Analysis of a Multistatic Adaptive Target Illumination and Detection Approach (MATILDA) to Time Reversal Imaging

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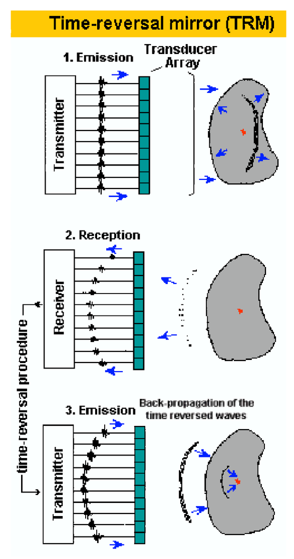
Overview of Time Reversal(TR)

Consider the wave equation

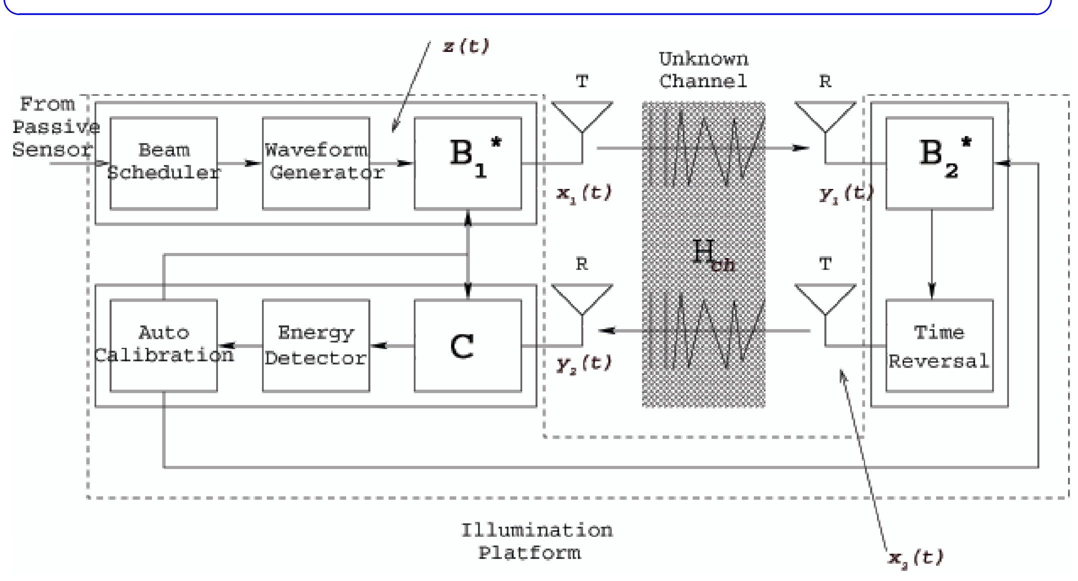
$$\kappa(r)\frac{\partial^2 P}{\partial^2 t^2} = \nabla^2\left(\frac{P}{\rho(r)}\right)$$

where $\kappa(r)$ is the compressibility, $\rho(r)$ is the density. This equation is time-reversal invariant. For every burst of sound $P(r,t)$ spreading from a source and possibly reflected, refracted or scattered at various locations in the medium, there exists, in theory an extended waveform $P(r,-t)$ that precisely retraces all of those complex paths and converges in synchrony at the original source, as if time were going backwards. This is the essence of time reversal [1].



- Use of iterative time reversaltechniques to focus energy in multitarget medium [2].
- Principal method: Diagonalization of time reversal operator (D.O.R.T Approach) [9]
- Applications of time reversal in
  - Target selection [5], Pulse-echo detection, Kidney stone destruction, Underwater acoustics[6], Communications, Detect cracks, defects in solids [4], Imaging([7,10]) unknown scattering environments.

MATILDA - A Generalized Approach to Time Reversal Imaging



- Box at upper left- transmitter incorporating beam scheduling, waveform selection, beamforming(BF) operator $\mathbf{B}_1$ and transmission into the medium, denoted by channel function $\mathbf{H}_{ch}$.
- Box on the right- receiver-transmitter for reinsertion using TR and spatial operator $\mathbf{B}_2$.
- Box on the lower left- receiver extracts the time reversed reinserted signal.

Standard time reversal - A specific case of this approache.i.e., when $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{I}$.

Mathematical Description

- N transducers, Imaging area divided into V voxels.
- Voxel characterized by its scattering coefficient.e.g., radar cross-section (RCS), denoted by $\{\sigma_v\}_{v=1}^V$.
- Channel model is narrowband far-field approximation [8]:

$$\underline{h}_{yv} = \left[\left(\frac{\exp(-j\omega/c\|r_{1k}-r_v\|)}{\|r_{1k}-r_v\|}\right)_{k=1\ldots N}\right]^T$$

- Channel between transmitted field and received backscattered field.

$$\begin{aligned}\mathbf{H}_y &= [\underline{h}_{y1}, \underline{h}_{y2}, \cdots, \underline{h}_{yV}]\\ \mathbf{D} &= \text{diag}\{\sigma_v\}_{v=1}^V\\ \mathbf{H}_{ch} &= \mathbf{H}_y\mathbf{D}\mathbf{H}_y^T\end{aligned}$$

Noise Analysis of Time Reversal Probing

$$\begin{aligned}\underline{x}_1 &= \mathbf{B}_1^*\underline{z}\\ \underline{y}_1 &= \mathbf{H}_{ch}\underline{x}_1 + n_1 = \mathbf{H}_y\mathbf{D}\mathbf{H}_y^T\mathbf{B}_1^*\underline{z} + n_1\\ \underline{x}_2 &= \mathbf{B}_2^*\underline{y}_1^* = \mathbf{B}_2^*\mathbf{H}_y^*\mathbf{D}^*\mathbf{H}_y^H\mathbf{B}_1\underline{z}^* + \mathbf{B}_2^*n_1^*\\ \underline{y}_2 &= \mathbf{H}_{ch}\underline{x}_2 + n_2 = \mathbf{H}_y\mathbf{D}\mathbf{H}_y^T\mathbf{B}_2^*\mathbf{H}_y^*\mathbf{D}^*\mathbf{H}_y^H\mathbf{B}_1\underline{z}^* + \mathbf{H}_y\mathbf{D}\mathbf{H}_y^T\mathbf{B}_2^*n_1^* + n_2\end{aligned}$$

where receiver noises n_1, n_2 are independent circularly symmetric complex Gaussian noise with variance $\zeta_n^2\mathbf{I}$.

GOAL :

1. To determine the Cramer Rao Bounds (CRB) on the scatter coefficients which gives lower bounds on the variance of any unbiased estimator.
2. Find the best possible beam steering operators that minimizes this bound.

Computing CRBs

- BF method: Bounds on estimation of scatter coefficients given measurement $\underline{y}_1$.
- TR method: Bounds on estimation of scatter coefficients given measurement $\underline{y}_2$.
- Starting Point:

  CRB for a set of parameters $\underline{\theta}$ given a vector of complex Normal observations with mean $\underline{\mu}(\underline{\theta})$ and covariance matrix $R(\underline{\theta})$ [3]

$$\begin{aligned}[I(\underline{\theta})]_{ij} &= 2Re\left\{\frac{\partial\underline{\mu}(\underline{\theta})^H}{\partial\theta_i}R^{-1}(\underline{\theta})\frac{\partial\underline{\mu}(\underline{\theta})}{\partial\theta_j}\right\}\\ &+tr\left[R^{-1}(\underline{\theta})\frac{\partial R(\underline{\theta})}{\partial\theta_i}R^{-1}(\underline{\theta})\frac{\partial R(\underline{\theta})}{\partial\theta_j}\right]\end{aligned}$$

where $Re\{\cdot\}$ indicates the real operator and $I(\underline{\theta})$ is the VxV Fisher Information Matrix.

CRB for a general beamforming case

- Compute and maximize FIM under constraint on energy transmitted: $\|\mathbf{B}_1^*\underline{z}\|^2 \leq E_1^2$
- Search for rank one beam steering matrices $\mathbf{B}_1$ (completely described by a beam steering vector $\underline{p}_1$) as a start.
- Best $\underline{p}_1$ would be given by,

$$\underline{p}_1 = E_1(eigvectmax(\tilde{\mathbf{H}}))$$

$$\tilde{\mathbf{H}} = \sum_{i=1}^V \underline{h}_{yi}\underline{h}_{yi}^H\|\underline{h}_{yi}\|^2$$

where $eigvectmax$ denotes the unit norm eigenvector corresponding to the maximum eigenvalue.

1. Can extend to multiple BF implementations: $\mathbf{B}_1$ has higher rank and is composed of eigenvectors of $\tilde{\mathbf{H}}$.
2. Idea very similar to D.O.R.T., where eigenvectors are used to focus on a scatterer.
3. To maximize FIM for one scatter coefficient v, best $\underline{p}_v$ is the Green's function between the voxel and the antennas, $\underline{h}_{yv}$.
4. Result suggests that D.O.R.T approach is optimal, or atleast nearly optimal in the sense of estimating scatter coefficients.

CRB for a general TR method

- Again assume initially $\mathbf{B}_1$ and $\mathbf{B}_2$ are rank one matrices completely defined by two beam forming vectors $\underline{p}_1$ and $\underline{p}_2$.
- Maximize $trace(FIM(\underline{p}_1, \underline{p}_2))$ subject to energy constraints: $\|\underline{p}_1\|^2 \leq E_1^2$, $\|\underline{p}_2\|^2 \leq E_2^2$ at high SNR.
- Solve using Lagrangian multipliers. Objective function L,

$$L = \frac{2|\underline{p}_2^H\mathbf{H}_{ch}\underline{p}_1^*|^2}{\zeta_n^2}(\underline{p}_2^H\tilde{\mathbf{H}}\underline{p}_2) + \lambda_1(E_1^2 - \|\underline{p}_1\|^2) + \lambda_2(E_2^2 - \|\underline{p}_2\|^2)$$

Optimal Solution:

$$\begin{aligned}\underline{p}_2 &= \mathbf{H}_{ch}\underline{p}_1^*\\ \underline{p}_1 &= eigvector\left(\tilde{\mathbf{H}} + \rho(\underline{p}_1)\mathbf{H}_{ch}\mathbf{H}_{ch}^*\right)\end{aligned}$$

1. Optimal solution would be to generate beam steering matrices iteratively.
2. If $\rho(\underline{p}_1) = 0$, then $\underline{p}_1$ is the BF solution.
3. When channel, $\mathbf{H}_{ch} = \mathbf{I}$, then the optimal solution is just $\underline{p}_1 = \underline{p}_2^*$.

Specific Time Reversal Probing System

- Use the optimal beam forming solution obtained above for TR.i.e., $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{\Pi}_{yv}$.

$$\mathbf{\Pi}_{yv} = \frac{\underline{p}_v\underline{p}_v^H}{\|\underline{p}_v\|^2}$$

- $\mathbf{\Pi}_{yv}$ - projects spatio-temporal waveform onto subspace spanned by conjugate Green's function.
- $\mathbf{\Pi}_{yv}$ - beamsteering operation that focuses on a voxel v for non dispersive channel.
- For this case can easily extract estimate of scattering attenuation of a particular voxel v. Can show when number of transducers is very large that,

$$\mathbf{\Pi}_{yv}\underline{y}^{(2)} = \frac{|\sigma_v|^2}{\|\underline{p}_v\|^2}\underline{y}^{(1)}$$

Compute the CRBs on unbiased estimators of scatter coefficients. Investigate advantages over conventional beam forming methods.

CALIBRATED CASE - Known sensor positions

Doing the math,

1. Conditions for this TR method to do better than the BF method on a particular voxel v at high SNR reduces to

$$\frac{|\underline{p}_v^T\mathbf{H}_{ch}\underline{p}_v|^2}{\|\underline{p}_v\|^2} \geq \frac{1}{V}\left(\sum_{w=1}^V\frac{|\underline{p}_w^T\mathbf{H}_{ch}\underline{p}_w|^2}{\|\underline{p}_w\|^2}\right)$$

$$\|\underline{p}_v\|^2 \leq \left(\frac{1}{V}\sum_{w=1}^V\|\underline{p}_w\|^2\right)$$

2. Implication: TR more informative than BF for strong scattering voxels far away from antennas.

Extend analysis to an uncalibrated case.i.e., when the sensor positions aren't known precisely.

UNCALIBRATED CASE - Unknown sensor positions

1. Set of unknown parameters now include scatter coefficients and unknown sensor locations.
2. Both TR and BF bounds are higher than calibrated case due to position uncertainty.
3. NOTE :Margin for improvement of TR vs BF increases for uncalibrated antenna.

Simulation Results

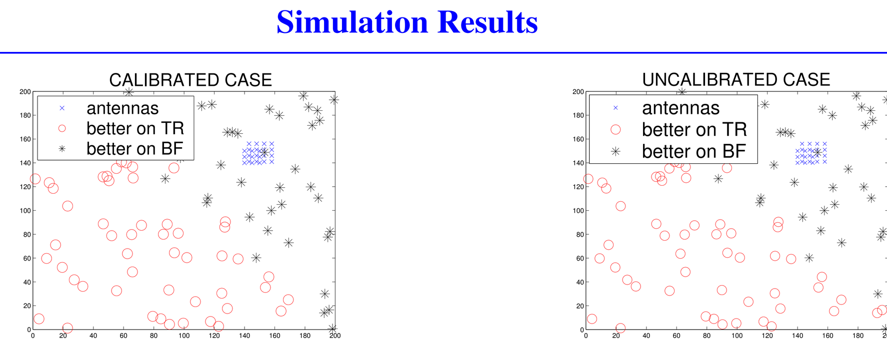


Figure 3 Results showing that TR performs better on voxels that are typically further away from the antennas at high SNR for calibrated and uncalibrated cases.

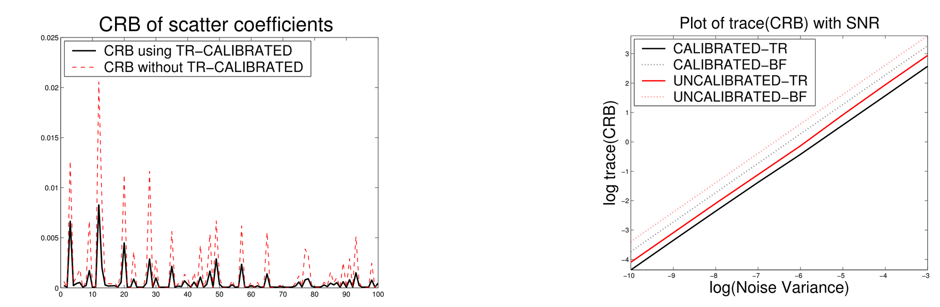


Figure 4 (a) Simulation plotting the exact CRB values on the scatter coefficients for calibrated case. (b) Simulation results plotting the trace(CRB) for the TR and the BF cases for both the Calibrated and Uncalibrated scenario. Plot indicates that TR has outperformed BF for both the calibrated and the uncalibrated cases and the bounds in the uncalibrated cases are simply scaled versions of their corresponding calibrated values.

Adaptive Autocalibration

- To suitably adapt the signal transmitted to calibrate the sensor locations.
- OUTLINE OF ALGORITHM

1. Create null at the unknown sensor locations, $\underline{r}$, using TR.
2. Measure total energy, $Q(\underline{r})$, received at the sensors.
3. Use Gradient Descent Algorithm to update the new sensor locations.

$$\underline{r}(k+1) = \underline{r}(k) - \mu\nabla_{\underline{r}}Q(\underline{r}(k))$$

4. Repeat until convergence.

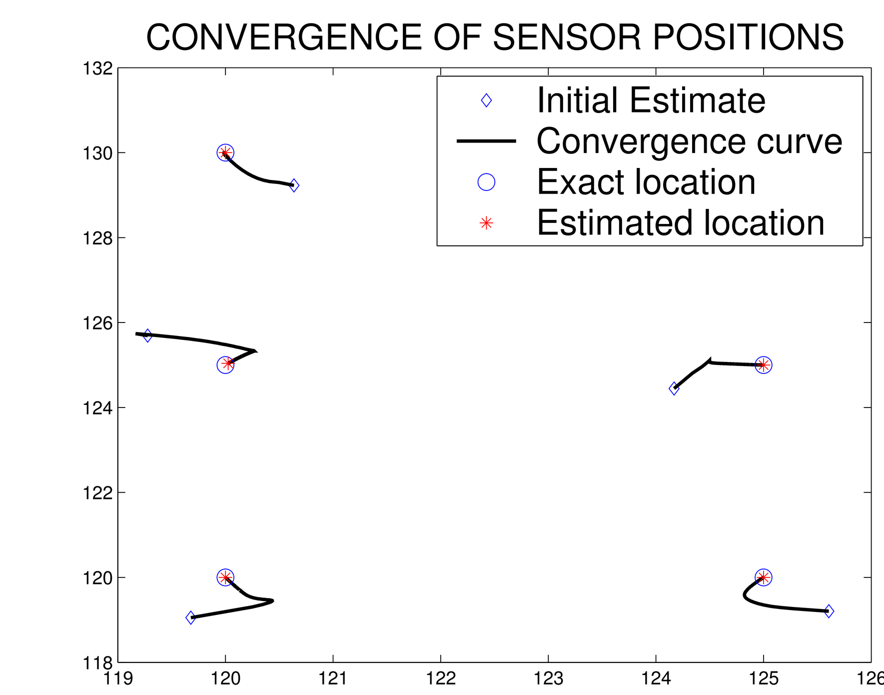


Figure 5 Five sensor locations were considered unknown with an uncertainty of 15% of their separation $\lambda/2$. Plot shows the autocalibration of the sensor positions at a SNR of $20dB$.

Conclusions and Further Study

CONCLUSIONS

- Studied general TR approach to imaging and concluded that the best solution for estimating the scatter reflectivities would be to generate the beam steering operators using an iterative process.
- Analyzed MATILDA TR method: showed advantages over the conventional methods both in the calibrated and uncalibrated cases.
- Developed TR autocalibration technique to calibrate unknown sensor locations.

FURTHER STUDY

- Look at adaptive implementation of optimal solution and analyze its performance.
- Implement efficient estimate of scatter coefficients.
- Extend optimality conditions to uncalibrated case.
- Compute and analyze bounds for dense media.

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