Deducing Size and Location based on limited scattering data

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Remote Sensing

- Inverse Source Problem for Helmholtz

$$(\Delta + k^2)u = f$$

- Inverse Scattering has an induced source

$$\begin{aligned} (\Delta + k^2 n^2(x))u &= 0 \\ (\Delta + k^2)u &= (1 - k^2 n^2(x))u \end{aligned}$$

- Born Approximation — source induced by free wave

$$(\Delta + k^2)u = (1 - k^2 n^2(x))u_0$$

Limited Data

- Limited Data means
  1. Far field radiated at one k, measured at all (a continuum of) angles of observation. For scattering, the field scattered by a single monochromatic incident wave. Also applies to backscatter.
  2. Far field measured at all (a continuum of) k's but only at a few angles.

- Motivation
  1. Need to work at fixed k when the radiation pattern of the source is time dependent or medium is dispersive.
  2. MUSIC, Time Reversal, utilize sensor arrays to locate point sources with this sort of data.

Outgoing Solutions and Far Fields

$$\begin{aligned} (k^2 - |\xi|^2)\widehat{u}(\xi, k) &= \widehat{F}(\xi) \\ \widehat{u}(\xi, k) &= \lim_{\varepsilon \downarrow 0} \frac{\widehat{F}(\xi)}{(k - i\varepsilon)^2 - |\xi|^2} \\ \widehat{u}(\rho\Theta, k) &= \frac{\widehat{F}(\rho\Theta)}{(k - i0)^2 - \rho^2} \end{aligned}$$

The far field (i.e. large x asymptotics of u) only depends on

$$\widehat{F}\Big|_{\rho=k} = \sum f_n(k) e^{in\theta}$$

so we may replace $\widehat{F}$ with

$$\widehat{F}_H = \sum f_n(k) \left(\frac{\rho}{k}\right)^n e^{in\theta}$$

The Far Field is the Restricted Fourier Transform

The inverse Fourier transform of

$$\left(\frac{\left(\frac{\rho}{k}\right)^n e^{in\theta}}{(k-i0)^2-\rho^2} \right)^{\vee} = H_n^-(kr)e^{in\phi}$$

so that, outside a ball,

$$\begin{aligned} u &= \sum f_n H_n^-(kr) e^{in\phi} \\ &\sim \frac{e^{-ikr}}{r^{\frac{1}{2}}} \sum f_n e^{in\phi} \\ &\sim \frac{e^{-ikr}}{r^{\frac{1}{2}}} \widehat{F}(k\Phi) \end{aligned}$$

Nonuniqueness

The far field data is the Fourier transform, $\widehat{F}(\rho\Theta)$, restricted to:

1. The sphere $\rho = k$
2. A few lines $\Theta = \theta_i$ through the origin

Because there are sources with arbitrary supports that produce zero data, we cannot find an upper bound on the support of F (unless we know something more about F).

For any Φ with compact support, the Fourier transform of

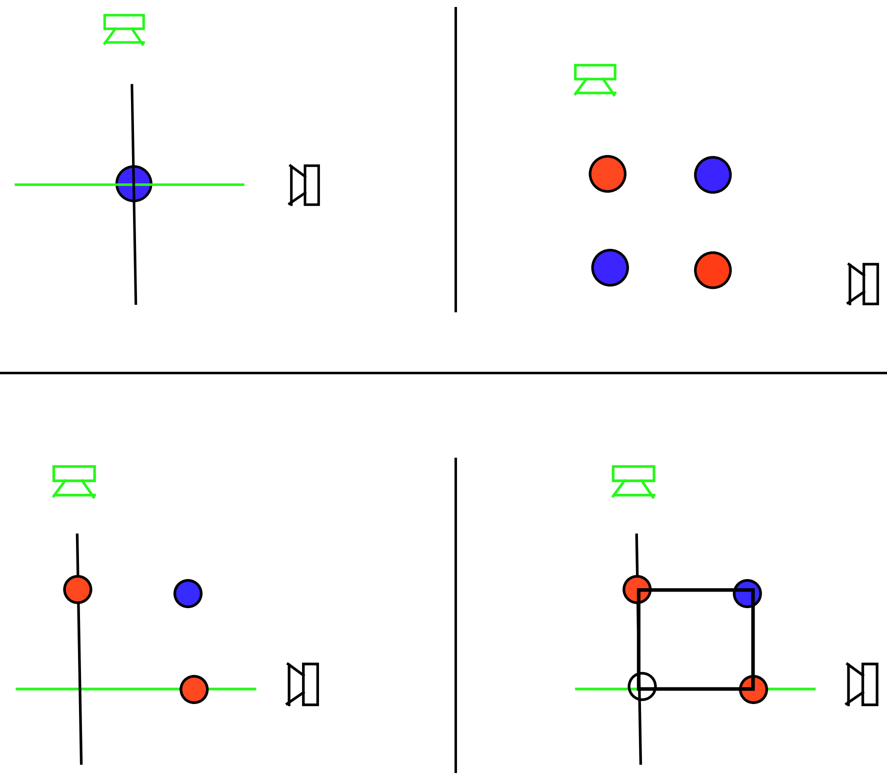
1. $F = (\Delta + k^2)\Phi \qquad \widehat{F} = (k^2 - \rho^2)\widehat{\Phi}$
2. $F = \prod_i \nabla_{\theta_i^\perp} \Phi \qquad \widehat{F} = \prod_i \rho(\theta \cdot \theta_i^\perp)\widehat{\Phi}$

have restricted Fourier transform equal to zero.

An Example

Sensors in the far field only transmit plane waves.

Sensors in the far field only receive averages over planes.



Support of the Data (Far Fields)

The *scattering support* of the data will be lower bound, not an upper bound, for the *support* of the source.

- "Any source or scatterer which radiates (has restricted Fourier transform equal to) the far field must *contain* the *scattering support* of the data."
- "For any far field, there exists a source which radiates the field, and is supported in (any neighborhood of) its scattering support."

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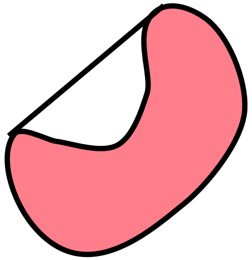
1. Single far field: *support* = convex hull of support
2. Broadband (many k's) at a few angles: *support* = convex polygon with normals equal to angles of observation

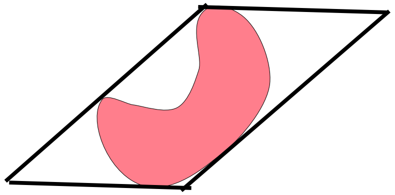
The Support of the Source

The convex Hull

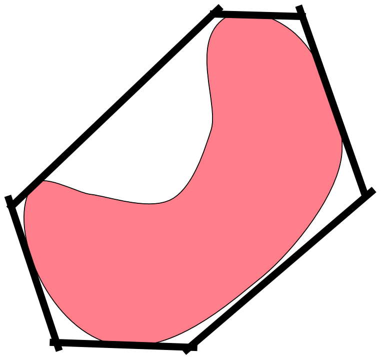


The Θ-convex Hull

Two θ's



Three θ's



Notice that Θ-convex hull always has *almost* 2N sides

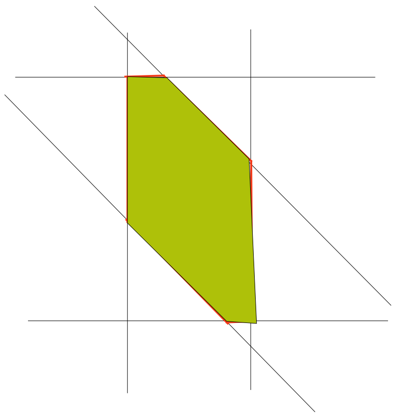
The Scattering Support of the Far Fields

Fixed Θ scattering support

$$\widehat{f_i}(\tau) = \int e^{i\tau\theta_i \cdot x} F(x) dx$$
$$= \int e^{i\tau s} \int_{\theta_i \cdot x = s} F(x) dS(x)\, ds$$

so each $f_i(s)$ supported in (B_i, T_i), and we can define

$$P_\Theta(f) := \bigcap_i \{B_i < x \cdot \theta_i < T_i\}$$



Fixed Θ Paley-Wiener

- Data polygon is always a subset of the source polygon

$$P_\Theta(f) \subset P_\Theta(F)$$

- $\widehat{f}_i$ have compatible Taylor expansions at the origin – $\widehat{f}_i$'s are $\widehat{F}$'s in polar coordinates (these are moment condtions for Radon transform).

Conversely, if

- $P_\Theta(f)$has *almost* 2N sides
- $\widehat{f}_i$ have compatible Taylor expansions at the origin

there exists F with

$$\widehat{F}(\tau\theta_i) = \widehat{f}_i(\tau)$$

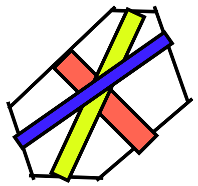
$$P_\Theta(F) \subset N_\varepsilon\left(P_\Theta(f)\right)$$

Sketch of Converse

- Get the Taylor series at the origin right, reducing to the case that $\widehat{f}_i$ vanish at the origin.
- Build an $\widehat{F}_i$ which gets one $\widehat{f}_i$ right and restricts to zero in the other directions.

$$\widehat{F}_i = \frac{\widehat{f}_i(\theta_1 \cdot \xi)}{(\theta_1 \cdot \xi)^{n-1}} \phi\left(\frac{\tilde{\theta}_1 \cdot \xi - c_i}{\varepsilon}\right) \prod_{k \neq i} \frac{\theta_k^\perp \cdot \xi}{\theta_k^\perp \cdot \theta_1}$$

- Use the choice of ϕ and the Paley-Wiener theorem to check

  that F_i is supported in one of the colored rectangles. 

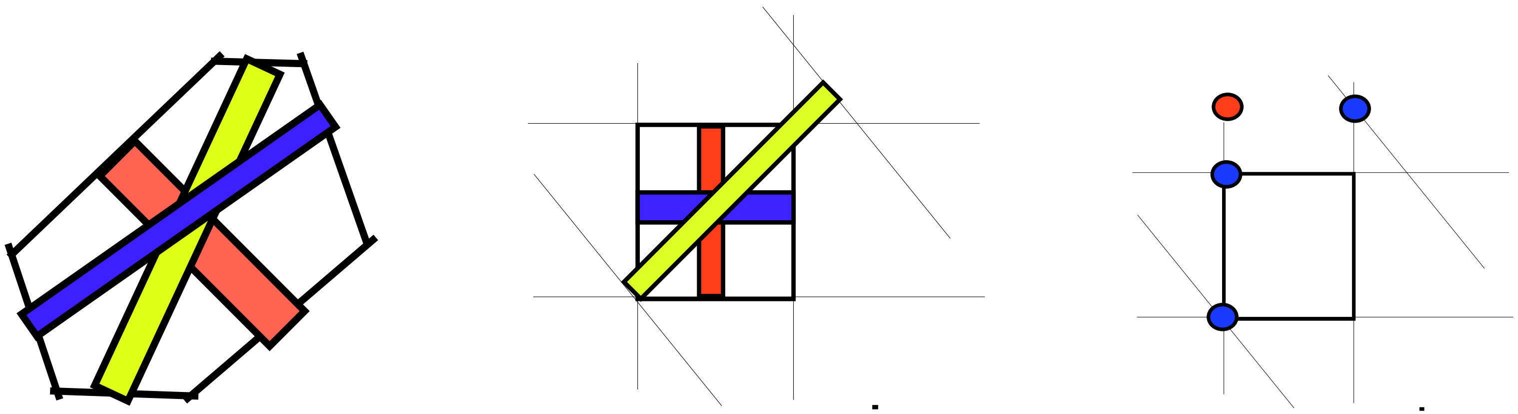
- Add them up. In general, they will be supported in an epsilon neighborhood of the union of the rectangles

$$\bigcup \{B_i < x \cdot \theta_i < T_i\} \cap \left\{-\varepsilon < x \cdot \theta_i^{\perp} - c_i < \varepsilon\right\}$$

  which is only contained in

$$P_\Theta(f) = \bigcap_i \{B_i < x \cdot \theta_i < T_i\}$$

  if the intersection has *almost* 2N sides.



Fixed k (circular) Paley-Wiener

$$\widehat{F}(k\Theta) = \sum f_n e^{in\theta}$$

There exists $F \in L^2(B_R)$ $\iff$ $\left\{ \frac{f_n}{\left(\int_0^R J_n^2(ks)sds\right)^{\frac{1}{2}}} \right\} \in l^2$

$$\Longrightarrow \quad f_n = \int F(r\Phi) J_n(kr) e^{in\phi} dV \quad \leq \quad ||F||_{L^2} \cdot ||J_n||_{L^2(\text{supp}F)}$$

$\Longleftarrow$ Extend $\sum f_n e^{in\theta}$ to $\sum f_n e^{in\theta} \frac{J_n(R\rho)}{J_n(Rk)}$

$$J_n(R\rho)e^{in\theta} = (\widehat{e^{in\phi}\delta_{r=R}})$$

Translation Formula

There is no natural origin in the Far Field.

The Far Field of the translation is the *translation* of the Far Field.

$$F_c = F(x-c)$$

$$\widehat{F}_c(k\Theta) = e^{ik\Theta\cdot c}\widehat{F}(k\Theta)$$

$$f_n^c = \sum J_{n-m}(k|c|)e^{-i(n-m)\theta_c} f_m$$

You can translate with a formula, you don't have to move the array.

Circular Paley-Wiener II

There exists

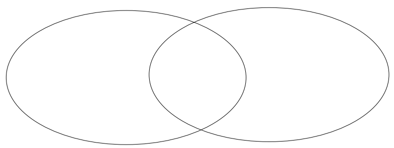
$$F \in L^2(\bigcap_{c\in C} B_{R_c}(c)) \Longleftrightarrow \left\{ \frac{f_n^c}{\left(\int_0^R J_n^2(ks)ds\right)^{\frac{1}{2}}} \right\} \in l^2 \quad \forall \quad c \in C$$

Intersection Lemma

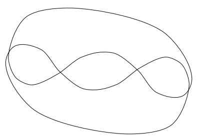
- supp $F_1 \subset \Omega_1$
- supp $F_2 \subset \Omega_2$
- $\widehat{F}_1(k\Theta) = \widehat{F}_2(k\Theta)$
- $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$ connected.

$\Longrightarrow$

- there exists F_3
- supp$F_3 \subset N_\varepsilon(\Omega_1 \cap \Omega_2)$
- $\widehat{F}_3(k\Theta) = \widehat{F}_1(k\Theta)$



and not

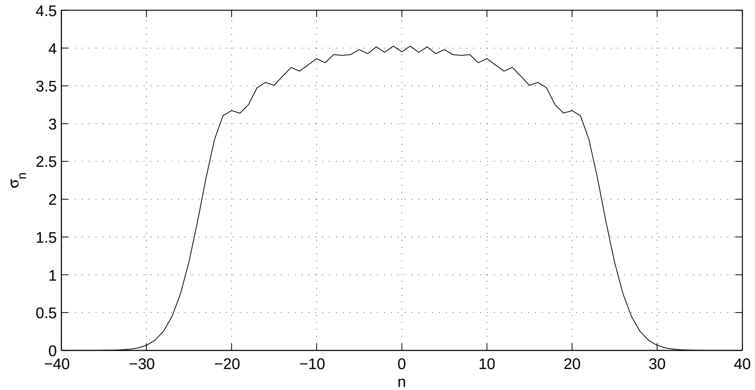


This requires convexity and unique continuation.

Finding the Convex Scattering Support

There exists $F \in L^2(B_R)$ $\iff$ $\left\{ \dfrac{f_n}{\left(\int_0^R J_n^2(ks)sds\right)^{\frac{1}{2}}} \right\} \in l^2$

$||J_n(r)||_{L^2(B_{25})}$



$$||J_n(r)||_{L^2(B_R)} \sim \begin{cases} R^{\frac{1}{2}}(R^2 - n^2)^{\frac{1}{4}} & |n| < R \\ \dfrac{1}{\sqrt{n+\frac{1}{2}}}\left(\dfrac{eR}{2(n+\frac{1}{2})}\right)^{(n+\frac{1}{2})} \sim 0 & |n| > R \end{cases}$$

Rapid Transition to Evanescence

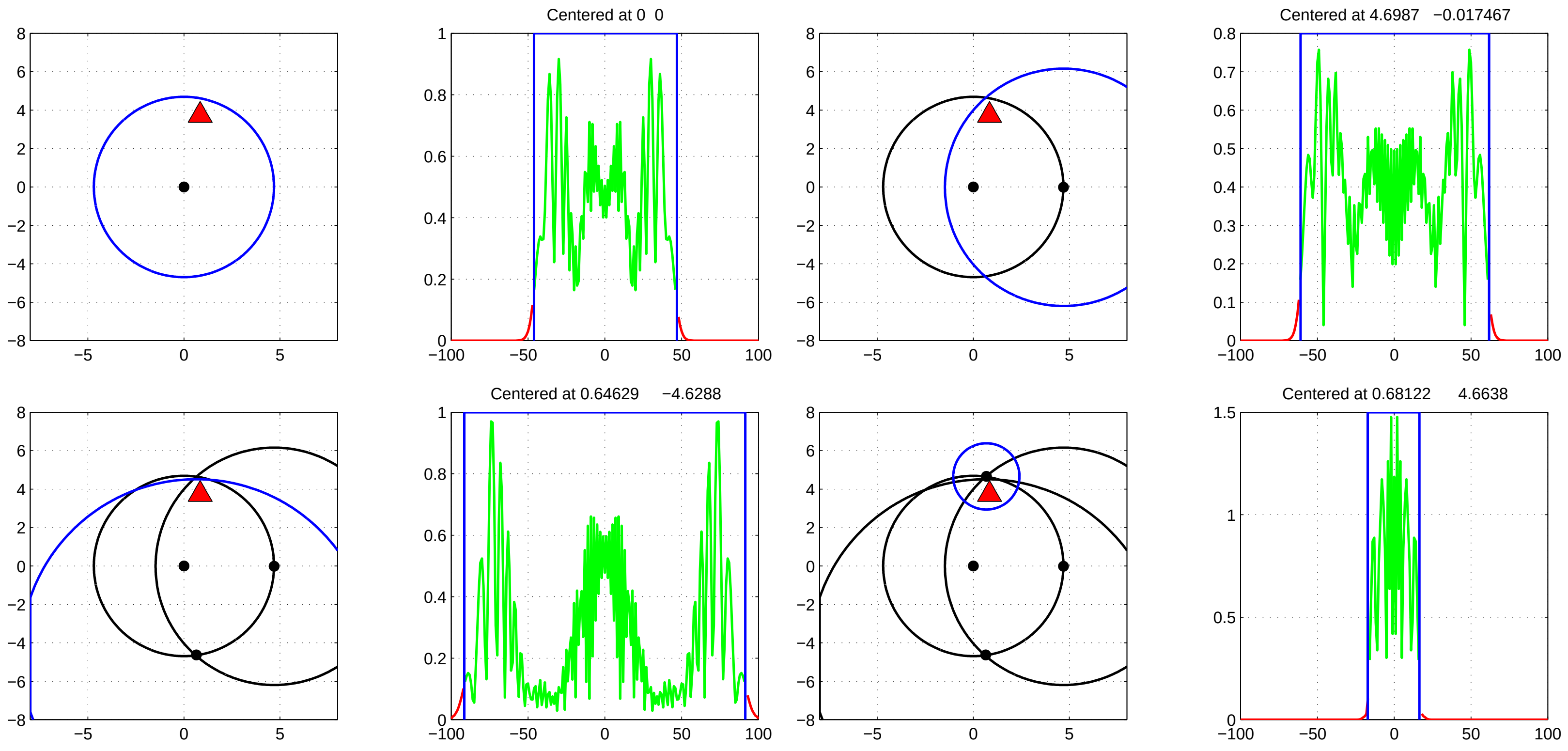
The homogeneous background exhibits a *rapid transition to evanescence.*

- $\sigma_n(R)$ uniformly big if $n < R$
- $\sigma_n(R)$ uniformly small if $n > R$
- Uniform contrast between big and small

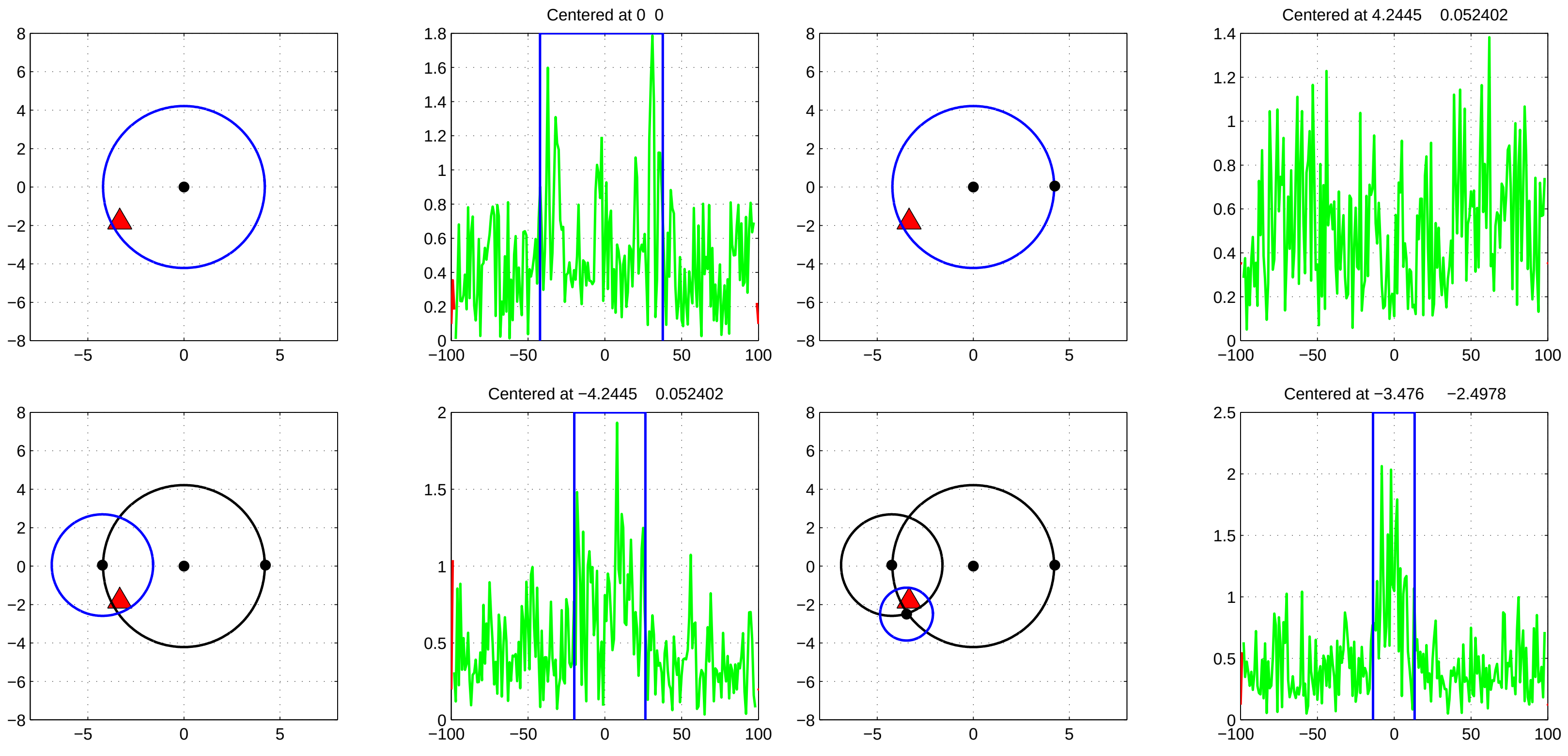
This makes the general criterion robust.

Next Slide: We will use the Rapid Transition to find a source triangle from its far field.

Locating the Scattering Support (k = 10)



70% additive random noise (k = 10)



Estimates of the Scattering Operator

Necessary conditions persist in the presence of multiple scattering.
Every reflected wave has a first bounce and a last bounce.

If supp $q \subset B_R$

$$\begin{aligned} (\mathcal{S}_q)_n^m &:= \left(e^{in\theta}, \mathcal{S}_q e^{im\theta}\right) \\ (\mathcal{S}_q)_n^m &\leq C_q \sigma_n(R) \sigma_m(R) \end{aligned}$$

and therefore

- The eigenvalues satisfy $|\lambda_n| \leq C_q \sigma_n(R)$
- The components of the eigenvectors satisfy $|v_n| \leq C_q \sigma_n(R)$
- The Fourier coefficients of the backscatttering kernel satisfy $|b_n| \leq C_q \sigma_n(2R)$

Factorization and Range of $\mathcal{S}_q$

	Send it back	bounce it around	Send it in
$\mathcal{S}_q\beta =$	$\mathcal{F}_0^+$	$\left(I - qG_0^+\right)^{-1} q$	${\mathcal{F}_0^+}^*$
$\mathcal{S}_q\beta =$	$\mathcal{F}_0^+$	$q\left(I - G_0^+ q\right)^{-1}$	${\mathcal{F}_0^+}^*\beta$
$\mathcal{S}_q\beta =$	$\left(\mathcal{F}_0^+\chi_{\text{supp}q}\right)$	$q\left(I - G_0^+ q\right)^{-1}$	$\left(\mathcal{F}_0^+\chi_{\text{supp}q}\right)^*\beta$

Linear Sampling Method (Colton-Kirsch) and subsequent factorization of scattering operator (Kirsch) motivated this work.