A Deterministic-Control-Based Approach to Motion by Curvature

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Joint work with Sylvia Serfaty

Preprint: on my web page

Crucial ingredient: G. Barles + F. Da Lio

Prior, related work: Catté, Dibos, Koepfler
Guichard
"Morphological evolutions"

INTERFACE MOTION

- MATERIALS SCIENCE: PHASE BOUNDARIES, GRAIN BOUNDARIES, ETC
- IMAGE PROCESSING: EDGES; DENOISING VIA NONLINEAR DIFFUSION
- HIGHER CODIMENSION ALSO INTERESTING: E.G. CURVES IN R^3

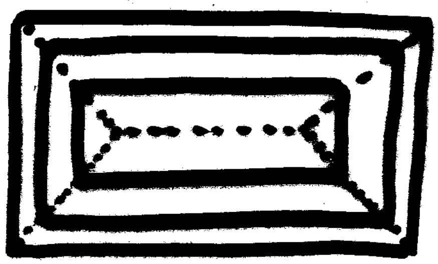
LEVEL SET METHOD

- FLEXIBLE NUMERICAL TOOL
- POWERFUL FOR ANALYSIS TOO

THIS TALK

- FRESH UNDERSTANDING VIA OPT'L CONTROL

REVIEW FAMILIAR FACTS ABOUT $V = 1$



Ω = RECTANGLE

ARRIVAL TIME — $u(x)$ = TIME OF ARRIVAL AT x
SOLVES $|\nabla u| = 1$, $u = 0$ AT $\partial\Omega$

LEVEL SET METHOD — IF $V_t / |\nabla V| = -1$ THEN EACH LEVEL SET OF V HAS VELOCITY 1

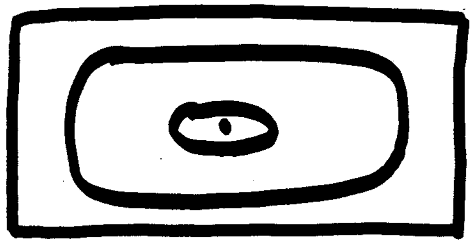
EQUIVALENCE — $V(x,t) = u(x) - t$

1st ORDER HJE — METHOD OF CHARACTERISTICS; NOTE SHOCKS

KINETIC — MODELS DECAY (OR GROWTH)

OPTIMAL CONTROL — $u(x) = \min\limits_{\substack{\text{VELOCITY} \le 1 \\ \text{STARTING AT } x}} \{\text{ARRIVAL TIME TO } \partial\Omega\}$

REVIEW FAMILIAR FACTS ABOUT $V = -K$



CONVEX Ω

ARRIVAL TIME $u(x)$ = TIME OF ARRIVAL AT x
SOLVES $\operatorname{div}(\nabla u/|\nabla u|) = -1/|\nabla u|$

LEVEL SET METHOD IF $V_t/|\nabla V| = \operatorname{div}(\nabla V/|\nabla V|)$ THEN
EACH LEVEL SET OF V HAS VELOCITY $= -K$

EQUIVALENCE $V(x,t) = u(x) - t$

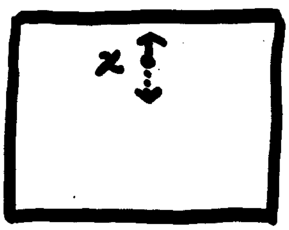
2^{nd} ORDER NO CHARACTERISTICS

THERMODYNAMIC STEEPEST DESCENT
FOR PERIMETER

OPTIMAL CONTROL 2^{nd} ORDER $\Longleftrightarrow$ STOCHASTIC CONTROL

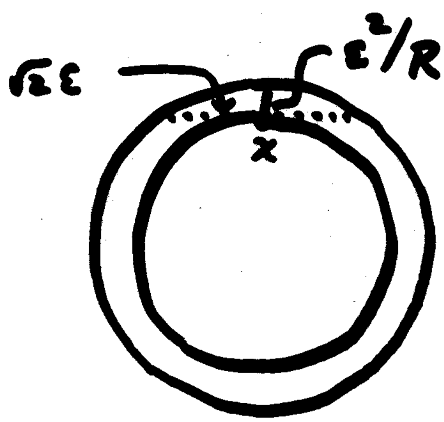
THE GAME

PAUL WANTS TO EXIT; **CAROL** WANTS TO STOP HIM; **STEP SIZE** IS $\varepsilon > 0$.

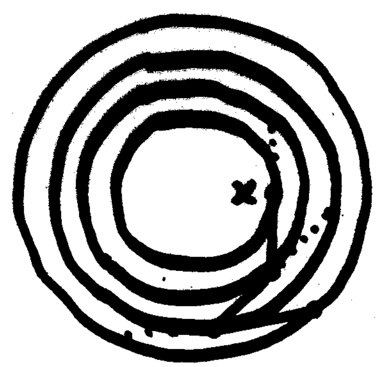


- PAUL CHOOSES DIR'N $|v| = 1$
- CAROL MAY REVERSE IT $b = \pm 1$
- PAUL GOES $x \to x + \sqrt{2}\,\varepsilon\, b v$

CAN PAUL EXIT? YES!



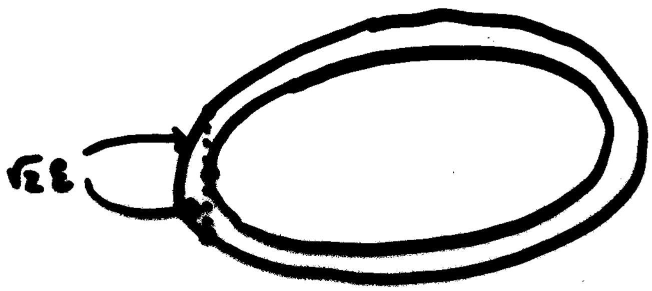
RADIUS R



- SET FROM WHICH PAUL CAN EXIT IN J STEPS HAS NORMAL VELOCITY $1/R_J$ (AFTER NORMALIZATION)

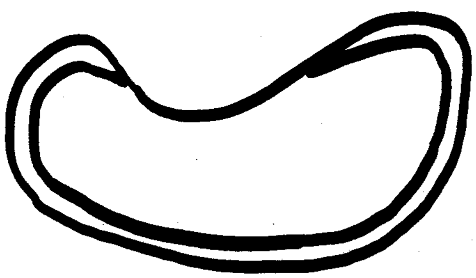
- CAROL CAN'T STOP PAUL, BUT SHE DOES SLOW HIM DOWN

GEOMETRIC INTERPRETATION



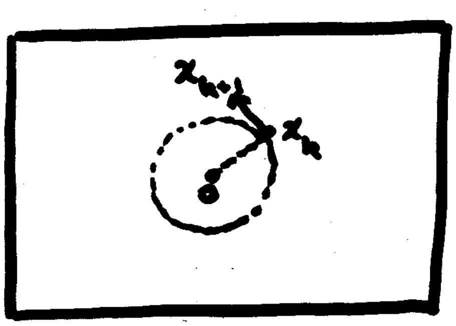
SET FROM WHICH PAUL NEEDS TWO STEPS TO EXIT IS LOCUS OF MIDPOINTS OF SECANTS

NONCONVEX CASE



SAME AS ABOVE, BUT HE CANNOT EXIT AT CONCAVE PART OF $\partial\Omega$

HE CAN EXIT IN $O(\varepsilon^{-2})$ STEPS



TRIAL STRATEGY: USE $v \perp x$. THEN

$$|x_{k+1}|^2 = x_k^2 + 2\varepsilon^2$$

GOING BEYOND PICTURES

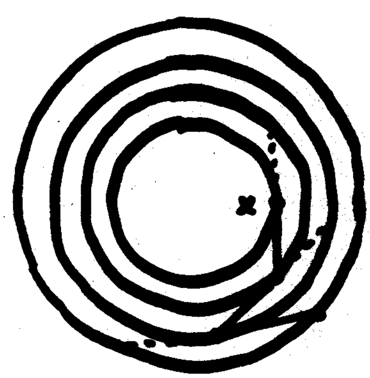
$$u_\varepsilon(x) = \varepsilon^2 \cdot \left[\begin{array}{l}\text{\# STEPS PAUL NEEDS TO} \\ \text{EXIT, STARTING AT } x\end{array}\right]$$

SIMPLEST THEOREM: FOR CONVEX PLANE DOMAINS

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \begin{array}{l}\text{ARRIVAL TIME OF } \partial\Omega \\ \text{UNDER MOTION BY CURVATURE}\end{array}$$

MAIN TOOL: DYNAMIC PROGRAMMING PRINCIPLE

$$u_\varepsilon(x) = \min_{|v|=1} \max_{b=\pm 1} \left\{ u_\varepsilon(x + \sqrt{2}\,\varepsilon b v) + \varepsilon^2 \right\}$$



"LEVEL-SET PDE" IS THE HJB EQN [redacted] TO OPT'L CONTROL PROBLEM. FORMALLY:

$$u_\varepsilon(x) = \min_{|v|=1} \max_{b=\pm 1} u_\varepsilon(x + \sqrt{2}\,\varepsilon b v) + \varepsilon^2$$

$$\Longrightarrow \quad u(x) \approx \min_{|v|=1} \max_{b=\pm 1} u(x) + \sqrt{2}\,\varepsilon\, b v \cdot \nabla u + \varepsilon^2 \langle D^2 u \cdot v, v \rangle + \varepsilon^2$$

$$\boxed{\max_{b=\pm 1} \varepsilon b v \cdot \nabla u = \varepsilon |v \cdot \nabla u| \quad \text{DOMINATES, UNLESS} = 0}$$

SO $$\cancel{u(x)} \approx \min_{v \perp \nabla u} \cancel{u(x)} + \varepsilon^2 \langle D^2 u \cdot v, v \rangle + \varepsilon^2$$

$$\left\langle D^2 u \cdot \frac{\nabla u^\perp}{|\nabla u|}, \frac{\nabla u^\perp}{|\nabla u|} \right\rangle + 1 = 0$$

$$\Longleftrightarrow \quad \Delta u - \left\langle D^2 u \cdot \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle + 1 = 0 \quad \text{IN } \mathbb{R}^2$$

$$\Longleftrightarrow \quad |\nabla u| \operatorname{div}\left(\nabla u / |\nabla u| \right) + 1 = 0$$

$$\boxed{\text{PDE IS } 2^{\text{nd}} \text{ ORDER BECAUSE } 1^{\text{st}} \text{ ORDER TAYLOR EXPANSION WAS NOT SUFFICIENT}}$$

EXTENSIONS

1) SAME GAME IN $\mathbb{R}^3$?

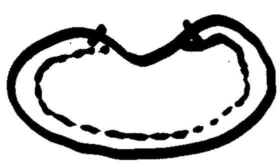
BOUNDARY MOVES WITH NORMAL VELOCITY
= LARGEST PRINCIPAL CURVATURE

2) CAN WE GET V = MEAN CURVATURE IN $\mathbb{R}^3$?

YES, WITH MODIFIED GAME

- PAUL CHOOSES TWO ORTHOG. DIRNS
  $|v| = |w| = 1$, $v \perp w$
- CAROL MAY REVERSE EITHER (OR BOTH)
  $b = \pm 1$, $\beta = \pm 1$
- PAUL GOES $x \to x + \sqrt{2}\,\varepsilon b v + \sqrt{2}\,\varepsilon \beta w$

3) WHAT IF Ω IS NOT CONVEX?

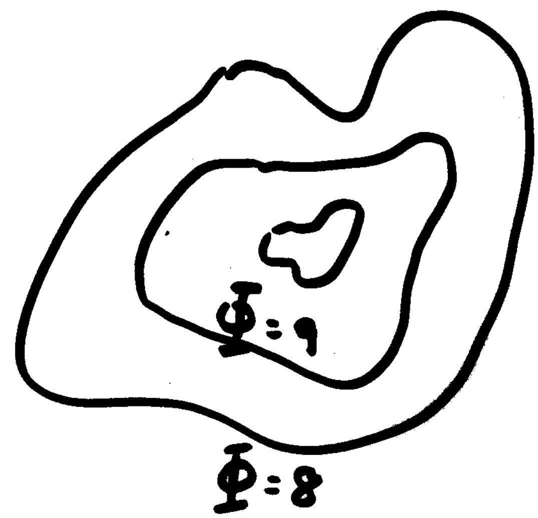


$\lim_{\varepsilon \to 0} u_\varepsilon(x)$ = ARRIVAL TIME OF
$\partial\Omega$ UNDER FLOW
BY VELOCITY = κ_+

4) THE ANALOGOUS FINAL-TIME GAME?

SAME RULE BUT PAUL HAS DIFFERENT GOAL:

$$V_\varepsilon(x,t) = \min_{\substack{\text{STARTING}\\ \text{AT } x \text{ AT}\\ \text{TIME } t}} \Phi(y_\varepsilon(T))$$



WHERE $y_\varepsilon(t)$ = PAUL'S PATH

Φ = FIXED OBJECTIVE

T = "MATURITY TIME"

THEN

$$V_t + \langle D^2 V \cdot \frac{\nabla V^\perp}{|\nabla V|}, \frac{\nabla V^\perp}{|\nabla V|} \rangle = 0 \qquad t < T$$

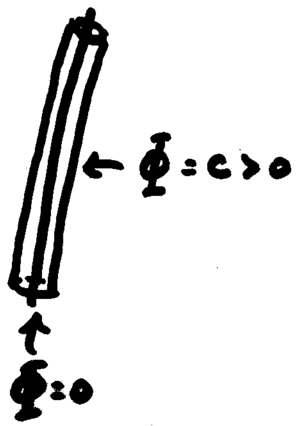
$$V = \Phi \qquad t = T$$

$$\boxed{\text{EACH LEVEL SET OF } V \text{ MOVES BY CURVATURE (BACKWARD IN TIME)}}$$

5) MOTION BY CURVATURE OF A CURVE IN $\mathbb{R}^3$?

USE TIME-DEPENDENT VERSION, WITH OBJECTIVE

$$\Phi(y) = \text{dist}(y, \text{CURVE})$$



$$v_t + \min_{\substack{\xi \perp \nabla v \\ |\xi| = 1}} \langle D^2 v \cdot \xi, \xi \rangle = 0 \qquad t < T$$

$$v = \Phi \qquad t = T$$

$$\boxed{\text{EACH LEVEL SET OF } v \text{ HAS VELOCITY = SMALLER PRIN. CURVATURE}}$$

AMBROSIO-SONER : ZERO-LEVEL SET "MOVES WITH VELOCITY = CURVATURE"

RIGOROUS ANALYSIS: TWO ALTERNATIVES

VISCOSITY SOLUTIONS OR **VERIFICATION ARG'T**

VISCOSITY SOLUTIONS	VERIFICATION ARG'T
• MORE GENERAL	• CLASSICAL + ELEMENTARY
• CONVERGENCE w/o RATE	• GETS CONV. RATE
• GUARANTEED TO SUCCEED	• NEEDS SMOOTHNESS

FOCUS HERE ON ONE RESULT, PROVED VIA VERIFICATION ARGUMENT

THEOREM: FOR SMOOTHLY BOUNDED CONVEX PLANE DOMAINS,

$$u(x) - C\varepsilon \le u_\varepsilon(x) \le u(x) + C\varepsilon$$

WHERE

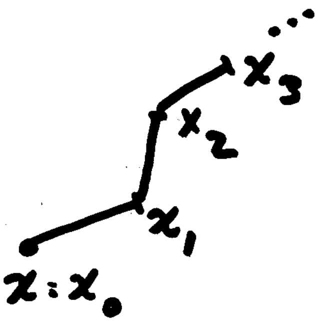
u_ε = PAUL'S SCALED ARRIVAL TIME

u : CURVATURE FLOW ARRIVAL TIME

NOTE:
- UPPER BOUND REQUIRES ONE GOOD STRATEGY
- LOWER BOUND MUST CONSIDER OPT'L STRATEGY
- PROOF RESEMBLES 1st ORDER THEORY

SKETCH OF UPPER BOUND $u_\varepsilon(x) \le u(x) + C\varepsilon$

TRIAL STRATEGY: ALWAYS CHOOSE $v = \pm \frac{\nabla u}{|\nabla u|}^{\perp}$



$$x_{k+1} = x_k + \sqrt{2}\,\varepsilon\, b_k v_k$$

UNTIL EXIT AT STEP K.

$$u_\varepsilon(x_k) = \min_{|v|=1} \max_{b=\pm 1} u_\varepsilon(x_k + \sqrt{2}\,\varepsilon\, b v) + \varepsilon^2$$

$$\le \max_{b=\pm 1} u_\varepsilon(x_k + \sqrt{2}\,\varepsilon\, b v_k) + \varepsilon^2 \qquad v_k = \frac{\nabla u}{|\nabla u|}$$

$$= u_\varepsilon(x_k + \sqrt{2}\,\varepsilon\, b_k v_k) + \varepsilon^2$$

$\Rightarrow$ $\boxed{u_\varepsilon(x_0) \le K\varepsilon^2 \quad \text{WHERE EXIT OCCURS AT STEP } K.}$

NOW CONSIDER $u(x)$: SINCE

$$x_{k+1} = x_k + \sqrt{2}\,\varepsilon\, b_k v_k \qquad v_k = \frac{\nabla u}{|\nabla u|}^{\perp}(x_k)$$

WE HAVE

$$u(x_{k+1}) = u(x_k) + \sqrt{2}\,\varepsilon\, \cancel{b_k v_k \cdot \nabla u}^{\,0}$$
$$+ \varepsilon^2 \underbrace{\langle D^2 u \cdot v_k, v_k \rangle}_{-1} + O(\varepsilon^3)$$

$$u(x_{k+1}) = u(x_k) - \varepsilon^2 + O(\varepsilon^3)$$

$$\Longrightarrow \quad \boxed{u(x_0) = K\varepsilon^2 + O(\varepsilon) \quad \text{WHERE EXIT IS AT STEP } K}$$

THUS $\qquad u_\varepsilon(x_0) \le K\varepsilon^2 \le u(x_0) + C\varepsilon \qquad$ QED

<u>NOTE</u>: ARGUMENT NEEDS THAT

$u(x) =$ ARRIVAL TIME OF $\partial\Omega$
UNDER CURVATURE FLOW

IS C^3. THAT'S TRUE BUT NONTRIVIAL.

SKETCH OF LOWER BOUND $u_\varepsilon(x) \geq u(x) - C\varepsilon$

THIS TIME CONSIDER PAUL'S OPTIMAL v_k,
AND A "TRIAL STRATEGY" FOR CAROL

$$x_{k+1} = x_k + \sqrt{2}\,\varepsilon\, b_k v_k$$

UNTIL EXIT AT STEP K

$$u_\varepsilon(x_k) = \max_{b=\pm 1} u_\varepsilon(x_k + \sqrt{2}\,\varepsilon\, b_k v_k) + \varepsilon^2$$

$$\geq u_\varepsilon(x_k + \sqrt{2}\,\varepsilon\, b_k v_k) + \varepsilon^2$$

FOR ANY STRATEGY USED BY CAROL

$\Rightarrow$ $\boxed{u_\varepsilon(x_0) \geq K\varepsilon^2 \quad \text{WHERE EXIT OCCURS AT STEP } K}$

CLAIM: FOR GOOD CHOICE OF CAROL'S STRATEGY

$$u(x_{k+1}) \geq u(x_k) - \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

IN FACT

$$\max_{b=\pm 1} u(x_k + \sqrt{2}\,\varepsilon b v) = u(x_k) + \sqrt{2}\,\varepsilon |v \cdot \nabla u| + \varepsilon^2 \langle D^2 u \cdot v, v\rangle + \mathcal{O}(\varepsilon^3)$$

- IF $v \cdot \nabla u \neq 0$, THAT TERM DOMINATES
- IF $v \cdot \nabla u = 0$, THEN $v = \pm \nabla u^{\perp}/|\nabla u|$
- BOTH CASES GIVE DESIRED INEQ

SO $\boxed{u(x_0) \leq K\varepsilon^2 + \mathcal{O}(\varepsilon) \quad \text{WHERE EXIT OCCURS AT } K}$

THUS $\quad u_\varepsilon(x_0) \geq K\varepsilon^2 \geq u(x_0) - C\varepsilon \qquad$ QED

CONCLUSIONS

- NEW UNDERSTANDING (BUT NOT PHYSICAL ?)
- SEMIDISCRETE NUMERICAL SCHEME (IS IT OF ANY PRACTICAL VALUE?)
- MANY OPEN QUESTIONS (WHICH 2^{nd} ORDER EQUATIONS HAVE DETERMINISTIC CONTROL INTERPRETATIONS?)