Tamsamani categories, Segal categories, and applications

Two talks by Bertrand Toën at the IMA 2004 Summer programme on *n-Categories: foundations and applications*. (Notes taken by Joachim Kock, revised by B.T.)

This version 2 (June 2005) is identical to the original except that a correction to 1.3 is given in a footnote.

1 'Segalic' definitions of higher categories

The 'Segalic' definitions of higher categories comprise the one of Tamsamani and the notion of Segal categories. The basic results have been worked out by Tamsamani, Simpson, Hirschowitz, and Pellissier.

1.1 Nerves. Let C be a small category, then we can construct its *nerve* NC: it is the simplicial set

$$\begin{array}{ccc} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{Set} \\ [n] & \longmapsto & \mathrm{Hom}([n], C) \end{array}$$

This defines a functor $N : \boldsymbol{Cat} \to \boldsymbol{sSet}$ which turns out to be fully faithful. The image of the nerve functor are the simplicial sets X for which the natural maps, called the *Segal maps*

$$X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are isomorphisms. Hence it makes sense to say that a category is a simplicial set such that the Segal maps are isomorphisms.

A *strict n-category* can be described inductively as a functor $X : \Delta^{\mathrm{op}} \to$ ***(n-1)Cat*** such that the Segal maps are isomorphisms, and such that X_0 is a discrete $(n-1)$-category, i.e. a set.

Tamsamani's definition of weak higher categories is just like that, except that the Segal maps are only required to be equivalences.

Tamsamani n-categories

1.2 Base of the induction. By definition, the category of 0-categories is

$$\boldsymbol{0Cat} := \boldsymbol{Set}.$$

A 0-equivalence between 0-categories is just a bijection of sets. There is a functor $\tau_0 : \boldsymbol{0Cat} \to \boldsymbol{0Cat}$ which is just the identity functor.

1.3 Induction hypothesis.[1] Suppose we have already defined ***(n-1)Cat*** as a category, and that we have defined the notion of $(n-1)$-equivalences between $(n-1)$-categories. Finally we suppose we have a functor τ_0 : ***(n-1)Cat*** $\rightarrow$ ***Set*** which sends $(n-1)$-equivalences to bijections and which has a fully faithful right adjoint ***Set*** $\hookrightarrow$ ***(n-1)Cat*** — the $(n-1)$-categories in the image of this right adjoint are called *discrete* — with the following two properties. First, it preserves products. Second, if $X \rightarrow A$ is a morphism of $(n-1)$-categories with A discrete, then there is an isomorphism $A \simeq \coprod_{x \in X} A_x$.

1.4 The induction step of the definition. (a) The category ***nCat*** is defined to be the category of functors

$$A : \Delta^{\text{op}} \rightarrow \textbf{\textit{(n-1)Cat}}$$

such that A_0 is discrete and such that for all $k \geq 1$ the Segal map

$$A_k \rightarrow A_1 \times_{A_0} \cdots \times_{A_0} A_1$$

is an $(n-1)$-equivalence. (Note that this fibre product exists because of the properties of τ_0.)

(b) Given an n-category A, let $\tau_1(A)$ denote the category whose object set is A_0, and whose arrows are $\tau_0(A)$. In other words, it is the category given by

$$\Delta^{\text{op}} \xrightarrow{A} \textbf{\textit{(n-1)Cat}} \xrightarrow{\tau_0} \textbf{\textit{Set}}.$$

This construction defines a functor τ_1 : ***nCat*** $\rightarrow$ ***Cat***.

(c) (Notation.) For each m there is a morphism $A_m \rightarrow A_0^{m+1}$ induced by the $m+1$ maps $[0] \rightarrow [m]$ in Δ. Since A_0^{m+1} is discrete we get a coproduct decomposition $A_m \simeq \coprod_{(a_0,\ldots,a_m)} A_{(a_0,\ldots,a_m)}$. The $(n-1)$-categories $A_{(a_0,\ldots,a_m)}$ should be thought of as generalised hom sets (more precisely, hom $(n-1)$-categories): it is equivalent to the $(n-1)$-category of strings of 1-arrows $a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_m$.

(d) A morphism $f : A \rightarrow B$ in ***nCat*** is called an *n-equivalence* if for all x, y in A_0 the induced map $A_{(x,y)} \rightarrow B_{(fx,fy)}$ is an $(n-1)$-equivalence and if $\tau_1 A \rightarrow \tau_1 B$ is an equivalence of categories. The first condition is that of being *fully faithful*. (Note that there are no separate notions of 'faithful' and 'full'.) The second condition is to be thought of as *essentially surjective*. However this is only the correct interpretation when combined with fully faithful.

(e) Put $\tau_0(A) := \tau_0\tau_1(A)$.

This completes the inductive definition of the category ***nCat***. Note that the morphisms of n-categories are just simplicial maps, i.e. natural transformations between the functors $\Delta^{\text{op}} \rightarrow$ ***(n-1)Cat***. These are sort of strict n-functors.

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[1] *JK (June 2005): The induction hypothesis is wrong as stated: we can not assume that τ_0 is left adjoint to the discrete objects functor. Already for* ***Cat*** *this would be wrong because the left adjoint is 'connected components' while τ_0 should be 'isomorphism classes'. The induction hypothesis should rather read as follows:*

Suppose we have already defined ***(n-1)Cat*** as a category, and that we have defined the notion of $(n-1)$-equivalences between $(n-1)$-categories. Next we suppose there is a fully faithful functor δ : ***Set*** $\hookrightarrow$ ***(n-1)Cat***; the $(n-1)$-categories in the image of δ are called *discrete* $(n-1)$-categories. Finally we assume there is a functor τ_0 : ***(n-1)Cat*** $\rightarrow$ ***Set*** with the following four properties: (i) the composite $\tau_0 \circ \delta$ is isomorphic to the identity functor on ***Set***; (ii) τ_0 sends $(n-1)$-equivalences to bijections; (iii) τ_0 preserves finite products; (iv) if $A \rightarrow O$ is a morphism of $(n-1)$-categories with O discrete, then there is an isomorphism $A \simeq \coprod_{x \in O} A_x$.

1.5 n-groupoids. The definition of groupoid is also by induction. By definition, all 0-categories are groupoids. Suppose the notion of $(n-1)$-groupoid has been defined. An n-category X is called an *n-groupoid* if for all $x, y \in A_0$, the 'hom $(n-1)$-cat' $A_{(x,y)}$ is an $(n-1)$-groupoid, and $\tau_1(A)$ is a groupoid (in the usual sense of categories). This defines ***nGrpd*** as a full subcategory of ***nCat***.

By definition the homotopy category Ho(***nGrpd***) is the localisation of ***nGrpd*** along the n-equivalences.

Let ***Top***$_n$ denote the category of truncated homotopy n-types.

1.6 Theorem. *(Tamsamani [10].) There is an equivalence of categories*

$$B : \mathrm{Ho}(\boldsymbol{nGrpd}) \leftrightarrow \mathrm{Ho}(\boldsymbol{Top}_n) : \Pi_n .$$

It is given by a pair of functors

$$B : \boldsymbol{nGrpd} \leftrightarrow \boldsymbol{Top}_n : \Pi_n$$

where B is geometric realisation, and Π_n is fundamental n-groupoid. These functors are not adjoint on this level, but there are n-equivalences

$$\begin{aligned} B \circ \Pi_n &\overset{\sim}{\to} \mathrm{id} \\ \Pi_n \circ B &\overset{\sim}{\to} \mathrm{id}\,. \end{aligned}$$

This is an important test for the 'correctness' of the notion of weak n-category. All theories of weak n-categories should feature such a result...

The main problem with Tamsamani's definition is that it does not give a notion of ∞-category. Most (if not all) higher categories appearing in algebraic geometry are actually ∞-categories. However, almost always the i-morphisms are invertible for sufficiently big i. The notion of n-Segal categories is a variation of Tamsamani categories which has invertible i-morphisms for $i > n$.

Segal n-categories

This notion is due to Hirschowitz and Simpson [5] and it was introduced to deal with moduli of complexes of sheaves.

Suppose A is an ∞-category whose i-morphisms are invertible for all $i > n$. This means that the ∞-categories of n-morphisms are ∞-groupoids. Now ∞-groupoids are essentially the same thing as simplicial sets. Since we already have a very good theory for simplicial sets, we don't want to create a separate theory for ∞-groupoids — it is more efficient just to use the notion of simplicial set, so the definition of Segal categories is a mixture of Tamsamani categories and simplicial sets. The only difference is that we start the induction from simplicial sets instead of starting it from sets.

So by definition, the category of 0-*Segal categories* is ***0SeCat*** := ***sSet***, the 0-*equivalences* are defined to be the weak homotopy equivalences, and τ_0 : ***0SeCat*** $\to$ ***Set*** is defined to be π_0 : ***sSet*** $\to$ ***Set***. The discrete objects are ***Set*** $\subset$ ***sSet***.

Inductively now, the category of *n-Segal categories* **nSeCat** is defined to be the category of functors $A : \Delta^{\mathrm{op}} \to$ ***(n-1)SeCat*** such that A_0 is discrete, and such that the Segal maps $A_k \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ are $(n-1)$-equivalences.

1.7 Remark. The category ***nCat*** is a full subcategory of ***nSeCat***. Also Ho(***nCat***) is a full subcategory of Ho(***nSeCat***).

1.8 1**-Segal categories.** The case of 1-Segal categories is particularly important. These are functors $A : \Delta^{\mathrm{op}} \to$ ***sSet*** such that A_0 is discrete (i.e., a constant simplicial set), and such that $A_k \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ are weak homotopy equivalences. The image of Ho(***1Cat***) $\to$ Ho(***1SeCat***) consists of those 1-Segal categories for which $\pi_i(A) = 0$ for all $i > 0$.

1.9 Remark. The image of the full embedding ***sCat*** $\to$ ***1SeCat*** consists of those 1-Segal categories for which Segal maps are isomorphism, not just weak equivalences.

1.10 Theorem. *(Dwyer-Kan-Smith [4].) The inclusion* Ho(***sCat***) $\hookrightarrow$ Ho(***1SeCat***) *is an equivalence.*

In conclusion, the homotopy theory of 1-Segal categories is essentially equivalent to the one of simplicial categories. However, we will see that the homotopy theory of 1-Segal category is better behaved than the one of simplicial categories.

2 Properties of Segal categories

The most important features is the model structure. We want to understand the homotopy theory of ***nSeCat***, and the idea is to put a model structure on it. Now this category cannot be a model category itself — it is too small, and in particular it does not have all limits and colimits.

Definition. The category ***nSePreCat*** of *n-Segal pre-categories* is defined inductively as the category of functors $A : \Delta^{\mathrm{op}} \to$ ***(n-1)SePreCat*** such that A_0 is discrete. (So compared to the definition of Segal category there is no condition on the Segal maps.)

The following theorem is due to Simpson and Pellissier. Pellissier found some gaps in the original proof of Simpson [9] and filled these gaps in his thesis [7].

2.1 Theorem. *There is a model structure on* **nSePreCat** *such that*

(i) The fibrant objects are n-Segal categories.

(ii) The cofibrations are the monomorphisms.

(iii) The weak equivalences between n-Segal categories are precisely the n-equivalences.

(iv) (**nSePreCat**$, \times, 1$) *is a monoidal model category in the sense of Hovey [6]. In particular, the functor* **nSePreCat** $\to$ **nSePreCat** *given by tensoring with an object X preserves weak equivalences.*

2.2 Remarks. Re (i): Not all n-Segal categories are fibrant. The fibrant objects seem impossible to describe explicitly, not unlike injective objects in an abelian category...

Re (iii): The weak equivalences for general objects are hard to describe.

2.3 Remark. Since ***nSeCat*** contains the category of fibrant objects, it has the same homotopy category as ***nSePreCat*** in the sense that the functor

$$\mathrm{Ho}(\boldsymbol{nSeCat}) \to \mathrm{Ho}(\boldsymbol{nSePreCat})$$

is an equivalence. This is used implicitly in the following corollaries.

2.4 Corollary. *All homotopy colimits exists in **nSeCat**.*

That is, for a small category I, consider the constant diagram functor

$$\mathrm{Ho}(\boldsymbol{nSeCat}^I) \leftarrow \mathrm{Ho}(\boldsymbol{nSeCat}).$$

Existence of homotopy colimits means that this functor has a left adjoint, and existence of homotopy limits means it has a right adjoint.

As an example, the category $I = \{\cdot \to \cdot \leftarrow \cdot\}$ gives rise to homotopy fibre products, while the category $I = \{\cdot \leftarrow \cdot \to \cdot\}$ gives homotopy pushouts.

2.5 Corollary. *Derived and internal homs exist in **nSeCat**.*

This means, given $A, B \in \mathrm{Ho}(\boldsymbol{nSeCat})$, there is an object $\mathbb{R}\mathrm{Hom}(A, B)$ in $\mathrm{Ho}(\boldsymbol{nSeCat})$ such that

$$\mathbb{R}\mathrm{Hom}(A \times C, B) \simeq \mathbb{R}\mathrm{Hom}(A, \mathbb{R}\mathrm{Hom}(C, B))$$

is an isomorphism in $\mathrm{Ho}(\boldsymbol{nSeCat})$. This is to say that $\mathrm{Ho}(\boldsymbol{nSeCat})$ is cartesian closed.

$\mathbb{R}\mathrm{Hom}(A, B)$ is the n-Segal category of 'weak' functors $A \to B$, in the sense that if R is a fibrant replacement functor, then

$$\mathbb{R}\mathrm{Hom}(A, B) \simeq \underline{\mathrm{Hom}}(A, RB)$$

where $\underline{\mathrm{Hom}}$ is the usual internal hom. One can think of RB as an injective resolution of B...

The fibrant n-Segal categories with their internal hom form an $(n+1)$-Segal category. In particular, there is a 2-Segal category of all 1-Segal categories.

Note that there is no internal model structure on simplicial categories with the correct equivalences. This is the main reason why 1-Segal categories are better behaved.

2.6 Application: Segal localisation. Let C be a category and let $W \subset C$ be a subcategory. We want to 'invert' the arrows of W in C in such a way as to produce a Segal category $L(C, W)$. This Segal category can be constructed using the homotopy pushout. Let $\Delta[1]$ denote the category $\{0 \to 1\}$, and let $\overline{\Delta[1]}$ denote its group completion $\{0 \leftrightarrow 1\}$. Now $L(C, W)$ is defined by this homotopy pushout square in $\mathrm{Ho}(\boldsymbol{1SeCat})$:

$$\begin{tikzcd}
C \arrow[r, "e"] & L(C,W) \arrow[dl, phantom, "\llcorner", very near start] \\
\coprod_{f \in W} \Delta[1] \arrow[u] \arrow[r] & \coprod_{f \in W} \overline{\Delta[1]} \arrow[u]
\end{tikzcd}$$

Now for every 1-Segal category A, the induced functor

$$\mathbb{R}\mathrm{Hom}(L(C,W),A) \xrightarrow{e^*} \mathbb{R}\mathrm{Hom}(C,A)$$

is fully faithful. Its image consists of those $f : C \to A$ such that $f(W) \subset \mathrm{Iso}(\tau_1 A) = \mathrm{Eq}(A)$. (Again, what is really meant here, cf. remark 2.3, is that the image are pseudo-functors $C \to RA$ where RA is a fibrant replacement of A.)

$L(C,W)$ is called the *Segal localisation* of C with respect to W. This construction gives lots of examples: take any model category M (with weak equivalences W), then $L(M,W)$ is a 1-Segal category. We will often suppress W from the notation, writing just LM.

For example, for $M = \boldsymbol{sSet}$, we get a 1-Segal category $L(\boldsymbol{sSet})$ which we call ***Top*** or $\boldsymbol{HOT}_0$. This is really the most fundamental example.

Intuitively, the Segal localisation formally adds inverses to all arrows in W, but inverses of the weakest possible kind. In the last example we are just turning weak equivalences into homotopy equivalences.

Another example is when W consists of the whole category C. In this case $L(C,W)$ is just the geometric realisation of C.

The homotopy pushout construction is rather abstract, but when C has a model structure then we know how actually to compute the localisation.

2.7 Proposition. *(Dwyer-Kan [3], [1], [2].) (a) Let M be a model category (for simplicity, a simplicial model category), and let* $\mathrm{Int}(M)$ *denote the simplicial category of fibrant-cofibrant objects in M (this is called the interior of M). Then there is an isomorphism in* $\mathrm{Ho}(\boldsymbol{1SeCat})$

$$LM \simeq \mathrm{Int}(M)$$

(b) For any (C,W), the Segal localisation $L(C,W)$ is equivalent to the hammock localisation (cf. [1]).

As a consequence of $LM \simeq \mathrm{Int}(M)$, we get

$$\boldsymbol{Top} \simeq \boldsymbol{Kan} \simeq \text{CW-complexes}.$$

2.8 Example. Let $M = C(k)$ be the category of unbounded chain complexes over a ring k. Let W be the class of quasi-isomorphisms. There are several model structures one can put on this M, one is the projective one (point-wise), another is the one where the cofibrations are the monomorphisms. Then LM has as objects the complexes over k (really the fibrant-cofibrant complexes). We have

$$\pi_i(LM)_{E,F} \simeq \mathbb{E}\mathrm{xt}^{-i}(E,F)$$

— this is hyper-ext.

2.9 Remark. In a sense the 1-Segal categories arising through Segal localisation are the only ones: every 1-Segal category A has a fully faithful embedding $A \hookrightarrow LM$ for some model category M. So the conclusion is that 1-Segal categories and model categories are 'essentially the same thing'. In practice this statement means that one can work with a 1-Segal category by embedding it into the localisation of a model category and work over in the model category. One main difference, however, is that we know how to define the Segal category $\mathbb{R}\mathrm{Hom}(LM, LN)$ for model categories M and N, but there is no model category of 'functors' $M \to N$. (This is not a precise statement, but it can be made precise.) So this is a situation where higher category theory is needed in order to speak about functors.

2.10 Example. (Cf. [11].) We have $\mathbb{R}\mathrm{Aut}(\boldsymbol{Top}) \simeq *$. This is what Grothendieck calls the 'hypothèse inspiratrice'. This statement can be proved with Segal categories but cannot be done with model categories. (This result is the analogue of the fact that the groupoid of auto-equivalences of ***Set*** is contractible.)

2.11 The strictification theorem. *Let M be a model category which we assume cofibrantly generated. Let I be a small category. Then*

$$L(M^I) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}(I, LM).$$

Here the equivalences in M^I are the level-wise ones. This theorem is due to Simpson and Hirschowitz [5] (and also to Rezk [8] for complete Segal spaces). The importance of this theorem is that it gives a way to 'compute' $\mathbb{R}\mathrm{Hom}$.

All of these properties allow one to develop the theory of 1-Segal categories in a fashion very similar to category theory. In particular, one can define the notion of limits and colimits, exact functors, adjunctions, etc. We will not describe these notions here, but just refer to [11]. However, this sort of constructions will be used implicitely in the next section.

3 Application: Stacks

The following application of Segal categories is an application to topology but it is inspired by algebraic geometry. It is motivated by Grothendieck's theory of Π_1.

3.1 Grothendieck's theory of Π_1. Let X be a CW-complex, and let $\boldsymbol{Sh}_{\mathrm{loc}}(X)$ denote the category of locally constant sheaves on X. A theorem of Grothendieck states that there is an equivalence

$$\Pi_1(X) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{geom}}(\boldsymbol{Set}, \boldsymbol{Sh}_{\mathrm{loc}}(X)).$$

Here $\Pi_1(X)$ denotes the fundamental groupoid of X, and $\mathrm{Hom}^{\mathrm{geom}}$ denotes the geometric morphisms, i.e., functors $\boldsymbol{Set} \to \boldsymbol{Sh}_{\mathrm{loc}}(X)$ that have an *exact* left adjoint. Given a point $x \in \Pi_1(X)$, the left adjoint provided by the theorem is

$$\begin{array}{rcl} \boldsymbol{Sh}_{\mathrm{loc}}(X) & \longrightarrow & \boldsymbol{Set} \\ E & \longmapsto & E_x. \end{array}$$

The importance of this theorem is that it *defines* Π_1 for algebraic varieties, where we haven't got a notion of paths.

The purpose is now to generalise this idea to higher homotopy types, with the motivation of defining homotopy types for more general objects like schemes and stacks. Instead of ***Set*** we are going to use ***Top***.

3.2 Pre-stacks and stacks. Let $\boldsymbol{O}(X)$ denote the category of open sets in X. Instead of considering the category $\{\boldsymbol{O}(X)^{\mathrm{op}} \to \textbf{\textit{Set}}\}$ of presheaves of sets, we will look at $\mathbb{R}\mathrm{Hom}(\boldsymbol{O}(X)^{\mathrm{op}}, \textbf{\textit{Top}})$, the 1-Segal category of *pre-stacks of* ∞*-groupoids*, which we will denote by $\textbf{\textit{PrSt}}(X)$. We are interested in its full subcategory of *stacks*: these are the pre-stacks $F : \boldsymbol{O}^{\mathrm{op}} \to \textbf{\textit{Top}}$ such that for every hypercover $U \to X$ the map $F(X) \to \operatorname{holim}_\Delta F(U_i)$ is a weak equivalence. (Recall that a hypercover is a simplicial object in the category of sheaves that in each degree is a disjoint union of open subsets and such that all stalks are contractible simplicial sets.) (For prestacks with trivial homotopy from a certain stage on, it would be enough to use Cech covers.)

Note that in order to describe $\mathbb{R}\mathrm{Hom}(\boldsymbol{O}(X)^{\mathrm{op}}, \textbf{\textit{Top}})$ you need either to use explicit fibrant replacements (cf. 2.3) or invoke the strictification theorem 2.11.

Now inside $\textbf{\textit{St}}(X)$ we consider the full subcategory of prestacks F with the property that there exists an open cover $\{U_i\}$ such that $F|_{U_i}$ is a constant stack. This is the category of locally constant stacks $\textbf{\textit{St}}_{\mathrm{loc}}(X)$.

3.3 Remark. $\textbf{\textit{St}}(X)$ has limits and colimits. The same is true for $\textbf{\textit{St}}_{\mathrm{loc}}(X)$.

3.4 Theorem. *(Cf. [11].) (a)* $\mathbb{R}\mathrm{Hom}^{geom}(\textbf{\textit{Top}}, \textbf{\textit{St}}_{loc}(X))$ *is a* 1*-Segal groupoid.*

(b) The geometric realisation of $\mathbb{R}\mathrm{Hom}^{geom}(\textbf{\textit{Top}}, \textbf{\textit{St}}_{loc}(X))$ *is equivalent to X in a natural way. In other words,* $\Pi_\infty(X) \simeq \mathbb{R}\mathrm{Hom}^{geom}(\textbf{\textit{Top}}, \textbf{\textit{St}}_{loc}(X))$.

Again, by $\mathbb{R}\mathrm{Hom}^{\mathrm{geom}}(\textbf{\textit{Top}}, \textbf{\textit{St}}_{\mathrm{loc}}(X))$ we mean morphisms $\textbf{\textit{Top}} \to \textbf{\textit{St}}_{\mathrm{loc}}(X)$ with an *exact* left adjoint.

3.5 Corollary. *If X and Y are CW-complexes, then*

$$|\mathbb{R}\mathrm{Hom}^{geom}(\textbf{\textit{St}}_{loc}(X), \textbf{\textit{St}}_{loc}(Y))| \simeq \mathrm{Map}(X, Y).$$

If C is a Grothendieck site, then the stacks on C form a 1-Segal category $\textbf{\textit{St}}(C)$. Consider the functor

$$\begin{aligned} H_C : \textbf{\textit{Top}} &\longrightarrow \textbf{\textit{Top}} \\ X &\longmapsto |\mathbb{R}\mathrm{Hom}^{\mathrm{geom}}(\textbf{\textit{St}}_{\mathrm{loc}}(C), \textbf{\textit{Top}}/X)| \end{aligned}$$

as an object in $\mathbb{R}\mathrm{Hom}(\textbf{\textit{Top}}, \textbf{\textit{Top}})$. Note that $\textbf{\textit{Top}}/X \simeq \textbf{\textit{St}}_{\mathrm{loc}}(|X|)$.

3.6 Theorem. *(Cf. [11]). The functor* H_C *is pro-representable.*

That is, there exists a pro-object $|C| \in$ ***Pro-sSet*** and a natural equivalence

$$H_C(X) \simeq \operatorname*{hocolim}_i \mathrm{Map}(|C|_i, X).$$

Definition. $|C|$ is called the *pro-homotopy* of C.

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