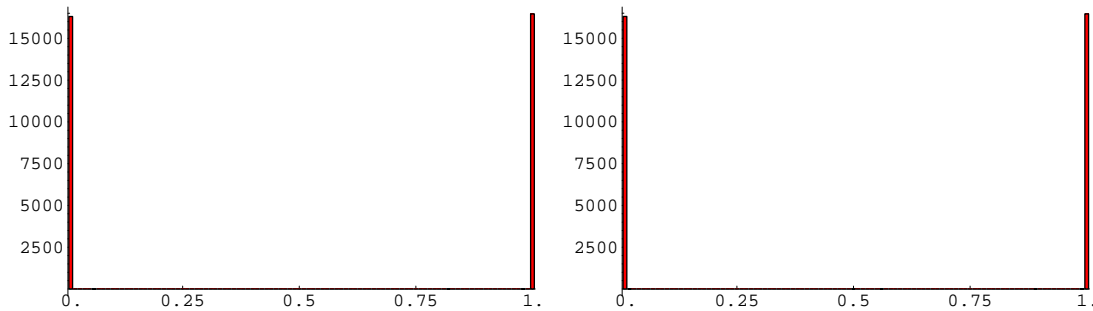


Notes:

- Extrinsic information (Q_1^*, Q_2^*) gets closer to $(0, 0)$ at moderately low SNRs.
- We say that these fixed points are “indecisive”.

Histograms of $\Pr_{Q_1^*}(b_i = 0)$ and $\Pr_{Q_2^*}(b_i = 0)$ (SNRs higher than S_h)

At SNR = 0.80 dB

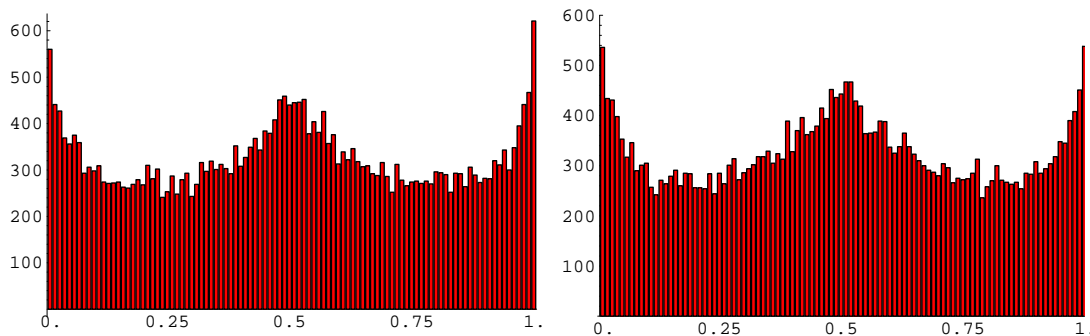


Notes:

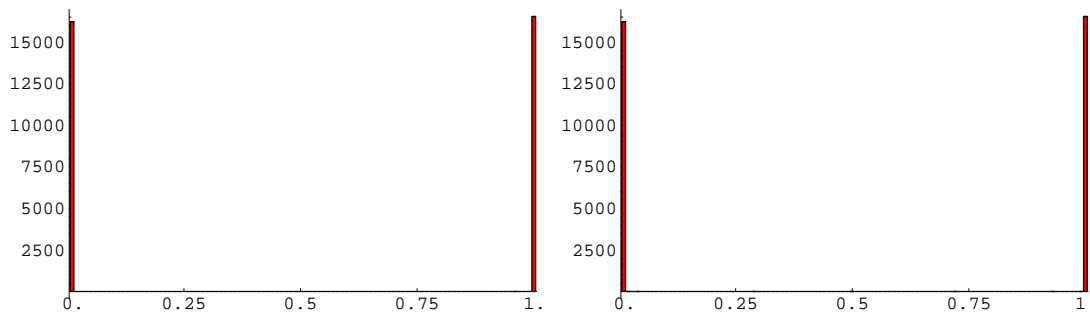
- For SNRs higher than S_h , extrinsic information (Q_1^*, Q_2^*) is unequivocal in its decisions on information bits.
- We call a fixed point with these characteristics as an “unequivocal fixed point”.

Histograms of $\Pr_{Q_1^*}(b_i = 0)$ and $\Pr_{Q_2^*}(b_i = 0)$ (SNRs between S_l and S_h)

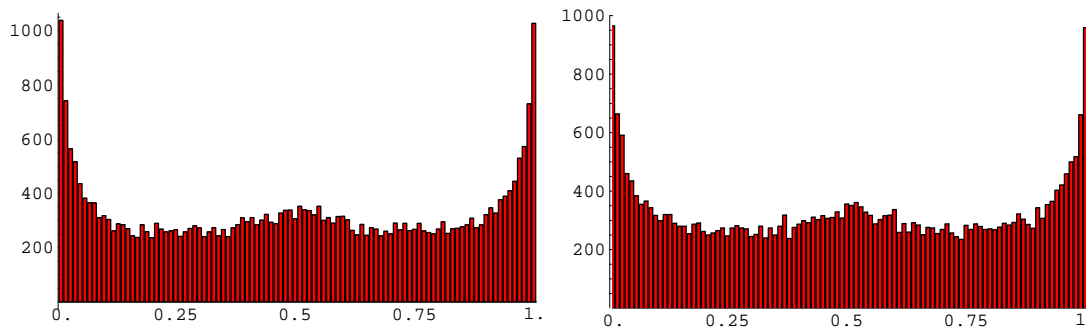
In some cases, it converges to an indecisive fixed point:



It may also converge to an unequivocal fixed point:



In others cases, it may never converge:



Phase Trajectories as a Function of SNR

- The turbo-decoding algorithm is parameterized by P_0 , P_1 , and P_2 .
- Given the transmitted codeword, P_0 , P_1 , and P_2 are functions of the noise-values z_1, z_2, \dots, z_{2n} .

- Fix the ratios,

$$\frac{z_1}{z_2}, \frac{z_2}{z_3}, \dots, \frac{z_{2n-1}}{z_{2n}}$$

and vary,

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{2n} z_i^2}{2n}$$

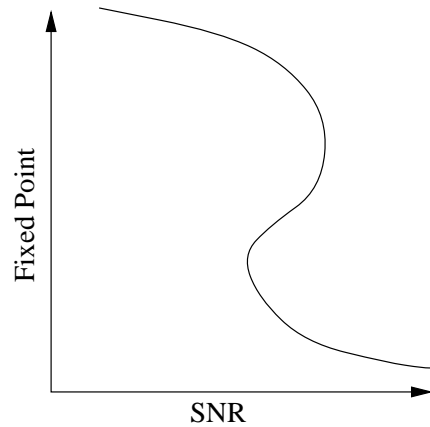
.

- With this,

$$(Q_1, Q_2) \leftarrow \mathcal{F}(Q_1, Q_2)$$

becomes a single parameter system, parameterized by SNR (approximately).

Continuation Principle



- Start from an SNR value, where the convergence, starting from the unbiased initialization, is guaranteed.
- Change the SNR in small increments.
- For each SNR value, find the fixed point by starting from the fixed point for the previous SNR value.

Continuation of Fixed Points

Unequivocal Fixed Points:

Even when the SNR is reduced substantially, unequivocal fixed points barely move. The log-likelihood ratios of information bits retain their large magnitudes.

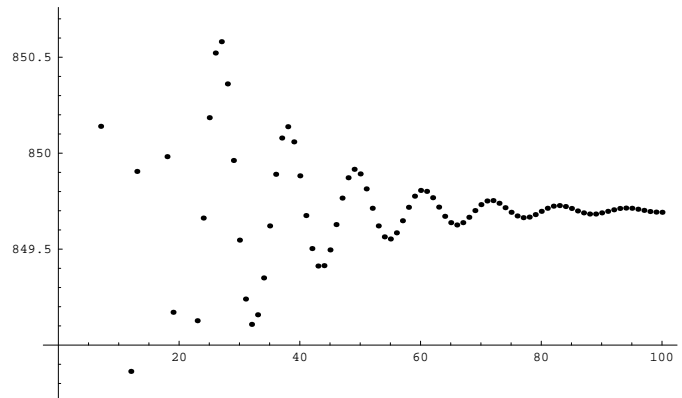
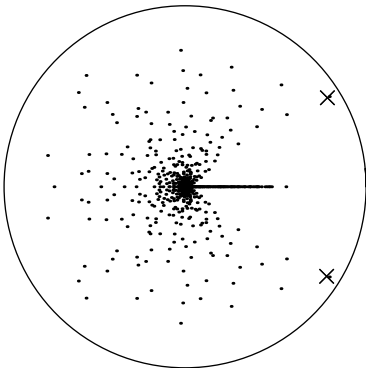
Indecisive Fixed Points:

On increasing the SNR, indecisive fixed points move substantially. They get less and less stable as the SNR is increased, and ultimately, they lose stability (bifurcate) in the waterfall region.

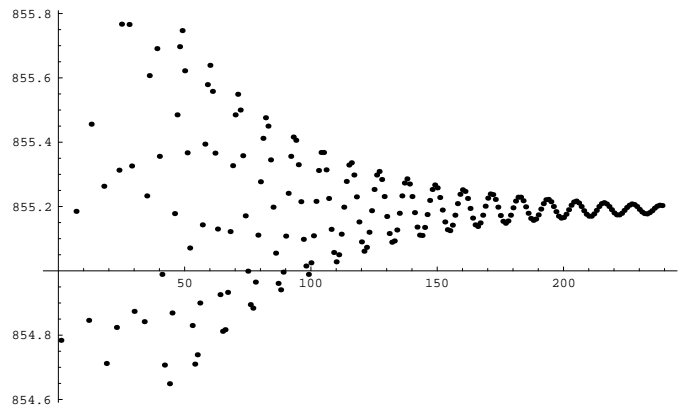
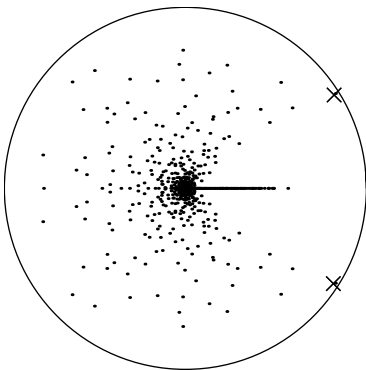
Notes:

- A fixed point can lose its stability only if an eigen value of the Jacobian evaluated at the fixed point has magnitude greater than one.
- There are three types of bifurcation:
 - Neimark-Sacker Bifurcation
 - Flip Bifurcation
 - Fold Bifurcation

Neimark-Sacker Bifurcation of Fixed Points

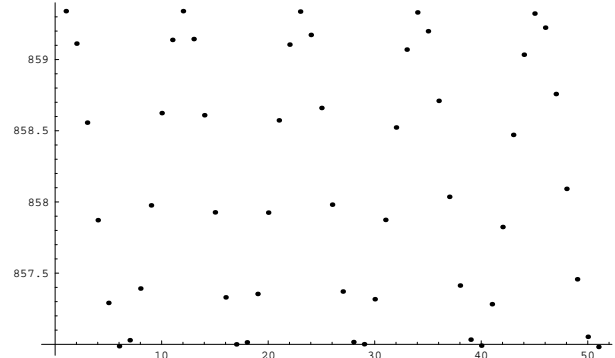
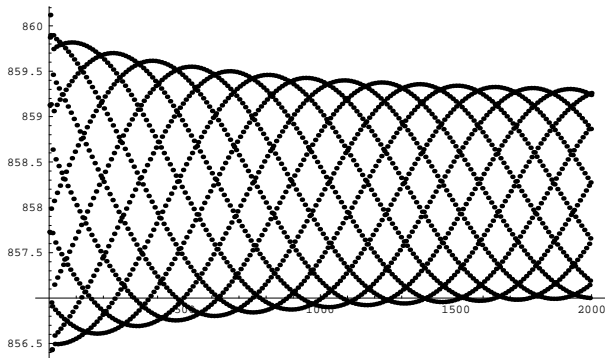


SNR = 0.65 dB

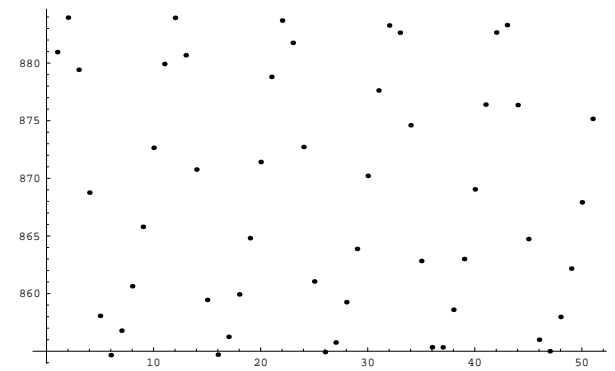
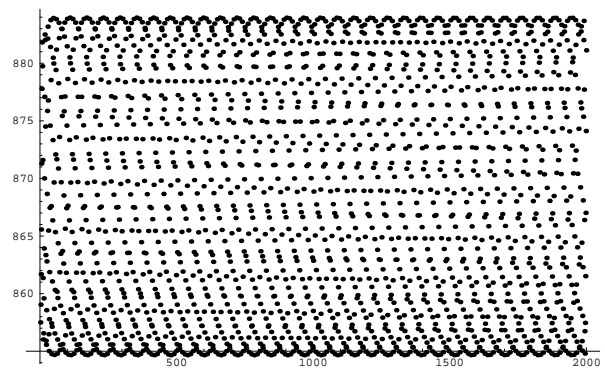


SNR = 0.67 dB

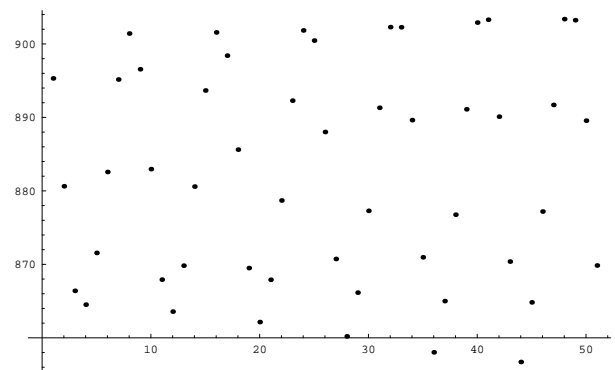
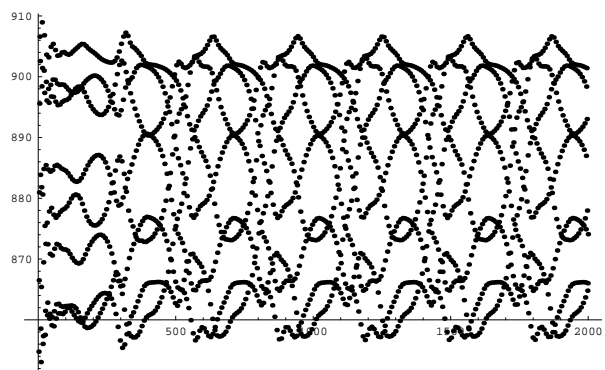
Neimark-Sacker Bifurcation of Fixed Points



SNR = 0.68 dB

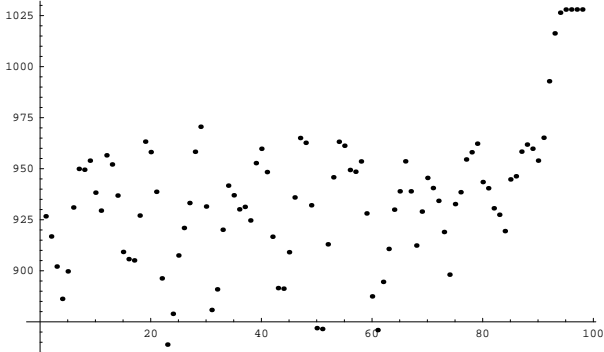


SNR = 0.71 dB



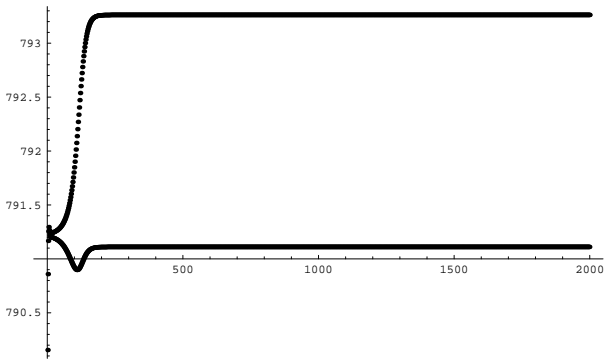
SNR = 0.75 dB

Neimark-Sacker Bifurcation of Fixed Points

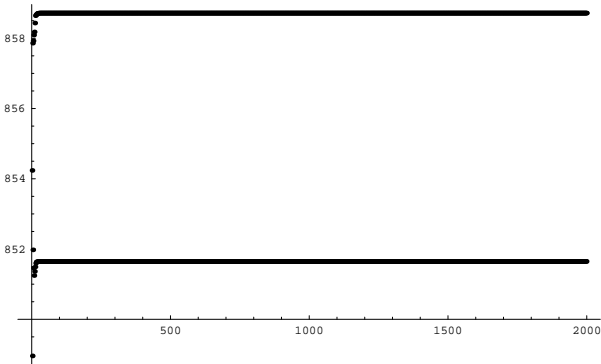


SNR = 0.85 dB

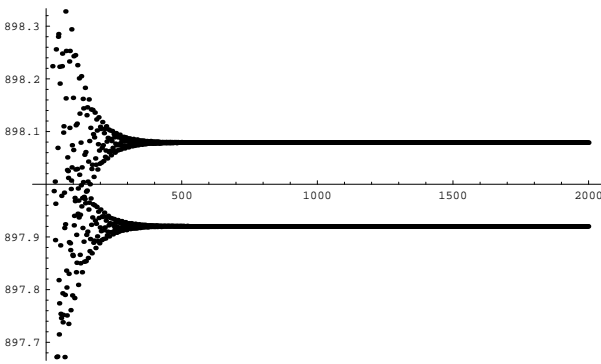
Flip Bifurcation of Fixed Points



SNR = 0.14 dB

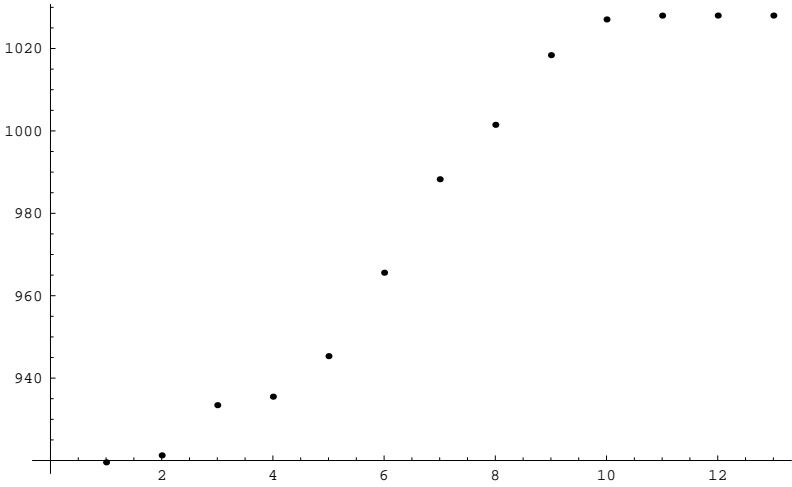


SNR = 0.40 dB



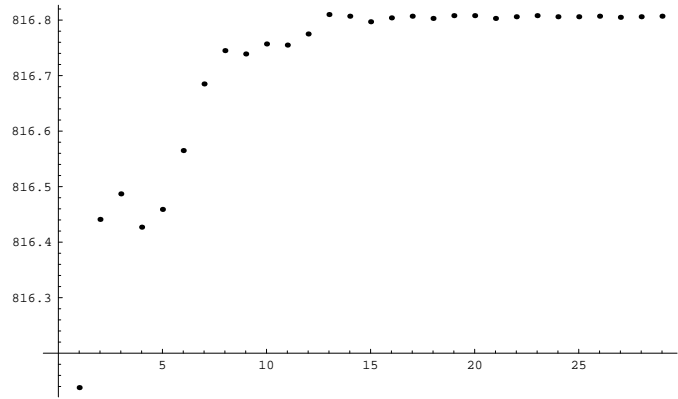
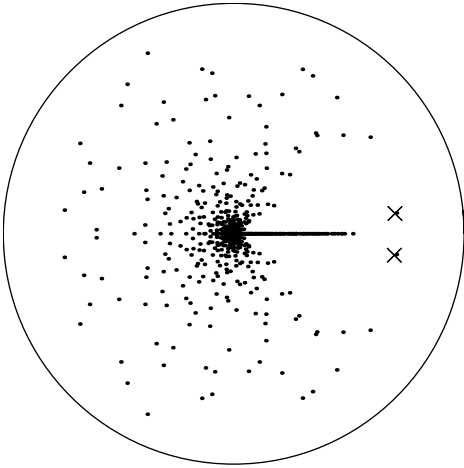
SNR = 0.60 dB

Flip Bifurcation of Fixed Points

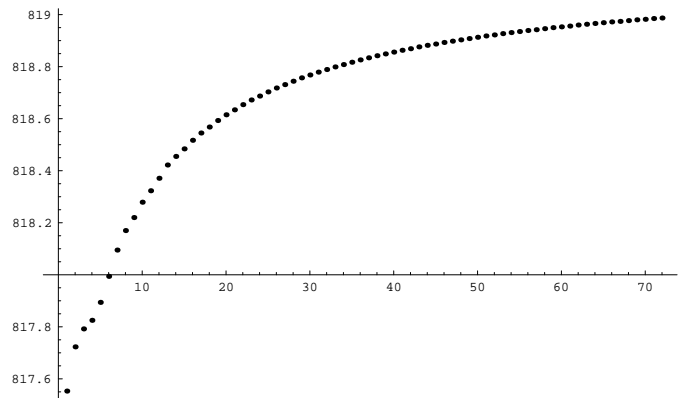
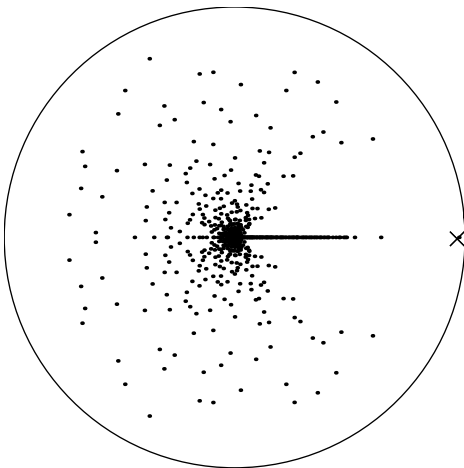


At SNR = 0.70 dB

Fold Bifurcation of Fixed Points

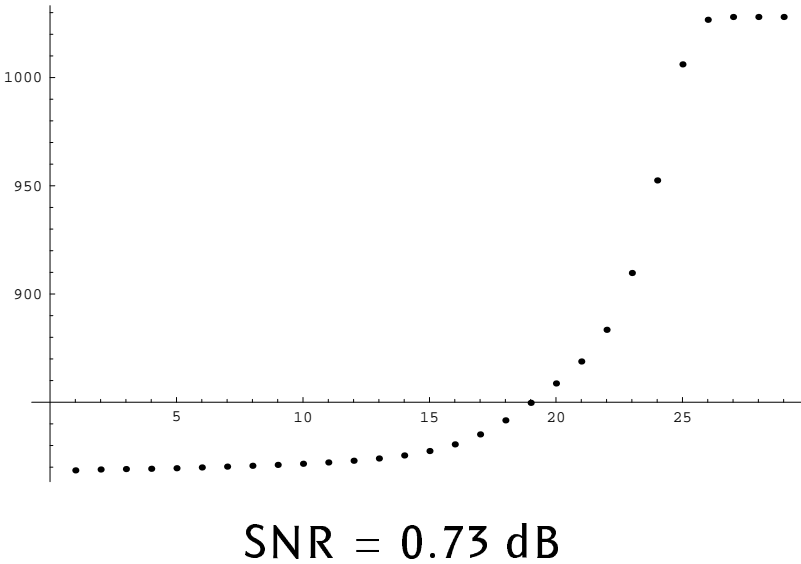


SNR = 0.72 dB



SNR = 0.7244 dB

Fold Bifurcation of Fixed Points



Conclusions

- Two main types of fixed points:
 - Indecisive fixed points
 - Unequivocal fixed points
- Indecisive fixed points bifurcate in the waterfall region, enabling the turbo-decoding algorithm to converge to an unequivocal fixed point.
- The transient behavior of phase trajectories can be effectively used to predict their asymptotic behavior.
- For a detailed discussion, see

<http://www.comm.csl.uiuc.edu/~dakshi>