OPTIMAL MEASUREMENTS, TIME-REVERSAL, and FREQUENCY TUNING

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Disclaimer: factors of 2π have been set equal to 1.

- Problem:
  - What measurements give the best signal-to-noise ratio?
  - What measurements are most different from predicted measurements?
- Precise formulation
  - "measurements"
  - criteria for "most" and "best"
- Adaptive algorithm
- The answer
- Experimental & numerical confirmation
- Discussion and open problems

Measurements

- Maxwell's equations in the time domain:

$$\nabla \times \boldsymbol{\mathcal{E}} = -\partial_t \boldsymbol{\mathcal{B}}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \partial_t \boldsymbol{\mathcal{D}}$$

- Half-space geometry

- Apply a downgoing wave $\boldsymbol{\mathcal{E}}^{\downarrow}$

- Measure the resulting upgoing wave $\boldsymbol{\mathcal{E}}^{\uparrow} = \mathcal{S}\boldsymbol{\mathcal{E}}^{\downarrow}$

- Define "upgoing" and "downgoing" by a horizontal Fourier transform

Upgoing and downgoing waves

Fourier transform into frequency domain:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = \mathbf{0}$$

then Fourier transform in horizontal variables:

$$\frac{\partial^2}{\partial z^2}\widetilde{\mathbf{E}} + k^2(1 - \mathbf{e}_{xy})\widetilde{\mathbf{E}} = \mathbf{0}$$

$\Rightarrow$

$$\widetilde{\mathbf{E}}(\mathbf{e}_{xy}, z, k) = \widetilde{\mathbf{E}}^{\uparrow}(\mathbf{e}_{xy}, k)e^{ike_z z} + \widetilde{\mathbf{E}}^{\downarrow}(\mathbf{e}_{xy}, k)e^{-ike_z z}$$

$$e_z(k) = \begin{cases} \sqrt{1 - e_{xy}^2} & \text{for } e_{xy} < 1 \text{ (propagating)} \\ i\text{sgn}(k)\sqrt{e_{xy}^2 - 1} & \text{for } e_{xy} > 1 \text{ (evanescent).} \end{cases}$$

$\Rightarrow$ "plane wave" decomposition is:

$$\boldsymbol{\mathcal{E}}^{\downarrow}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \widetilde{\mathbf{E}}^{\downarrow}(\mathbf{e}_{xy}, k)e^{ik\mathbf{e}^{-}\cdot\mathbf{r}}e^{-ic_0 kt}\, k^2 d\mathbf{e}_{xy} c_0 dk$$

$$\boldsymbol{\mathcal{E}}^{\uparrow}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \iint_{\mathbb{R}^2} \widetilde{\mathbf{E}}^{\uparrow}(\mathbf{e}_{xy}, k)e^{ik\mathbf{e}^{+}\cdot\mathbf{r}}e^{-ic_0 kt}\, k^2 d\mathbf{e}_{xy} c_0 dk$$

where $\mathbf{e}^{\pm} = \mathbf{e}_{xy} \pm \widehat{\mathbf{z}} e_z$,

Criteria

- To maximize signal-to-noise:

$$\max_{\mathcal{E}^{\downarrow}} \frac{\|\mathcal{S}\mathcal{E}^{\downarrow}\|}{\|\mathcal{E}^{\downarrow}\|}$$

- To maximize the difference with respect to a predicted measurement:

$$\max_{\mathcal{E}^{\downarrow}} \frac{\|\mathcal{S}\mathcal{E}^{\downarrow} - \mathcal{S}_0\mathcal{E}^{\downarrow}\|}{\|\mathcal{E}^{\downarrow}\|}$$

Size, or what is $\|\cdot\|$?

$$\|\boldsymbol{\mathcal{E}}\| = \int_{-\infty}^{\infty} \iint_{z=0} -\hat{\mathbf{z}} \cdot (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \; dx dy \, dt$$

(**weighted** L^2 norm)

Parseval's identity $\Rightarrow$

$$\|\boldsymbol{\mathcal{E}}\| = \|\mathcal{F}\boldsymbol{\mathcal{E}}\| = \|\mathbf{E}\|$$

Assume separation between sources and scatterers

$\Rightarrow$

upgoing and downgoing evanescent waves are not both present at the plane $z = 0$

$\Rightarrow$

$$\|\mathbf{E}^{\downarrow} + \mathbf{E}^{\uparrow}\| = \|\mathbf{E}^{\downarrow}\| + \|\mathbf{E}^{\uparrow}\|$$

Algorithm

1. Begin with any downgoing wave $\boldsymbol{\mathcal{E}}_0^{\downarrow}$; let $j = 0$.

2. Send $\boldsymbol{\mathcal{E}}_j^{\downarrow}$ into the lower half-space; measure the resulting upgoing field $\boldsymbol{\mathcal{E}}_j^{\uparrow}(t, \mathbf{r}) = \mathcal{S}\boldsymbol{\mathcal{E}}_j^{\downarrow}(t, \mathbf{r})$.

3. Calculate the corresponding scattering from the reference configuration $\mathcal{S}_0\boldsymbol{\mathcal{E}}_j^{\downarrow}(t, \mathbf{r})$. Calculate the difference field $\boldsymbol{\mathcal{E}}_j^{\uparrow}(t, \mathbf{r}) = \boldsymbol{\mathcal{E}}_j^{\uparrow}(t, \mathbf{r}) - \mathcal{S}_0\boldsymbol{\mathcal{E}}_j^{\downarrow}(t, \mathbf{r})$.

4. The next downgoing wave is the (normalized) time-reversed difference $\boldsymbol{\mathcal{E}}_{j+1}^{\downarrow}(t, \mathbf{r}) = \left[\boldsymbol{\mathcal{E}}_j^{\uparrow}(-t, \mathbf{r}) - (\mathcal{S}_0\boldsymbol{\mathcal{E}}_j^{\downarrow})(-t, \mathbf{r})\right]/c_j$. Add one to j.

5. Go to 2.

Why does this work?

$$\max_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{\|\mathcal{S}\boldsymbol{\mathcal{E}}^{\downarrow}\|}{\|\boldsymbol{\mathcal{E}}^{\downarrow}\|} = \max_{\boldsymbol{\mathcal{E}}^{\downarrow}} \frac{\langle \boldsymbol{\mathcal{E}}^{\downarrow}, \mathcal{S}^{\dagger}\mathcal{S}\boldsymbol{\mathcal{E}}^{\downarrow}\rangle}{\langle \boldsymbol{\mathcal{E}}^{\downarrow}, \boldsymbol{\mathcal{E}}^{\downarrow}\rangle}$$

- $\mathcal{S}^{\dagger} = \mathcal{T}\mathcal{S}\mathcal{T}$
  $\mathcal{T}$ is the operator of time reversal

- This only works for the inner product corresponding to the energy norm (not for the usual L^2 norm).

- Maximize quotient by power method
  (Apply $(\mathcal{S}^{\dagger}\mathcal{S})^n = (\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S})^n$ to anything)

Structure of the scattering operator $\mathcal{S}$

- Problem: $\mathcal{S}$ has continuous spectrum.
- But: Fourier transform into frequency domain
- At each frequency, S is compact, so it has discrete eigenvalues and corresponding eigenfunctions
- If S were a matrix,

$$(S^\dagger S)^n = P^{-1} \begin{pmatrix} \lambda_1^n & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^n & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^n & \cdots & 0 \\ \vdots & \vdots & & \ddots & \lambda_n^n \end{pmatrix} P$$

- Everything depends on frequency ω.

To what does the algorithm converge?

- Normalize $\Rightarrow \begin{array}{l} \lambda_1^n \to \delta(\omega - \omega_*) \\ \lambda_2^n \to 0 \\ \lambda_3^n \to 0 \\ \vdots \end{array}$

- Find the frequency at which the largest eigenvalue is biggest.

- The iterative algorithm converges to a **fixed-frequency** wave with that frequency.

- The iterative algorithm converges to a fixed-frequency wave with the **spatial shape** given by the corresponding eigenfunction.

Scope

Best field and best frequency can be found **without knowing what the scatterer is**.

Restrictions:

- sources and scatterers must be separated
- medium must be linear and reciprocal
- dissipation and dispersion OK
- applies to band-limited data

Discussion

- The best field is not a finite-energy field (as expected).

- This approach decouples the signal-to-noise issue from the resolution issue. (Compare with chirps and pulses.)

- Fink's iterative time-reversal experiments are the answer to the question "What incident wave results in the biggest received scattered wave?".

- This theory explains the pulse-broadening and frequency-shifting seen in the work of Fink and others.

Open questions

- OK for limited aperture?
- How can we trade off signal-to-noise and resolution?
- What is a full set of optimal measurements for forming an image?
- What does knowledge of the best frequency tell us about the scatterers? (resonances?)
- How is information about the scatterers encoded in the eigenfunctions?