Near Field imaging : a study of the SNR issue

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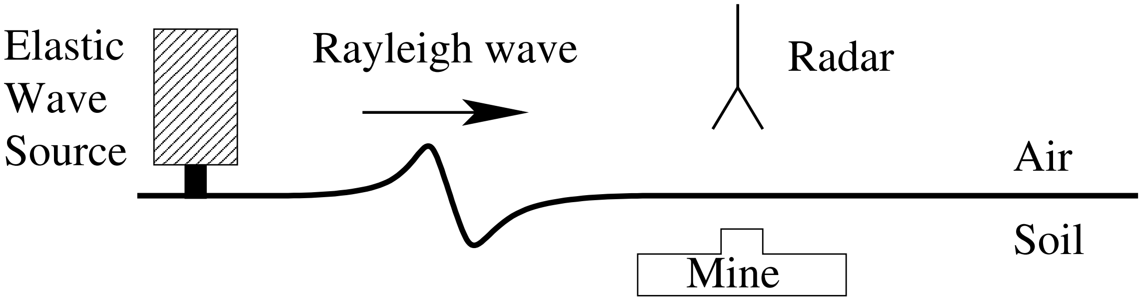
Context: seismic detection of landmines

Experimental setup developped at Georgia Tech. by Waymond R. Scott *et al.*



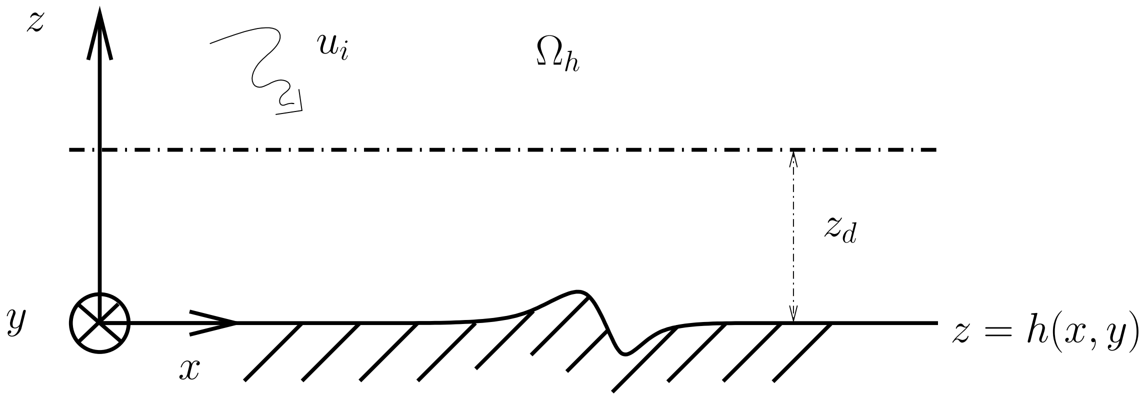
- Measurement of the surface displacement induced by the propagation of a seismic wave with a GPR operating at 8GHz ($\lambda = 3.25$ cm)
- Observation of particular resonnances around the mine
- New tool to help to distinguish between mine and clutter objects

To have good cross range resolution, the radar must be placed very close to the surface (3-4 cm)

Investigation of the use of Near-Field data to enhance the resolution

Presentation of the model

- Scalar wave equation
- Surface displacement described by $h(x, y), x, y \in \mathbb{R}^2$
- Dirichlet boundary condition on $z = h(x, y)$
- Frequency domain with angular frequency $\omega = ck$



For an incident wave u_i, we want to recover the scattered field u_s such that:

$$\left\{\begin{array}{lll} (a) & \Delta u_s + k^2 u_s = 0, & \text{for } z > h(x, y) \\ (b) & u_s(x, y, h(x, y)) = -u_i(x, y, h(x, y)), & \forall x, y \in \mathbb{R}^2, \\ (c) & \text{Sommerfeld Radiation condition.} & \end{array}\right.$$

Linearization and position of the problem

Linearization of (b)

+ Born approximation

+ Substraction of the scattered field u_s^0 produced by a flat surface

$\Longrightarrow$ New problem with unknown $u = u_s - u_s^0$:

$$\begin{cases} (a') & \Delta u + k^2 u = 0, & \text{for } z > 0, \\ (b') & u(x,y,0) = -h(x,y)\partial_z u_i(x,y,0) \overset{\text{def}}{=} f(x,y), & \forall x,y \in \mathbb{R}^2, \\ (c') & \text{Sommerfeld Radiation condition.} & \end{cases}$$

Measurements are made at distance z_d

Simulation of actual measurements by adding *i.i.d.* Gaussian noise :

$$g(x,y) = u(x,y,z_d) + w(x,y)$$

$$\boxed{\text{Given the data } g(x,y) \text{ we seek to recover the displacement } h(x,y)}$$

Spectral representation

- Perform the Fourier Transform in the transverse coordinate $\underline{\rho} = (x, y)$:

$$\begin{cases} \partial_z^2 \hat{u}(\underline{\xi}, z) + (k^2 - |\underline{\xi}|^2)\hat{u}(\underline{\xi}, z) = 0, \quad \forall \underline{\xi} \in \mathbb{R}^2 \text{ and } \forall z > 0, \\ \hat{u}(\underline{\xi}, 0) = \hat{f}(\underline{\xi}), \\ \text{Radiation condition} \end{cases}$$

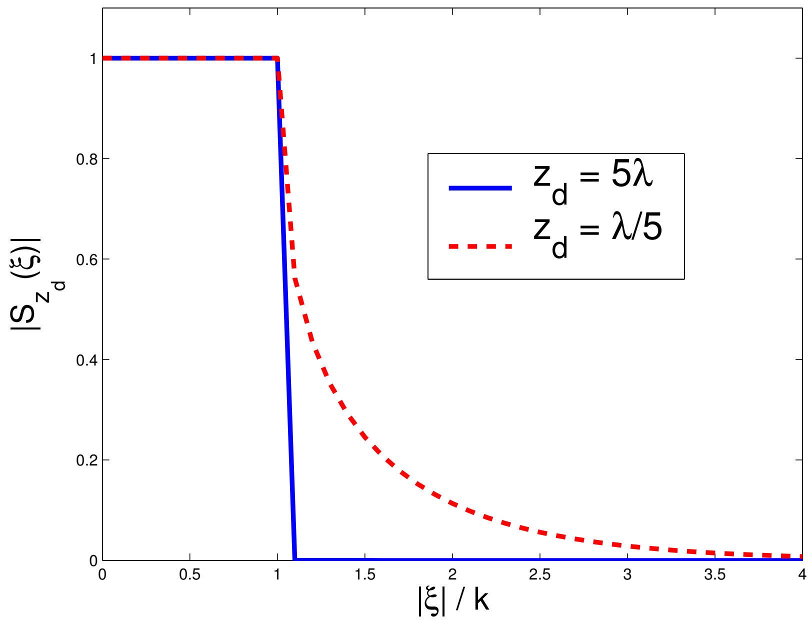
- The solution can be written as a superposition of plane waves:

$$u(\underline{\rho}, z_d) = \frac{1}{2\pi} \int_{\underline{\xi}} \hat{f}(\underline{\xi}) e^{i(\underline{\rho}.\underline{\xi} + z_d\sqrt{k^2 - |\underline{\xi}|^2})} d\underline{\xi} = \frac{1}{2\pi} \int_{\underline{\xi}} \hat{S}_{z_d}(\underline{\xi}) \hat{f}(\underline{\xi}) e^{i(\underline{\rho}.\underline{\xi})} d\underline{\xi}$$

- The transfert function of the system is $\hat{S}_{z_d}(\underline{\xi}) = e^{iz_d\sqrt{k^2 - |\underline{\xi}|^2}}$

  Propagating wave for $|\underline{\xi}| \leq k$ and **evanescent wave** for $|\underline{\xi}| > k$

Far-Field region version Near-Field region



Modulus of $\hat{S}_{z_d}(\underline{\xi})$ *versus* $|\underline{\xi}|/k$.

Near Field Region: $z_d < \lambda$, the contribution of evanescent modes is important

Far Field Region: $z_d \gg \lambda$, the contribution of evanescent modes can be neglected

The propagation acts as a low frequency filter

Inversion with Far-Field data

Evanescent waves are neglected. Inverse of $\hat{S}_{z_d}$ is approximated by:

$$\hat{S}_{z_d}^{FF-}(\underline{\xi}) = \begin{cases} e^{-ia\sqrt{k^2-\left|\underline{\xi}\right|^2}} & \text{for } \left|\underline{\xi}\right| \leq k \\ 0 & \text{else.} \end{cases}$$

which leads to the following "Far-Field" estimation of $\hat{f}$:

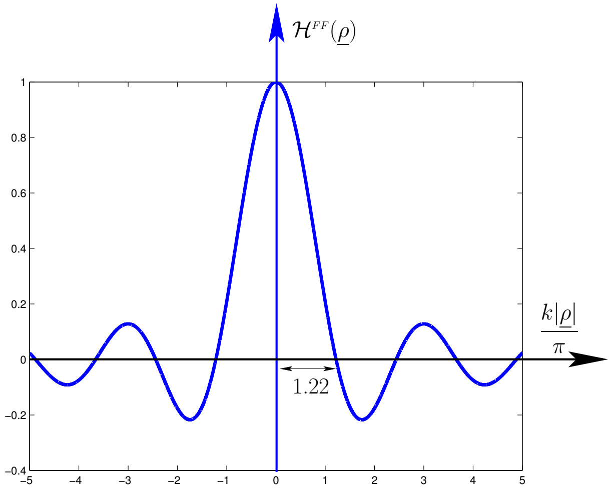
$$\hat{f}^{FF}(\underline{\xi}) = \mathbf{1}_k(\underline{\xi})\hat{f}(\underline{\xi}) + \hat{S}_{z_d}^{FF-}(\underline{\xi})\hat{w}(\underline{\xi}),$$

where $\mathbf{1}_k$ is the characteristic function of the frequency band $\mathcal{B}_k = \{\underline{\xi}, |\underline{\xi}| \leq k\}$

Resolution analysis in the Far-Field case

- In the space domain, the first term is a convolution with kernel

$$\mathcal{H}^{FF}(\underline{\rho}) = \frac{k}{2\pi} \frac{J_1(k|\underline{\rho}|)}{|\underline{\rho}|}$$



$\Longrightarrow$ The resolution is given by the **Rayleigh criterion**: $\boxed{R_{FF} \geq 1.22\frac{\lambda}{2}.}$

- $|\hat{S}^{FF-}_{z_d}| \leq 1 \quad \Longrightarrow \quad$ the **noise is NOT amplified**

Inversion with Near-Field data

Evanescent waves "greater than noise" are taken into account. If SNR is an estimation of the Signal-to-Noise Ratio, the inverse of $\hat{S}_{z_d}$ is approximated by:

$$\hat{S}_{z_d}^{NF-}(\underline{\xi}) = \begin{cases} e^{-ia\sqrt{k^2-\left|\underline{\xi}\right|^2}} & \text{for } \left|\underline{\xi}\right| \leq k_{NF} \\ 0 & \text{else,} \end{cases}$$

where $k_{NF} = \max\left\{\left|\underline{\xi}\right|, \ |\hat{S}_{z_d}(\underline{\xi})| \geq 1/\mathrm{SNR}\right\}$,

which leads to the following **"Near-Field" estimation of $\hat{f}$** :

$$\hat{f}^{NF}(\underline{\xi}) = \mathbf{1}_{k_{NF}}(\underline{\xi})\hat{f}(\underline{\xi}) + \hat{S}_{z_d}^{NF-}(\underline{\xi})\hat{w}(\underline{\xi}),$$

Resolution analysis in the Near-Field case

- **The improvement of the resolution is given by :**

$$\frac{R_{FF}}{R_{NF}} = \left[1 + \frac{1}{4\pi^2}\left(\frac{\lambda}{z_d}\right)^2 \log^2 \mathrm{SNR}\right]^{\frac{1}{2}}$$

	Distance of observation z_d/λ					
SNR	$1/100$	$1/10$	$1/5$	$1/2$	1	2
20 dB	73.300	7.397	3.798	1.774	1.239	1.065
10 dB	36.660	3.798	2.087	1.239	1.065	1.016
6 dB	22.086	2.422	1.489	1.093	1.024	1.006

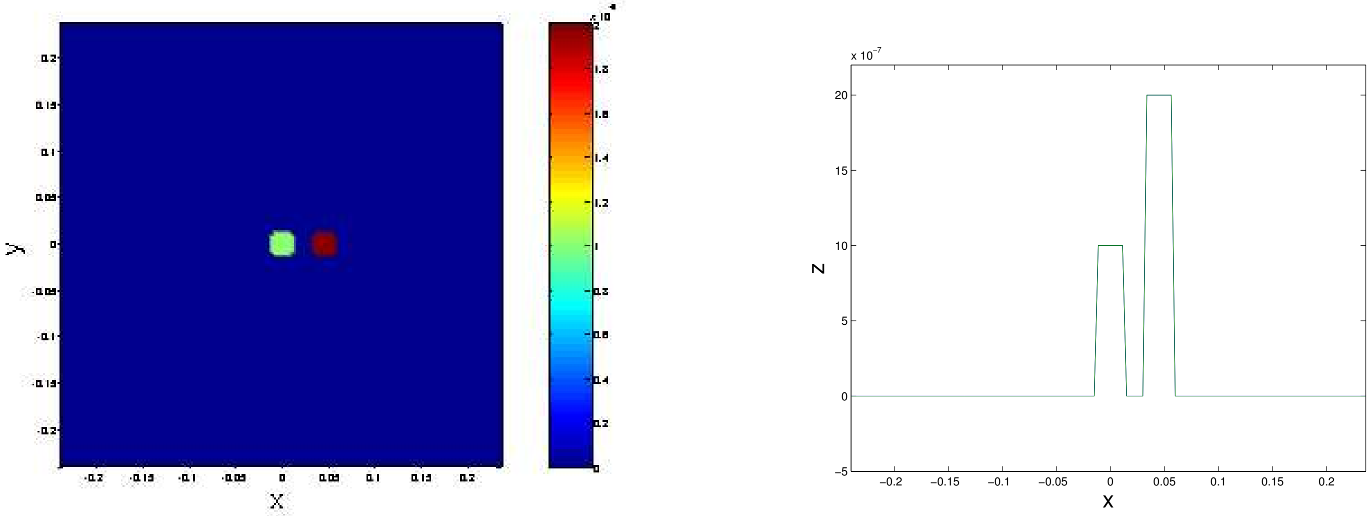
values of R_{FF}/R_{NF} *versus different values of noise and observation distance* z_d.

- **The noise is amplified** in the Fourier range $[k, k_{NF}]$

A numerical example

Central Frequency :	5 GHz ($\lambda = 6$ cm)
Measurements:	128 x 128 array
Spacing between points:	$\lambda/16$

Single Monochromatic plane wave with normal incidence

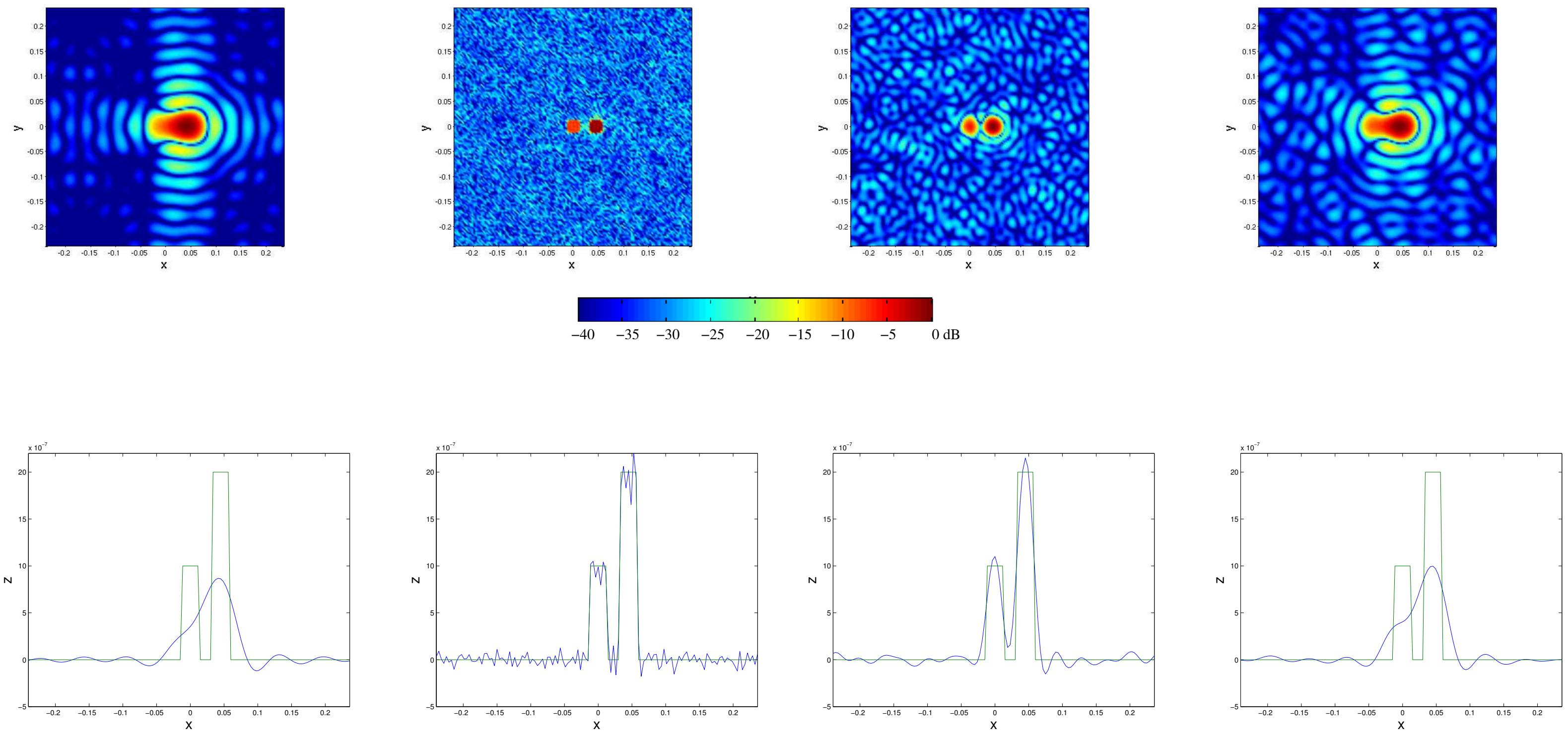


2D linear scale

1D section in the plane $y = 0$

Original shape: two disks of diameter $\lambda/2$ and separated by a distance of $\lambda/4$.

Monochromatic plane wave — SNR = 20 dB



far-field

near-field
$z_d = \lambda/10$

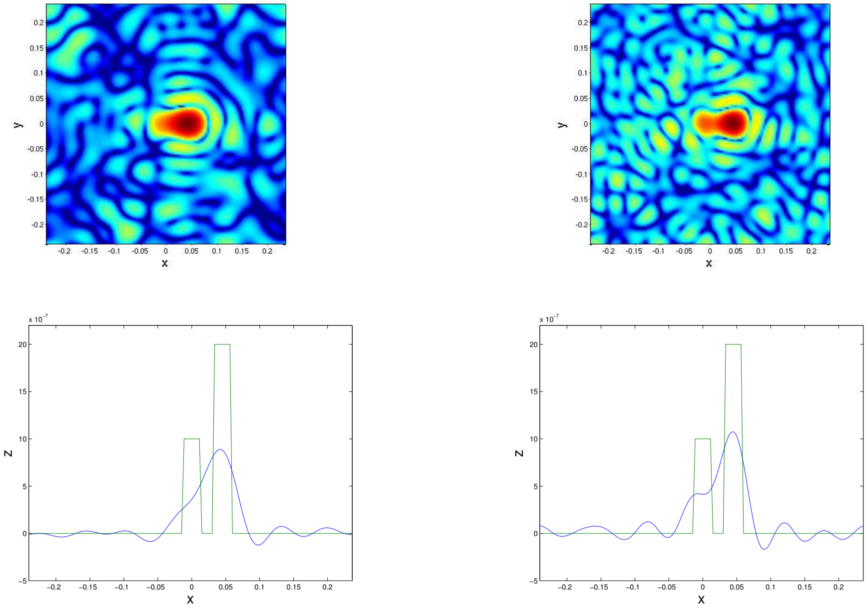
near-field
$z_d = \lambda/2$

near-field
$z_d = 2\lambda$

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$.

Monochromatic plane wave — SNR = 6 dB



far-field

near-field

$z_d = \lambda/2$

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$.

Use of a broadband incident wave

Incident broadband plane wave with direction of incidence $\underline{\hat{k}}_i$:

$$u_i(\underline{r},t) = \psi(\frac{1}{c}\underline{\hat{k}}_i.\underline{r} - t) = \frac{1}{2\pi}\int_\omega e^{i(\frac{\omega}{c}\underline{\hat{k}}_i.\underline{r}-\omega t)}\tilde{\psi}(\omega)d\omega$$

where $\tilde{\psi}$ denotes the Fourier Transform in time of ψ.

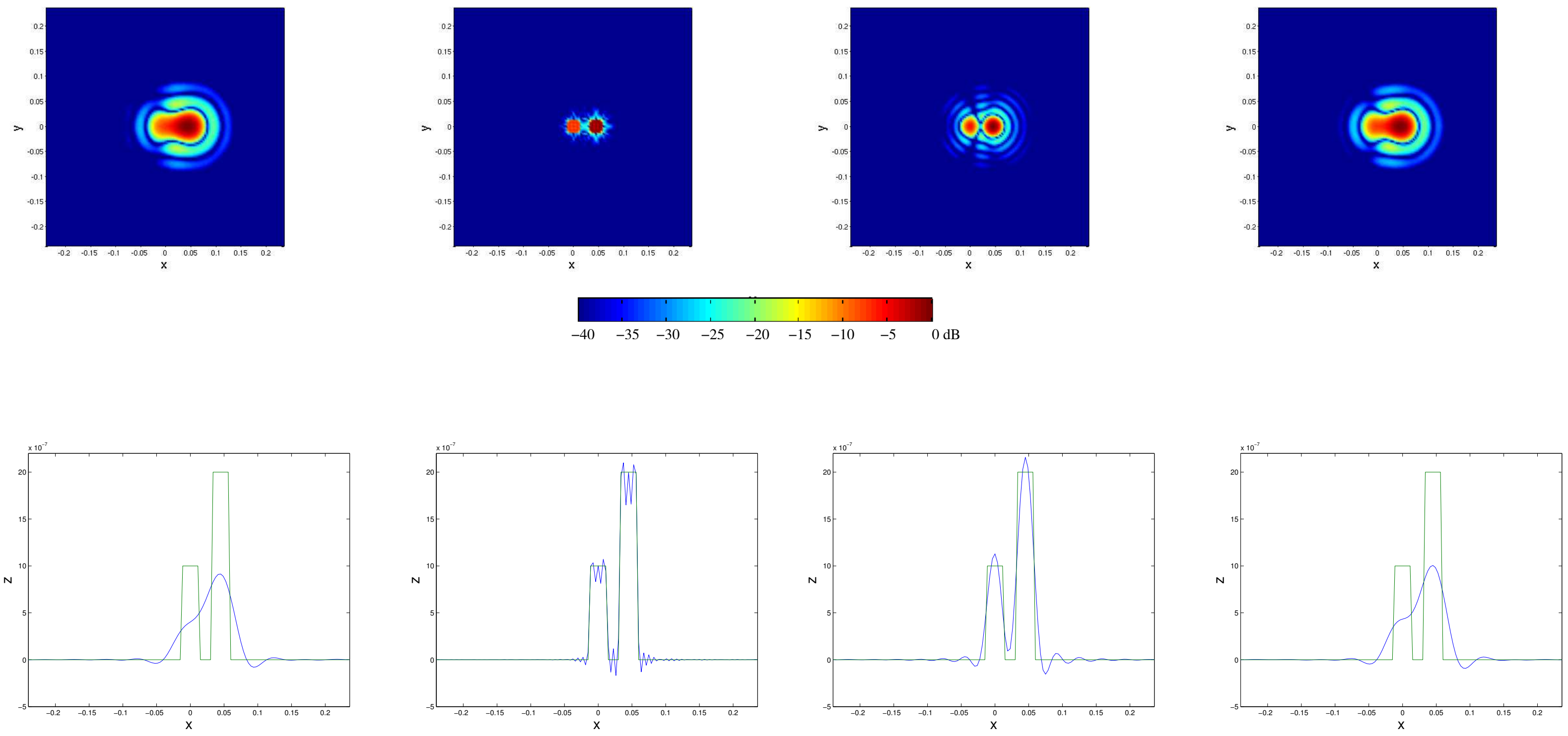
For each angular frequency ω one obtains an image of the shape denoted $h_\omega(\underline{\rho})$

Then **compute the average value** using the Power Spectrum of ψ as weight:

$$\boxed{h_{BB}(\underline{\rho}) = \frac{1}{\|\tilde{\psi}\|_2^2}\int_\omega h_\omega(\underline{\rho})|\tilde{\psi}(\omega)|^2 d\omega}$$

In the following, we note $\varphi(\omega) = \dfrac{|\tilde{\psi}(\omega)|^2}{\|\tilde{\psi}\|_2^2}$

Broadband plane wave - SNR= 20dB



far-field

near-field
$z_d = \lambda/10$

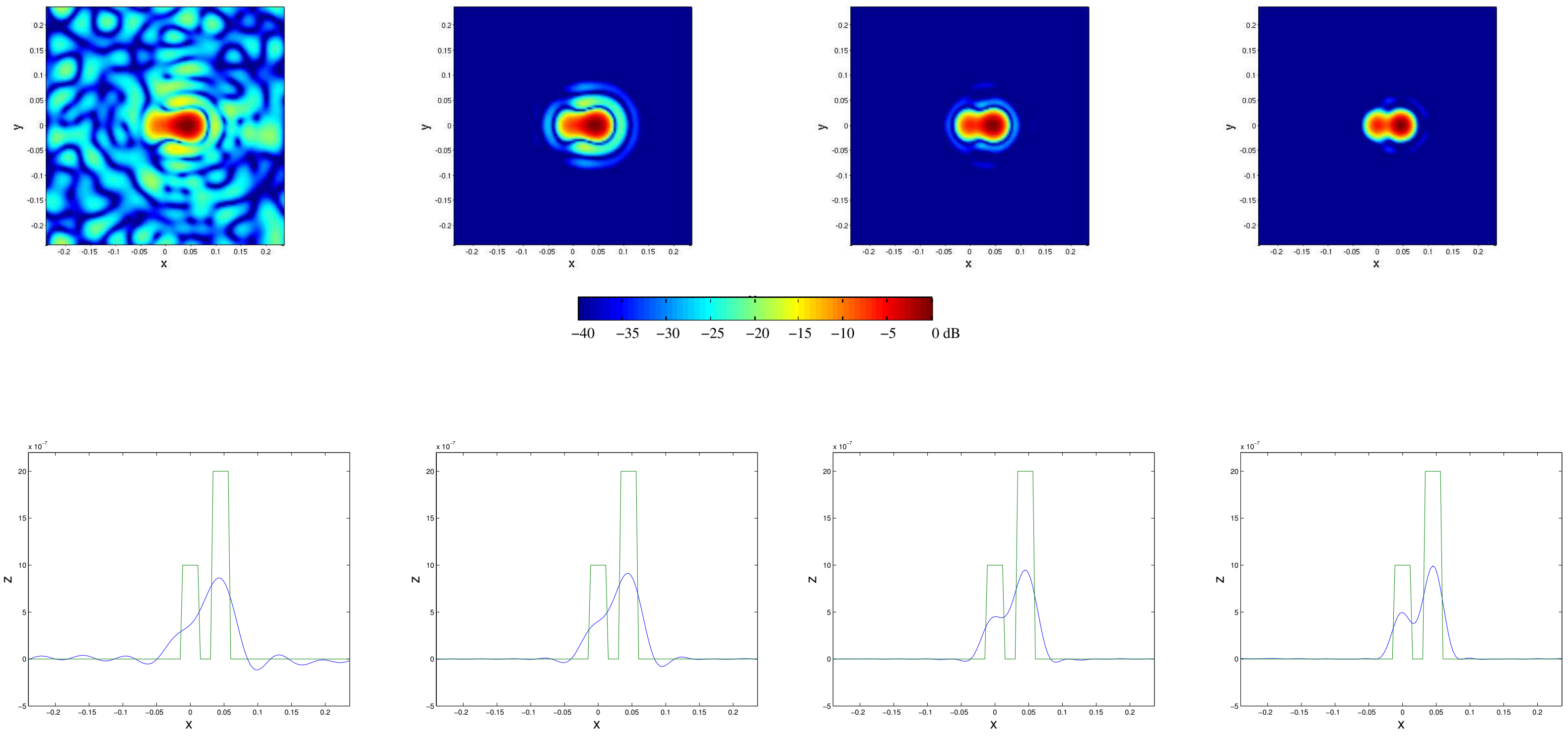
near-field
$z_d = \lambda/2$

near-field
$z_d = 2\lambda$

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$*.*

Broadband plane wave - Far Field - SNR= 6dB

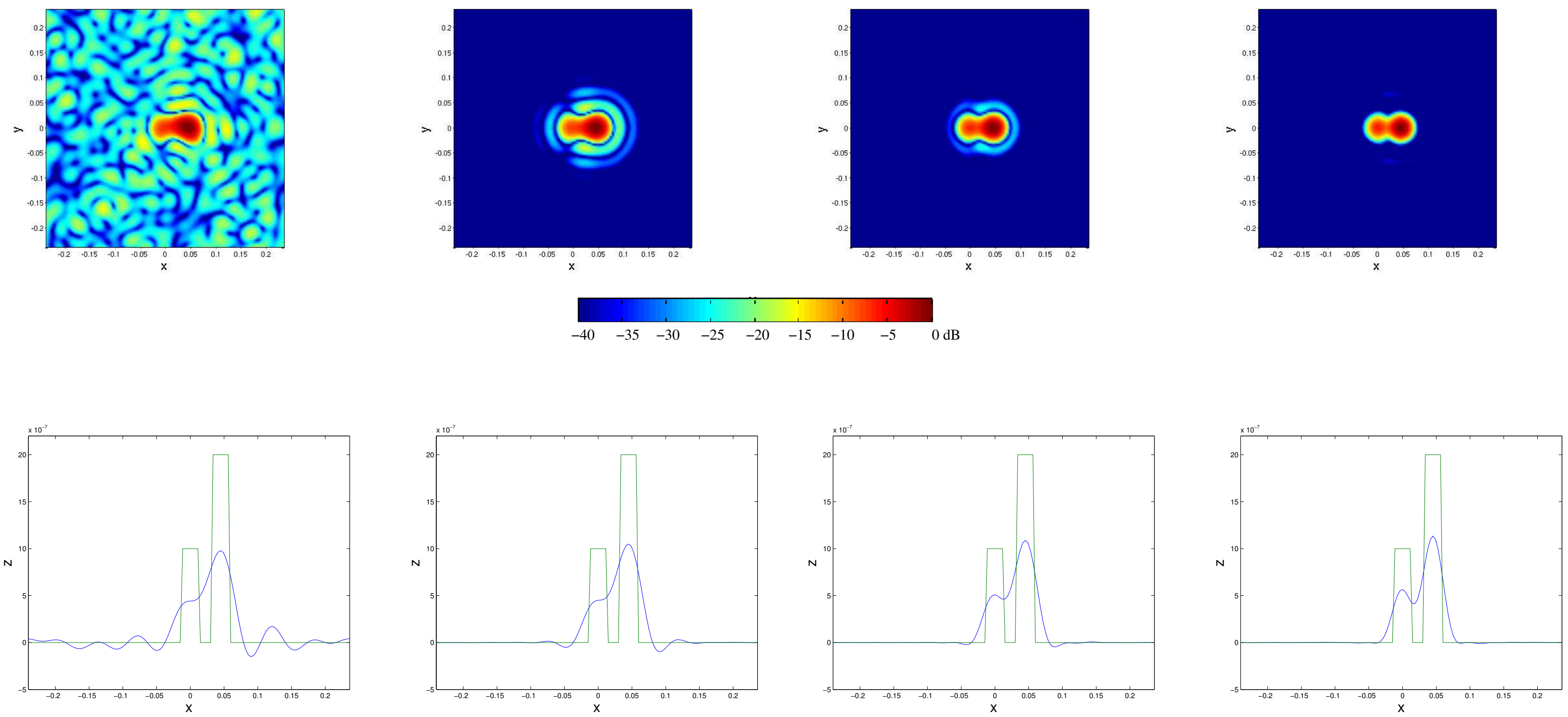


narrowband *50%* *100%* *150%*

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$.

Broadband plane wave - Near Field - SNR= 6dB - $z_d = \lambda/2$ (No correction on k_{NF})

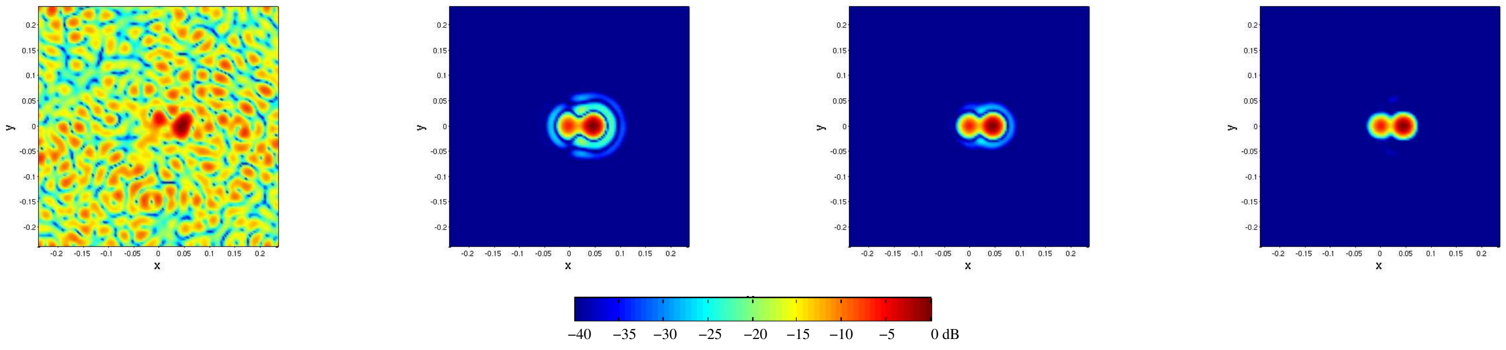


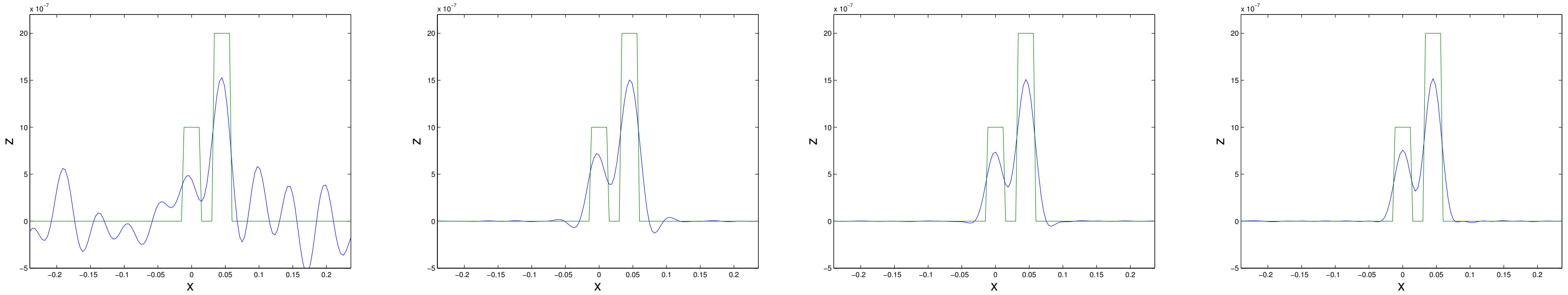
narrowband *50%* *100%* *150%*

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$.

Broadband plane wave - Near Field - SNR= 6dB - $z_d = \lambda/2$ (k_{NF} is 10% greater)





narrowband *50%* *100%* *150%*

Above: 40dB scale.

Bottom: comparison between original and reconstructed shape in the plane $y = 0$.

Conclusion

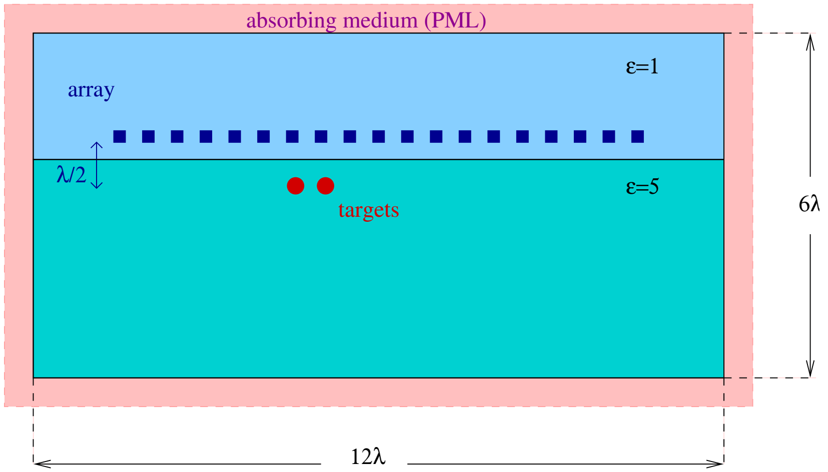
- Imaging with Near-Field data is a way to enhance the resolution
- SNRissue : important trade-off between resolution and noise amplification
- Interest of a broadband signal

Work in progress: Investigation of Near-Field inversion for imaging the underground with a GPR.

Work in progress

Investigation of Near-Field inversion for imaging the underground with a GPR

Typical setup for numerical simulations



- Resolution of the scalar wave equation with a 2D finite element code.
- The size of the computational domain is 12λ x 6 λ with 64 points per wavelength, surrounded by perfectly matched layers. The central wavelength is $\lambda = 30cm$ (at central frequency $f_0 = 1GHz$ with $c_0 = 3\,10^8 m.s{-}1$).
- The array is above the surface (in the air, $\epsilon_r = 1$). It has 18 elements (source-receivers) separated by $\lambda/2$ located at $(x_p)_{1\leq p\leq 18}$. The array data is the response matrix $(P_{sr}(t))$ measured for all elements (x_s, x_r) at time t.
- 2 single points scatterers are in the soil. They are situated at distance of $\lambda/2$ from the array and close to each other (typically $\lambda/4$ appart).

Zero offset migration and Near-Field migration

In a homogeneous medium, the imaging functional for zero-offset Kirchhoff Migration at a search point y^S is:

$$I^{ZOM}(y^S) = \sum_p P_{pp}(2\tau(x_p, y^S)) = \sum_p \frac{1}{2\pi} \int \hat{P}_{pp}(\omega) e^{-i\omega\tau(x_p, y^S)} d\omega$$

where $\tau(x_p, y^s) = |x_p - y^S|/c0$ is the travel time.

In the Near-Field case, amplitudes become important so we introduce the correction (we use the Born approximation here):

$$I^{NFM}(y^S) = \sum_p \frac{1}{2\pi} \int \hat{P}_{pp}(\omega) \hat{G}(x_p, y^S, \omega) \hat{G}(y^S, x_p, \omega) d\omega$$

where $\hat{G}(x_p, y^S, \omega)$ is the Green function for a homogeneous medium.

Layered background

For a layered medium, we use the spectral decomposition of the Green function in plane waves and introduce a cutoff for regularization.

$$\hat{G}(x,z,x',z',\omega) = \frac{i}{2(2\pi)} \int (1+R(\kappa)e^{2iz'\sqrt{k_0-\kappa^2}}) \frac{e^{i\kappa(x-x')+i(z-z')\sqrt{k_0^2-\kappa^2}}}{\sqrt{k_0^2-\kappa^2}} d\kappa$$

where $k_0 = \omega/c_0$ is the wave number and $R(\kappa)$ is the reflection coefficient for the index of refraction n^2:

$$R(\kappa) = \frac{\sqrt{k_0^2-\kappa^2} - \sqrt{n^2k_0^2-\kappa^2}}{\sqrt{k_0^2-\kappa^2} + \sqrt{n^2k_0^2-\kappa^2}}$$

Questions that we want to address

- Near-field correction: We can use many different imaging functionals. How can we take advantage of the measured Near-Field information ?
- Clutter: What is the effect of a random soil-air interface ?
- Born approximation: The targets are very close to the surface. Can multiple reflections be neglected ?
- Signal to Noise Ratio: The noise plays a very important role in Near-Field.
- Use of broadband signals.